Discontinuous Galerkin Finite Element Methods for Quasilinear PDEs

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Joint work with
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Universität Bern, 2013

Outline

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1. Introduction
2. Two-Grid Energy Norm Based Adaptivity
   - Two-grid methods for quasilinear elliptic PDEs
   - *hp*-Mesh adaptation
   - Two-grid methods based on a single Newton iteration
3. Non-Newtonian Fluids
   - A priori error bounds
   - A posteriori error bounds and adaptivity
   - Two-grid methods for non-Newtonian fluids
4. Two-Grid DWR Based Adaptivity for Quasilinear Elliptic PDEs

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- They have recently been extended to DGFEMs (Bi & Ginting 2011), which covered *a priori* error analysis.
- *A posteriori* error analysis and, hence, automatic mesh refinement has not been developed. This is the area we are interested in.

Standard Formulation

Nonlinear Problem

Given a semi-linear form $\mathcal{N}(\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u, v) = 0 \qquad \forall v \in V.$$

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Given a semi-linear form $\mathcal{N}(\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u,v) = 0 \qquad \forall v \in V.$$

Create a mesh on the domain and define V_h be the FE space on that mesh, then:

(Standard) Discretisation Method

Find $u_h \in V_h$ such that

$$\mathcal{N}_h(u_h,v_h) = 0 \qquad \forall v_h \in V_h.$$

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Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

Two-Grid Discretisation Method

Find $u_H \in V_H$ such that

$$\mathcal{N}_H(u_H, v_H) = 0 \qquad \forall v_H \in V_H,$$

find $u_{2G} \in V_h$ such that

$$\mathcal{B}_h[u_H](u_{2G}, v_h) = 0 \qquad \forall v_h \in V_h.$$

where, for fixed φ, $\mathcal{B}_h[\varphi](\cdot,\cdot)$ is a linearised approximation to $\mathcal{N}_h(\cdot,\cdot)$.

The nonlinear problem is only solved on a coarse mesh and the fine mesh involves only solving a linear problem; hence, the computational expense of the two grid method should be lower than solving the nonlinear problem on the fine mesh.

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Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Assumption

1. *$\mu \in C(\bar{\Omega} \times [0, \infty))$ and*
2. *there exists positive constants m_μ and M_μ such that*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

- $\mathcal{T}_h$ is a mesh consisting of triangles, quadrilaterals and hexahedral of granularity h.
- *hp*-DG finite element space:

$$V(\mathcal{T}_h, \boldsymbol{k}) = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_{k_\kappa}(\kappa), \forall \kappa \in \mathcal{T}_h\},$$

- $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{B}} \cup \mathcal{F}_h^{\mathcal{I}}$ denotes the set of all faces in the mesh $\mathcal{T}_h$.
- Trace operators

$$\{\!\{\cdot\}\!\} : \text{Average Operator} \qquad [\![\cdot]\!] : \text{Jump Operator}.$$

(Standard) Interior Penalty Method

Find $u_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{h,k}; u_{h,k}, v_{h,k}) = F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

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$$A_{h,k}(\psi; u, v) = \int_\Omega \mu(|\nabla_h \psi|)\nabla_h u \cdot \nabla_h v \, \mathrm{d}\mathbf{x} - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mu(|\nabla \psi|)\nabla u\}\!\} \cdot [\![v]\!] \, \mathrm{d}s$$

$$+\theta \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mu(h_F^{-1}|[\![\psi]\!]|)\nabla v\}\!\} \cdot [\![u]\!] \, \mathrm{d}s + \sum_{F \in \mathcal{F}_h} \int_F \sigma_{h,k} [\![u]\!] \cdot [\![v]\!] \, \mathrm{d}s,$$

$$F_{h,k}(v) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v \, \mathrm{d}\mathbf{x}.$$

where $\theta \in [-1, 1]$. Note: $\theta = 1$ is NIP, $\theta = 0$ is IIP and $\theta = -1$ is SIP.

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Find $u_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

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for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

Interior penalty parameter:

$$\sigma_{h,k} = \gamma \frac{k_F^2}{h_F},$$

where $k_F = \max(k_{\kappa_1}, k_{\kappa_2})$ and h_F is the diameter of the face.

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References:

Bustinza & Gatica 2004, Gatica, Gonzáles & Meddahi 2004, Houston, Robson & Süli 2005, Bustinza, Cockburn & Gatica 2005, Houston, Süli & Wihler 2007, Gudi, Nataraj & Pani 2008

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

$$V(\mathcal{T}_H, \boldsymbol{K}) \subseteq V(\mathcal{T}_h, \boldsymbol{k})$$

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2. Compute the coarse grid approximation $u_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$ such that

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for all $v_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$.

3. Determine the fine grid approximation $u_{2G} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{H,K}; u_{2G}, v_{h,k}) = F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

Theorem (Standard DGFEM)

The following bound holds:

$$\left\| u - u_{h,k} \right\|_{h,k}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2r_\kappa - 2}}{k_\kappa^{2s_\kappa - 3}} \|u\|_{H^{s_\kappa}(\kappa)}^2$$

with $1 \leq r_\kappa \leq \min(k_\kappa + 1, s_\kappa)$*,* $k_\kappa \geq 1$*, for* $\kappa \in \mathcal{T}_h$*.*

Proof.

See Houston, Robson & Süli 2005. ∎

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Theorem (Two-Grid Approximation)

The following bounds hold:

$$\left\| u_{h,k} - u_{2G} \right\|_{h,k}^2 \leq C_2 \sum_{\kappa \in \mathcal{T}_H} \frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|u\|_{H^{S_\kappa}(\kappa)}^2$$

$$\|u - u_{2G}\|_{h,k}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2r_\kappa - 2}}{k_\kappa^{2s_\kappa - 3}} \|u\|_{H^{s_\kappa}(\kappa)}^2 + C_2 \sum_{\kappa \in \mathcal{T}_H} \frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|u\|_{H^{S_\kappa}(\kappa)}^2$$

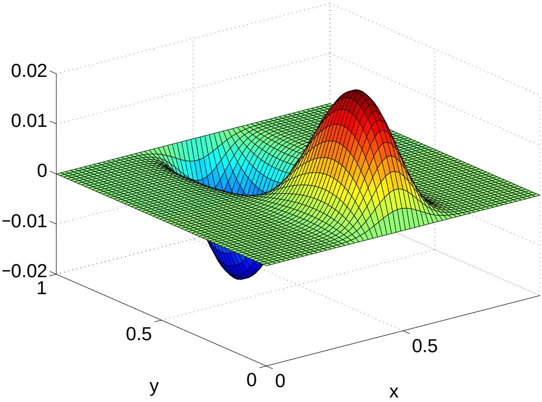
with $1 \leq r_\kappa \leq \min(k_\kappa + 1, s_\kappa)$, $k_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$, *and* $1 \leq R_\kappa \leq \min(K_\kappa + 1, S_\kappa)$, $K_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_H$

Proof.

