Error Analysis of a Two-Grid Nonlinear *hp*–Version Discontinuous Galerkin Finite Element Method

Scott Congreve

School of Mathematical Sciences, University of Nottingham, UK

Joint work with
Paul Houston (University of Nottingham) and Thomas Wihler (University of Bern)

ENUMATH 2011

Outline

Standard Formulation

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u;u,v) = 0 \qquad \forall v \in V.$$

Nonlinear Problem

Given a semilinear form $\mathcal{N}(\cdot;\cdot,\cdot)$, find $u \in V$ such that

$$\mathcal{N}(u;u,v) = 0 \qquad \forall v \in V.$$

Define V_h be the FE space on the mesh, then:

(Standard) Discretization Method

Find $u_h \in V_h$ such that

$$\mathcal{N}_h(u_h;u_h,v_h) = 0 \qquad \forall v_h \in V_h.$$

Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

Two-Grid Discretization Method

Find $u_H \in V_H$ such that

$$\mathcal{N}_H(u_H; u_H, v_H) = 0 \qquad \forall v_H \in V_H,$$

Two-Grid Methods

Create a mesh which is 'coarser' than the original mesh and define V_H as the FE space on this mesh, then:

Two-Grid Discretization Method

Find $u_H \in V_H$ such that

$$\mathcal{N}_H(u_H; u_H, v_H) = 0 \qquad \forall v_H \in V_H,$$

find $u_{2G} \in V_h$ such that

$$\mathcal{N}_h(u_H; u_{2G}, v_h) = 0 \qquad \forall v_h \in V_h.$$

Xu 1992, 1994, 1996, Xu & Zhou 1999, Axelsson & Layton 1996, Dawson, Wheeler & Woodward 1998, Utnes 1997, Marion & Xu 1995, Wu & Allen 1999, Bi & Ginting 2007, 2011

Quasilinear Problem

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and $f \in L^2(\Omega)$, find u such that

$$\begin{aligned} -\nabla \cdot \{\mu(\boldsymbol{x}, |\nabla u|)\nabla u\} &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

Assumption

1. $\mu \in C(\bar{\Omega} \times [0, \infty))$ *and*
2. *there exists positive constants* m_μ *and* M_μ *such that*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

- $\mathcal{T}_h$ is a mesh consisting of triangles/parallelograms of granularity h.
- *hp*-DG finite element space:

$$V(\mathcal{T}_h, \boldsymbol{k}) = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{S}_{k_\kappa}(\kappa), \forall \kappa