Based on an extension of the analysis in Houston, Robson & Süli 2005 and Bi & Ginting 2011. ∎

We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2}.$$



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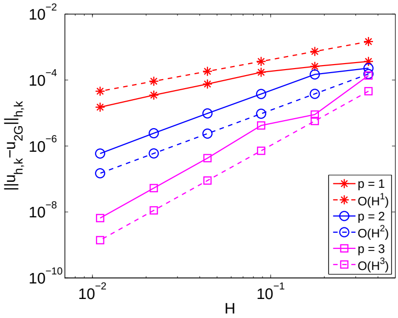

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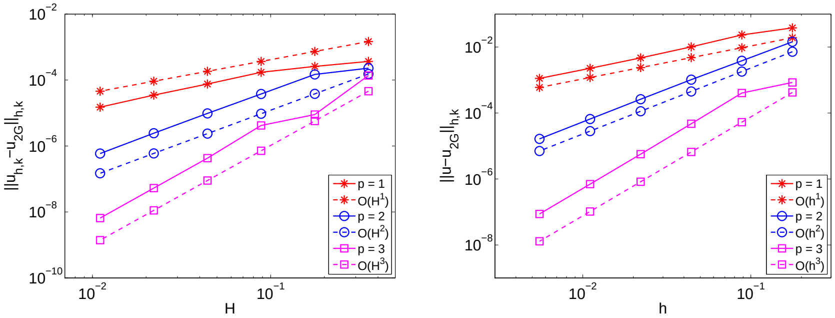


Theorem (Standard Quasilinear DGFEM)

The following bound holds:

$$\left\|u - u_{h,k}\right\|_{h,k}^2 \leq C_3 \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 .$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2} \left\|f + \nabla \cdot \{\mu(|\nabla u_{h,k}|)\nabla u_{h,k}\}\right\|_{L^2(\kappa)}^2 + \frac{h_\kappa}{k_\kappa} \left\|[\![\mu(|\nabla u_{h,k}|)\nabla u_{h,k}]\!]\right\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma^2 \frac{k_\kappa^3}{h_\kappa} \left\|[\![u_{h,k}]\!]\right\|_{L^2(\partial\kappa)}^2$$

Proof.

See Houston, Süli & Wihler 2008. ∎

Theorem (Two-Grid Quasilinear Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{h,k}^2 \leq C_4 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 \right).$$

Here the local fine grid error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2} \left\| f + \nabla \cdot \{\mu(|\nabla u_{H,K}|)\nabla u_{2G}\} \right\|_{L^2(\kappa)}^2 + \frac{h_\kappa}{k_\kappa} \left\| [\![\mu(|\nabla u_{H,K}|)\nabla u_{2G}]\!] \right\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma^2 \frac{k_\kappa^3}{h_\kappa} \left\| [\![u_{2G}]\!] \right\|_{L^2(\partial\kappa)}^2$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \left\| (\mu(|\nabla u_{H,K}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G} \right\|_{L^2(\kappa)}^2.$$

Two-Grid *hp*-Adaptivity

1. Construct the initial coarse and fine FE *hp*-mesh ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation $u_{H,K}$ and two-grid solution u_{2G}.
3. Evaluate the elemental error indicators η_κ and ξ_κ.
4. Select elements in both meshes for refinement/derefinement based on some strategy using both η_κ and ξ_κ.
5. Decide in the marked elements whether to perform *h*- or *p*-refinement/derefinement.
6. Construct the new coarse and fine *hp*-mesh performing smoothing to ensure the coarse space is a subset of the fine space.
7. Goto 2.

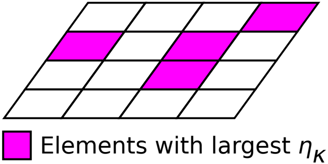
Two strategies have been considered for Step 4.

hp-Mesh Adaptation

- The local fine grid error indicators η_κ are similar to the local error indicators that occur in the standard DGFEM.
  - This suggests that these indicators model the error in the method on the fine grid; hence,
  - these indicators should be used to refine the fine grid.
- The local two-grid error indicators ξ_κ appear to model the error in using the coarse grid solution $u_{H,K}$ in the nonlinearity.
  - This suggests these indicators model the error committed in the difference between the fine and coarse meshes; hence,
  - these indicators should be used to refine the coarse grid.

hp-Mesh Adaptation (Strategy 1)

- Perform standard refinement on the fine mesh based on η_κ



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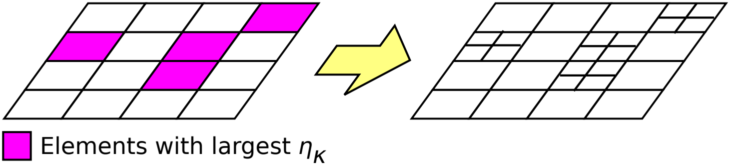

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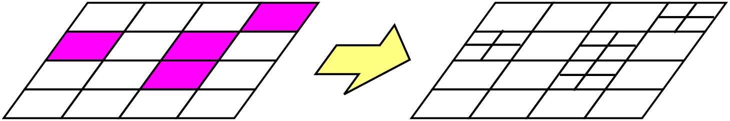
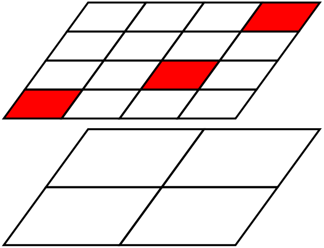


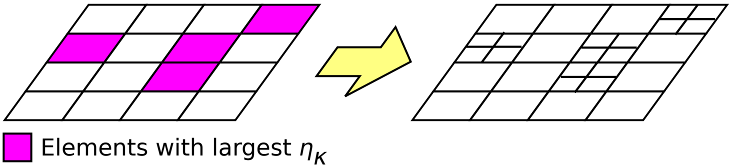
Elements with largest η_K

- For each fine element $\kappa \in \mathcal{T}_h$ where $\lambda\xi_\kappa \geq \eta_\kappa$, $\lambda \geq 0$ refine the coarse element $\kappa_H \in \mathcal{T}_H$ where $\kappa \subseteq \kappa_H$.



Elements with $\lambda\xi_K \geq \eta_K$

- Perform standard refinement on the fine mesh based on η_κ



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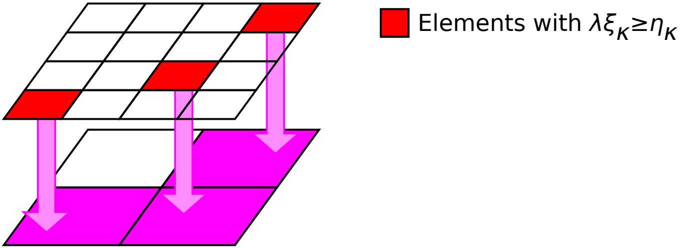
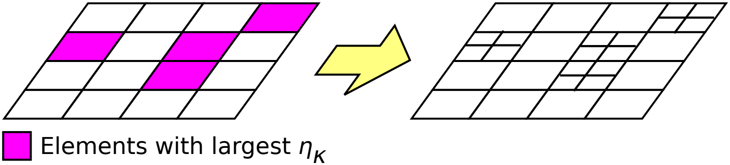


hp-Mesh Adaptation (Strategy 1)

- Perform standard refinement on the fine mesh based on η_κ



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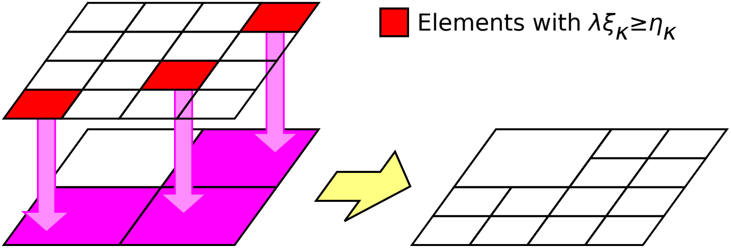
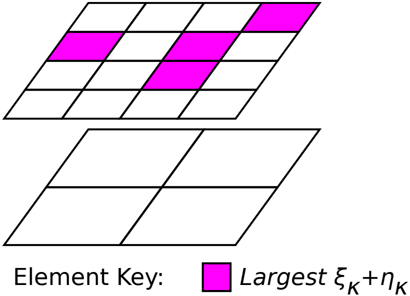
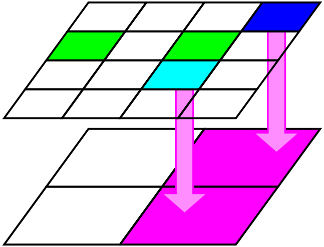




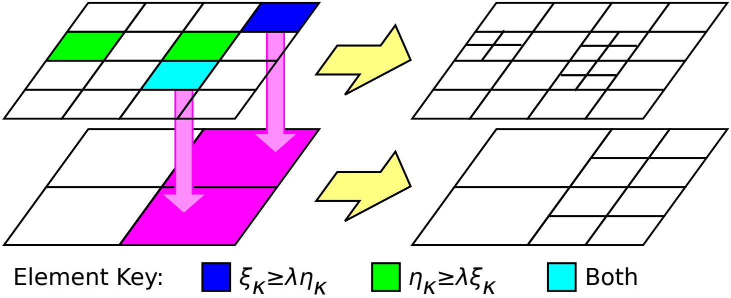
- Use $\eta_\kappa + \xi_\kappa$ to calculate the 'fine' elements which need refining.



Element Key: $\xi_\kappa \geq \lambda \eta_\kappa$ $\eta_\kappa \geq \lambda \xi_\kappa$ Both

- Use $\eta_\kappa + \xi_\kappa$ to calculate the 'fine' elements which need refining.
- For each 'fine' element $\kappa \in \mathcal{T}_h$ marked for refinement decide whether to refine that element or the 'parent' coarse element:
  - if $\lambda_F \xi_\kappa \leq \eta_\kappa$ select the fine element, and/or,
  - if $\lambda_C \eta_\kappa \leq \xi_\kappa$ select the coarse element,

  where $\lambda_C, \lambda_F \in (0, 1]$.



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  - if $\lambda_C \eta_\kappa \leq \xi_\kappa$ select the coarse element,

  where $\lambda_C, \lambda_F \in (0, 1]$.
- Refine the meshes.

We let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$,

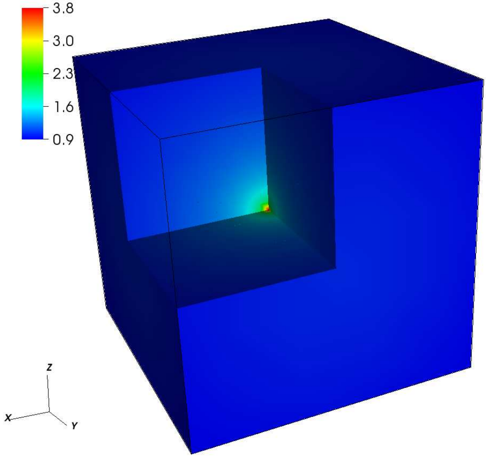
$$\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1 + |\nabla u|^2}$$

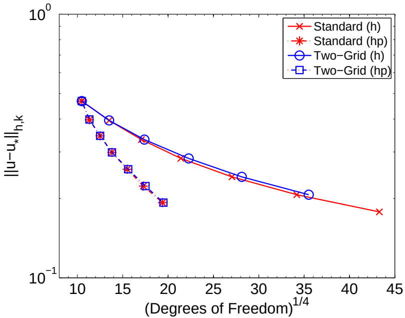
and select f so that

$$u(\boldsymbol{x}) = (x^2 + y^2 + z^2)^{q/2}, \quad q \in \mathbb{R};$$

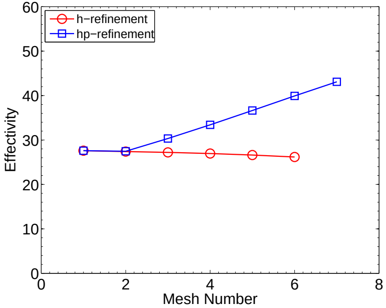
for $q > -1/2, u \in H^1(\Omega)$. Here, we select $q = -1/4$.

Beilina, Korotov & Křížek 2005



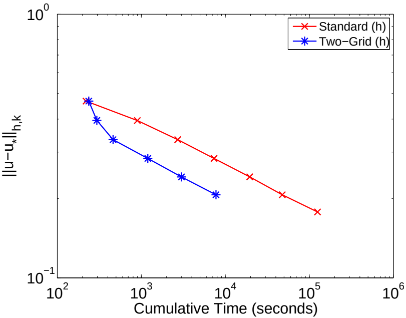


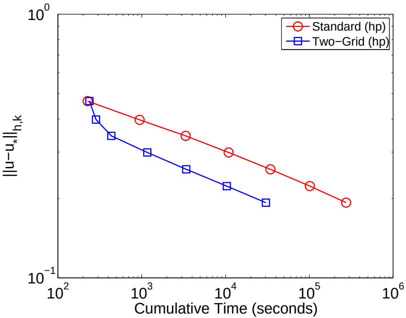
10^0
10^{-1}
$\|u-u_*\|_{h,k}$
10
15
20
25
30
35
40
45
(Degrees of Freedom)$^{1/4}$
Standard (h)
Standard (hp)
Two−Grid (h)
Two−Grid (hp)



h−refinement
hp−refinement
Effectivity
0
10
20
30
40
50
60
0
2
4
6
8
Mesh Number

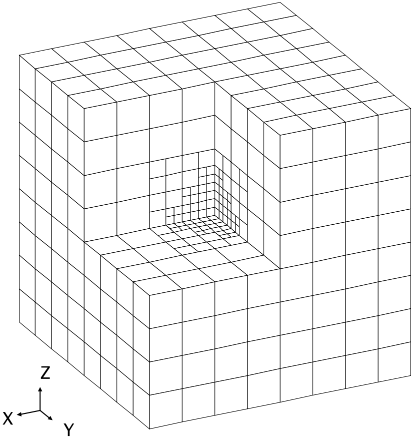
Quasilinear PDE: Singular Solution



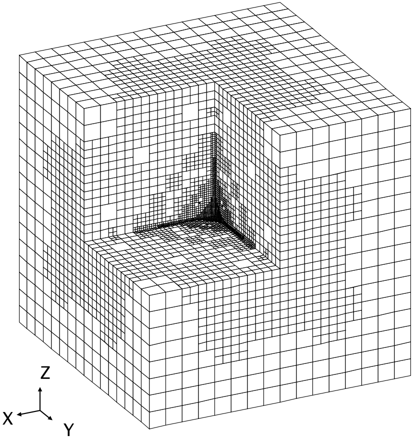


Quasilinear PDE: Singular Solution

h–Mesh after 5 adaptive refinements

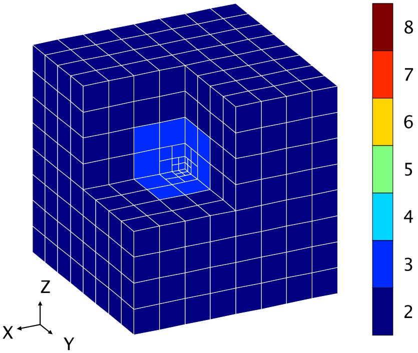


Coarse Mesh

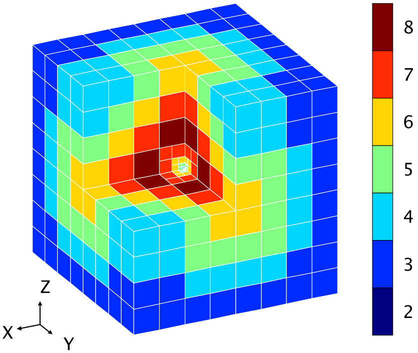


Fine Mesh

hp–Mesh after 6 adaptive refinements



Coarse Mesh



Fine Mesh

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

$$V(\mathcal{T}_H, \boldsymbol{K}) \subseteq V(\mathcal{T}_h, \boldsymbol{k})$$

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

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2. Compute the coarse grid approximation $u_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$ such that

$$A_{H,K}(u_{H,K}, v_{H,K}) = F_{H,K}(v_{H,K})$$

for all $v_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$.

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Two-Grid Approximation

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for all $v_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$.

3. Determine the fine grid approximation $u_{2G} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A'_{h,k}[u_{H,K}](u_{2G}, v_{h,k}) = A'_{h,k}[u_{H,K}](u_{H,K}, v_{h,k}) - A_{h,k}(u_{H,K}, v_{h,k}) + F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

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Theorem (Two-Grid based on a Single Newton Iteration)

On a uniform mesh of size h, with polynomial degree k the following bounds hold:

$$\|u_{h,k} - u_{2G}\|_{h,k} \leq C_5 \frac{k^{7/2}}{h} \frac{H^{2R-2}}{K^{2S-3}} \|u\|^2_{H^S(\Omega)}$$

$$\|u - u_{2G}\|_{h,k} \leq C_1 \frac{h_\kappa^{s-1}}{k^{s-3/2}} \|u\|_{H^s(\Omega)} + C_5 \frac{k^{7/2}}{h} \frac{H^{2R-2}}{K^{2S-3}} \|u\|^2_{H^S(\Omega)}$$

with $1 \leq r \leq \min(k+1, s)$ *and* $1 \leq R \leq \min(K+1, S)$.

Proof.

See C., & Houston 2013. ■

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Theorem (Two-Grid based on a Single Newton Iteration)

$$\|u - u_{2G}\|_{h,k}^2 \leq C_6 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2\right).$$

Here the local fine grid error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = h_\kappa^2 k_\kappa^{-2} \left\| f + \nabla \cdot \{\mu(|\nabla u_{h,k}|)\nabla u_{2G}\}\right\|_{L^2(\kappa)}^2 + h_\kappa k_\kappa^{-1} \left\| [\![\mu(|\nabla u_{h,k}|)\nabla u_{2G}]\!]\right\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma^2 k_\kappa^3 h_\kappa^{-1} \left\|[\![u_{2G}]\!]\right\|_{L^2(\partial\kappa)}^2$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \left\|(\mu(|\nabla u_{H,K}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G}\right\|_{L^2(\kappa)}^2 + \left\|\left(\mu'_{\nabla u}(|\nabla u_{H,K}|) \cdot (\nabla u_{2G} - u_{H,K})\right) \nabla u_{H,K}\right\|_{L^2(\kappa)}^2 + h_\kappa k_\kappa^{-1} \left\|\left(\mu'_{\nabla u}(|\nabla u_{H,K}|) \cdot (\nabla u_{2G} - u_{H,K})\right) \nabla u_{H,K}\right\|_{L^2(\partial\kappa)}^2.$$

Outline

1. Introduction
2. Two-Grid Energy Norm Based Adaptivity
   - Two-grid methods for quasilinear elliptic PDEs
   - *hp*-Mesh adaptation
   - Two-grid methods based on a single Newton iteration
3. Non-Newtonian Fluids
   - A priori error bounds
   - A posteriori error bounds and adaptivity
   - Two-grid methods for non-Newtonian fluids
4. Two-Grid DWR Based Adaptivity for Quasilinear Elliptic PDEs

Non-Newtonian Fluid Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $\boldsymbol{f} \in L^2(\Omega)^d$, find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, |\underline{e}(\boldsymbol{u})|)\underline{e}(\boldsymbol{u})\} + \nabla p = \boldsymbol{f} & \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \quad \text{on } \Gamma, \end{aligned}$$

where $\underline{e}(\boldsymbol{u})$ is the *symmetric* $d \times d$ *strain tensor* defined by

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Assumption

1. *$\mu \in C(\bar{\Omega} \times [0, \infty))$ and*
2. *there exists positive constants m_μ and M_μ such that*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

- *hp*-DG finite element space:

$$\boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_\kappa \in \mathcal{S}_{k_\kappa}(\kappa)^d, \forall \kappa \in \mathcal{T}_h\},$$
$$Q(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L_0^2(\Omega) : q|_\kappa \in \mathcal{S}_{k_\kappa - 1}(\kappa), \forall \kappa \in \mathcal{T}_h\}.$$

- Jump operator: $\underline{[\![\boldsymbol{v}]\!]} = \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-$

- *hp*-DG finite element space:

$$\boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_\kappa \in \mathcal{S}_{k_\kappa}(\kappa)^d, \forall \kappa \in \mathcal{T}_h\},$$
$$Q(\mathcal{T}_h,\boldsymbol{k}) = \{q \in L^2_0(\Omega) : q|_\kappa \in \mathcal{S}_{k_\kappa-1}(\kappa), \forall \kappa \in \mathcal{T}_h\}.$$

- Jump operator: $\underline{[\![\boldsymbol{v}]\!]} = \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-$

(Standard) Interior Penalty Method

Find $(\boldsymbol{u}_{h,k}, p_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$ such that

$$\begin{aligned} A_{h,k}(\boldsymbol{u}_{h,k};\boldsymbol{u}_{h,k},\boldsymbol{v}_{h,k}) + B_{h,k}(\boldsymbol{v}_{h,k},p_{h,k}) &= F_{h,k}(\boldsymbol{v}_{h,k}) \\ -B_{h,k}(\boldsymbol{u}_{h,k},q_{h,k}) &= 0 \end{aligned}$$

for all $(\boldsymbol{v}_{h,k}, q_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$.

Theorem (Well-Posedness)

Provided that the penalty parameter γ is chosen sufficiently large, and the inf-sup condition,

$$\inf_{0\neq q\in Q(\mathcal{T}_h,\boldsymbol{k})} \sup_{\boldsymbol{0}\neq\boldsymbol{v}\in\boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k})} \frac{B_h(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{h,k}\,\|q\|_{0,\Omega}} \geq c\left(\max_{\kappa\in\mathcal{T}_h} k_\kappa\right)^{-1},$$

holds then exactly one solution $(\boldsymbol{u}_{h,k}, p_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$ of the above hp-DGFEM exists.

Proof.

As the inf-sup condition can be shown to hold (Schotzau, Schwab & Toselli (2002)), then existence of a unique solution follows, see C., Houston, Süli & Wihler (2013). ■

Theorem (Standard Non-Newtonian DGFEM)

Providing the inf-sup condition is valid the following bound holds:

$$\left\|(\boldsymbol{u} - \boldsymbol{u}_{h,k}, p - p_{h,k})\right\|_{DG}^2 \leq C_7 k_{\max}^4 \sum_{\kappa \in \mathcal{T}_h} \left(\frac{h_\kappa^{2r_\kappa - 2}}{k_\kappa^{2s_\kappa - 3}} \|\boldsymbol{u}\|_{H^{s_\kappa}(\kappa)}^2 + \frac{h_\kappa^{2r_\kappa - 2}}{k_\kappa^{2s_\kappa - 2}} \|p\|_{H^{s_\kappa - 1}(\kappa)}^2 \right),$$

with $1 \leq r_\kappa \leq \min(k_\kappa + 1, s_\kappa)$, $k_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$.

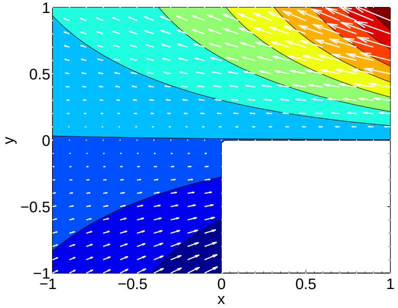
Proof.

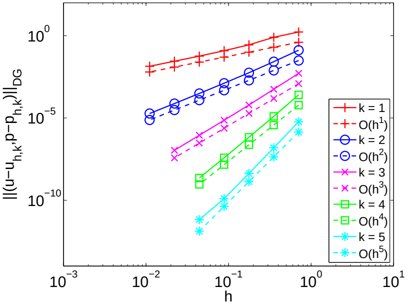
See C., Houston, Süli & Wihler (2013). ∎

We let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select $\boldsymbol{f}$ so that

$$\boldsymbol{u}(x,y) = \begin{pmatrix} -e^x(y\cos y + \sin y) \\ e^x y \sin y \end{pmatrix},$$

$$p(x,y) = 2e^x \sin y - \frac{2(1-e)(\cos 1 - 1)}{3}.$$





Theorem (Standard Non-Newtonian DGFEM)

The following bound holds:

$$\left\|(\boldsymbol{u}-\boldsymbol{u}_{h,k},p-p_{h,k})\right\|_{DG}^2 \leq C_8 \sum_{\kappa\in\mathcal{T}_h} \eta_\kappa^2.$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2}\left\|\boldsymbol{f}+\nabla\cdot\{\mu(|\underline{e}(\boldsymbol{u}_{h,k})|)\underline{e}(\boldsymbol{u}_{h,k})\}-\nabla p_{h,k}\right\|_{L^2(\kappa)}^2 + \left\|\nabla\cdot\boldsymbol{u}_{h,k}\right\|_{L^2(\kappa)}^2$$
$$+\frac{h_\kappa}{k_\kappa}\left\|[\![p_{h,k}]\!]-[\![\mu(|\underline{e}(\boldsymbol{u}_{h,k})|)\underline{e}(\boldsymbol{u}_{h,k})]\!]\right\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma^2\frac{k_\kappa^3}{h_\kappa}\left\|\underline{[\![\boldsymbol{u}_{h,k}]\!]}\right\|_{L^2(\partial\kappa)}^2$$

Proof.

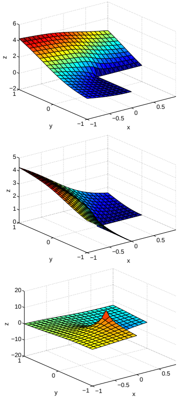
See C., Houston, Süli & Wihler (2013). ∎

Non-Newtonian Fluid: Singular Solution

Let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu = 1 + e^{-|\underline{e}(\boldsymbol{u})|}$ and select $\boldsymbol{f}$ so that

$$\boldsymbol{u}(x,y) = r^{\lambda} \begin{pmatrix} (1+\lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi) \\ \sin(\varphi)\Psi'(\varphi) - (1+\lambda)\cos(\varphi)\Psi(\varphi) \end{pmatrix},$$

$$p(x,y) = -r^{\lambda-1}\left\{(1+\lambda)^2\Psi'(\varphi) + \Psi'''(\varphi)\right\}/(1-\lambda),$$



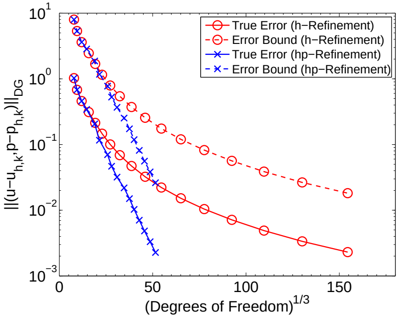
where (r,φ) denotes polar coordinates,

$$\Psi(\varphi) = \frac{\sin((1+\lambda)\varphi)\cos(\lambda\omega)}{1+\lambda} - \cos((1+\lambda)\varphi) - \frac{\sin((1-\lambda)\varphi)\cos(\lambda\omega)}{1-\lambda} + \cos((1-\lambda)\varphi),$$

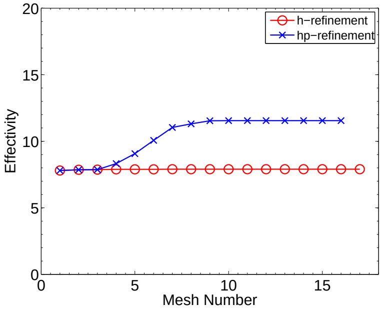
and $\omega = \frac{3\pi}{2}$. Here, the exponent λ is the smallest positive solution of

$$\sin(\lambda\omega) + \lambda\sin(\omega) = 0;$$

thereby, $\lambda \approx 0.54448373678$. Note that $\boldsymbol{u} \notin H^2(\Omega)^2$ and $p \notin H^1(\Omega)$.

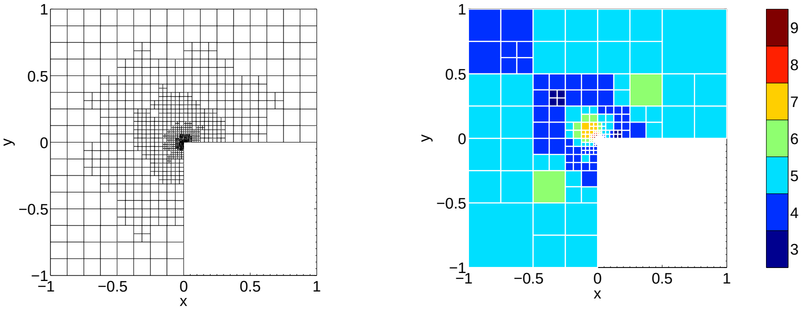


True Error (h−Refinement)
Error Bound (h−Refinement)
True Error (hp−Refinement)
Error Bound (hp−Refinement)
$\|(u-u_{h,k}, p-p_{h,k})\|_{DG}$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
50
100
150
(Degrees of Freedom)$^{1/3}$



h−refinement
hp−refinement
Effectivity
20
15
10
5
0
0
5
10
15
Mesh Number

Non-Newtonian Fluid: Singular Solution



Two-Grid Approximation

1. Construct $\boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K})$, $Q(\mathcal{T}_H,\boldsymbol{K})$, $\boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k})$ and $Q(\mathcal{T}_h,\boldsymbol{k})$ such that

$$\boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K}) \subseteq \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \quad \text{and} \quad Q(\mathcal{T}_H,\boldsymbol{K}) \subseteq Q(\mathcal{T}_h,\boldsymbol{k})$$

2. Compute $(\boldsymbol{u}_{H,K}, p_{H,K}) \in \boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K}) \times Q(\mathcal{T}_H,\boldsymbol{K})$ such that

$$\begin{aligned} A_{H,K}(\boldsymbol{u}_{H,K};\boldsymbol{u}_{H,K},\boldsymbol{v}_{H,K}) + B_{H,K}(\boldsymbol{v}_{H,K},p_{H,K}) &= F_{H,K}(\boldsymbol{v}_{H,K}), \\ -B_{H,K}(\boldsymbol{u}_{H,K},q_{H,K}) &= 0 \end{aligned}$$

   for all $(\boldsymbol{v}_{H,K}, q_{H,K}) \in \boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K}) \times Q(\mathcal{T}_H,\boldsymbol{K})$.

3. Determine $(\boldsymbol{u}_{2G}, p_{2G}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$ such that

$$\begin{aligned} A_{h,k}(\boldsymbol{u}_{H,K};\boldsymbol{u}_{2G},\boldsymbol{v}_{h,k}) + B_{h,k}(\boldsymbol{v}_{h,k},p_{2G}) &= F_{h,k}(\boldsymbol{v}_{h,k}), \\ -B_{H,K}(\boldsymbol{u}_{2G},q_{h,k}) &= 0 \end{aligned}$$

   for all $(\boldsymbol{v}_{h,k}, q_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$.

Theorem (Standard Non-Newtonian DGFEM)

Providing the inf-sup condition is valid the following bound holds:

$$\left\|(\boldsymbol{u}-\boldsymbol{u}_{h,k},p-p_{h,k})\right\|_{DG}^2 \leq C_6 k_{\max}^4 \sum_{\kappa\in\mathcal{T}_h}\left(\frac{h_\kappa^{2r_\kappa-2}}{k_\kappa^{2s_\kappa-3}}\|\boldsymbol{u}\|_{H^{s_\kappa}(\kappa)}^2 + \frac{h_\kappa^{2r_\kappa-2}}{k_\kappa^{2s_\kappa-2}}\|p\|_{H^{s_\kappa-1}(\kappa)}^2\right),$$

with $1 \leq r_\kappa \leq \min(k_\kappa+1, s_\kappa)$, $k_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$.

Proof.

See C., Houston, Süli & Wihler (2013). ■

Theorem (Two-Grid Non-Newtonian DGFEM)

Providing the inf-sup condition is valid the following bounds hold:

$$\|\mathbf{u}_{h,k} - \mathbf{u}_{2G}\|_{h,k}^2 \leq C_8 k_{\max}^4 \sum_{\kappa \in \mathcal{T}_h} \left(\frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|\mathbf{u}\|_{H^{S_\kappa}(\kappa)}^2 + \frac{h_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 2}} \|p\|_{H^{S_\kappa - 1}(\kappa)}^2 \right),$$

$$\|p_{h,k} - p_{2G}\|_{L^2(\Omega)}^2 \leq C_9 k_{\max}^6 \sum_{\kappa \in \mathcal{T}_h} \left(\frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|\mathbf{u}\|_{H^{S_\kappa}(\kappa)}^2 + \frac{h_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 2}} \|p\|_{H^{S_\kappa - 1}(\kappa)}^2 \right),$$

$$\|(\mathbf{u} - \mathbf{u}_{2G}, p - p_{2G})\|_{DG}^2 \leq C_6 k_{\max}^4 \sum_{\kappa \in \mathcal{T}_h} \left(\frac{h_\kappa^{2r_\kappa - 2}}{k_\kappa^{2s_\kappa - 3}} \|\mathbf{u}\|_{H^{s_\kappa}(\kappa)}^2 + \frac{h_\kappa^{2r_\kappa - 2}}{k_\kappa^{2s_\kappa - 2}} \|p\|_{H^{s_\kappa - 1}(\kappa)}^2 \right)$$

$$+ C_{10} k_{\max}^6 \sum_{\kappa \in \mathcal{T}_h} \left(\frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|\mathbf{u}\|_{H^{S_\kappa}(\kappa)}^2 + \frac{h_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 2}} \|p\|_{H^{S_\kappa - 1}(\kappa)}^2 \right),$$

with $1 \leq r_\kappa \leq \min(k_\kappa + 1, s_\kappa)$, $k_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$, *and* $1 \leq R_\kappa \leq \min(K_\kappa + 1, S_\kappa)$, $K_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_H$

Theorem (Standard Non-Newtonian DGFEM)

The following bound holds:

$$\left\|(\boldsymbol{u}-\boldsymbol{u}_{h,k},p-p_{h,k})\right\|_{DG}^2 \leq C_7 \sum_{\kappa\in\mathcal{T}_h} \eta_\kappa^2 \quad .$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2}\left\|\boldsymbol{f}+\nabla\cdot\{\mu(|\underline{e}(\boldsymbol{u}_{h,k})|)\underline{e}(\boldsymbol{u}_{h,k})\}-\nabla p_{h,k}\right\|_{L^2(\kappa)}^2+\left\|\nabla\cdot\boldsymbol{u}_{h,k}\right\|_{L^2(\kappa)}^2$$
$$+\frac{h_\kappa}{k_\kappa}\left\|[\![p_{h,k}]\!]-[\![\mu(|\underline{e}(\boldsymbol{u}_{h,k})|)\underline{e}(\boldsymbol{u}_{h,k})]\!]\right\|_{L^2(\partial\kappa\setminus\Gamma)}^2+\gamma^2\frac{k_\kappa^3}{h_\kappa}\left\|\underline{[\![\boldsymbol{u}_{h,k}]\!]}\right\|_{L^2(\partial\kappa)}^2$$

Proof.

See C., Houston, Süli & Wihler (2013). ■

Theorem (Two-Grid Non-Newtonian DGFEM)

The following bound holds:

$$\|(\boldsymbol{u}-\boldsymbol{u}_{2G}, p-p_{2G})\|_{DG}^2 \leq C_{11} \sum_{\kappa\in\mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2\right).$$

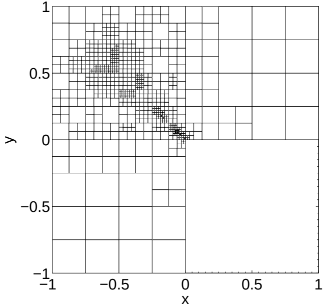
Here the local fine grid error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2}\left\|\boldsymbol{f} + \nabla\cdot\{\mu(|\underline{e}(\boldsymbol{u}_{H,K})|)\underline{e}(\boldsymbol{u}_{2G})\} - \nabla p_{2G}\right\|_{L^2(\kappa)}^2 + \|\nabla\cdot\boldsymbol{u}_{2G}\|_{L^2(\kappa)}^2$$
$$+ \frac{h_\kappa}{k_\kappa}\left\|[\![p_{2G}]\!] - [\![\mu(|\underline{e}(\boldsymbol{u}_{H,K})|)\underline{e}(\boldsymbol{u}_{2G})]\!]\right\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma^2\frac{k_\kappa^3}{h_\kappa}\left\|\underline{[\![\boldsymbol{u}_{2G}]\!]}\right\|_{L^2(\partial\kappa)}^2$$

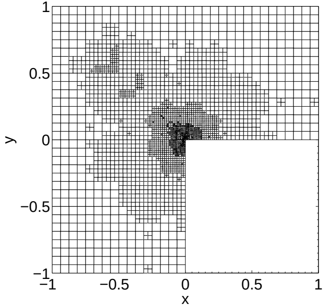
and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\xi_\kappa^2 = \left\|(\mu(|\underline{e}(\boldsymbol{u}_{H,K})|) - \mu(|\underline{e}(\boldsymbol{u}_{2G})|))\underline{e}(\boldsymbol{u}_{2G})\right\|_{L^2(\kappa)}^2.$$

h–Mesh after 11 adaptive refinements



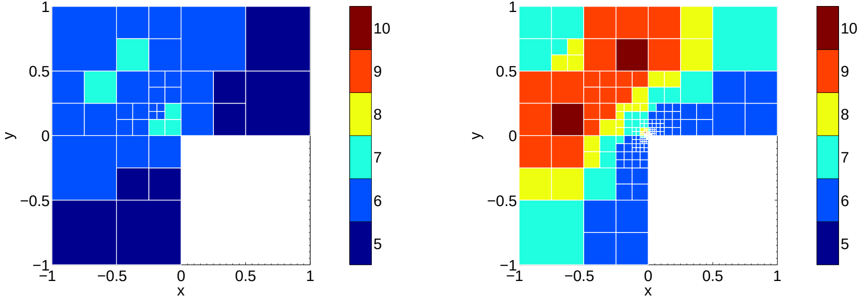
Coarse Mesh



Fine Mesh

Non-Newtonian Fluid: Singular Solution

hp–Mesh after 11 adaptive refinements



Coarse Mesh

Fine Mesh

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- A posteriori error bounds and adaptivity
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4 Two-Grid DWR Based Adaptivity for Quasilinear Elliptic PDEs

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, u, \nabla u)\nabla u\} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Remark

Here we do not enforce any condition on the nonlinearity μ*.*

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

$$V(\mathcal{T}_H, \boldsymbol{K}) \subseteq V(\mathcal{T}_h, \boldsymbol{k})$$

2. Compute the coarse grid approximation $u_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$ such that

$$A_{H,K}(u_{H,K}, v_{H,K}) = F_{H,K}(v_{H,K})$$

   for all $v_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$.

3. Determine the fine grid approximation $u_{2G} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A'_{h,k}[u_{H,K}](u_{2G}, v_{h,k}) = A'_{h,k}[u_{H,K}](u_{H,K}, v_{h,k}) - A_{h,k}(u_{H,K}, v_{h,k}) + F_{h,k}(v_{h,k})$$

   for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

We can find a *a posteriori* error estimate by introducing the dual:

(Fine Grid) Dual Problem

Find $\varphi \in H_0^1(\Omega)$ such that

$$A'_{h,k}[u_{H,K}](v, \varphi) = J(v)$$

for all $v \in H_0^1(\Omega)$, where $J(\cdot)$ is a linear functional.

We can find a *a posteriori* error estimate by introducing the dual:

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with it's associated approximation:

(Fine Grid) Dual Approximation

Find $\varphi_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A'_{h,k}[u_{H,K}](v_{h,k},\varphi_{h,k}) = J(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

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Theorem

For a given linear functional $J(\cdot)$ we can estimate the error in the two grid approximation with:

$$J(u) - J(u_{2G}) \approx F_{h,k}(\varphi - \varphi_{h,k}) + A'_{h,k}[u_{H,K}](u_{H,K} - u_{2G}, \varphi - \varphi_{h,k}) - A_{h,k}(u_{H,K}, \varphi - \varphi_{h,k}) - \mathcal{Q}(u_{H,K}, u_{2G}, \varphi)$$

where

$$\mathcal{Q}(v, w, \varphi) = \int_0^1 (1-t) A''_{h,k}[v + t(w-v)](w-v, w-v, \varphi)\, \mathrm{d}t.$$

Remark

We note that $\mathcal{Q}(v, w, \varphi)$ is the remainder from a 1st order Taylor's expansion, about 0*, of the function $\eta(t) = A_{h,k}(v + t(w-v), \varphi)$ evaluated at* 1*.*

Selecting $\mu = |\nabla u|^{p-2}$, for $p \in (0, \infty)$ gives rise to the *p*-Laplacian:

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d, d = 2, 3$ and a smooth given data $f \in L^2(\Omega)$, find u such that

$$\begin{aligned} -\nabla \cdot \{|\nabla u|^{p-2} \nabla u\} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $p \in (0, \infty)$.

We consider the domain $\Omega = (0, 1)^2$ with $p = 3$ and select the forcing function such that the analytical solution is $u = r^{3/4}$. This results in a singularity at the origin.

Ainsworth & Kay 1999

We select the linear functional as a point functional near the singularity,

$$J(u) = u(0.01, 0.01).$$

DoFs (F)	DoFs (C)	$J(u) - J(u_{2G})$	$\mathcal{E}(u, u_{H,K}, u_{2G})$	Eff.
144	144	0.3718×10^{-3}	0.2306×10^{-2}	6.20
252	144	0.1649×10^{-4}	0.2198×10^{-2}	133.23
387	252	-0.1000×10^{-2}	0.1228×10^{-3}	-0.12
657	360	-0.1801×10^{-3}	-0.6972×10^{-3}	3.87
1008	603	0.1506×10^{-2}	0.1507×10^{-2}	1.00
1575	1062	0.1676×10^{-2}	0.1306×10^{-2}	0.78
2574	1548	0.2524×10^{-3}	0.2264×10^{-3}	0.90
4356	2439	0.2977×10^{-3}	0.2640×10^{-3}	0.89
7785	3789	0.1221×10^{-3}	0.1138×10^{-3}	0.93
13293	6732	0.2540×10^{-4}	0.2741×10^{-4}	1.08
22986	11673	0.8779×10^{-5}	0.9309×10^{-5}	1.06
41130	20556	0.3225×10^{-5}	0.3361×10^{-5}	1.04
73692	36243	0.1158×10^{-5}	0.1192×10^{-5}	1.03
132498	64620	0.4171×10^{-6}	0.4258×10^{-6}	1.02
244035	120942	0.1668×10^{-6}	0.1692×10^{-6}	1.01

- Summary:
  - *A priori* and *a posteriori* error analysis for non-Newtonian fluids
  - Two-grid *h*-/*hp*-DGFEMs proposed for quasilinear/non-Newtonian.
  - Energy norm *a priori* and *a posteriori* error analysis of two-grid method.
  - Dual weighted residual *a posteriori* error analysis for two-grid.
  - Two-grid *h*-/*hp*-adaptive algorithms developed to control the discretization error in both the coarse and fine grid solutions.

Summary and Future Work

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- Future Work:
  - 3D two-grid dual weighted residual
  - Two-grid dual weighted residual for non-Newtonian fluids.
  - Compressible flows.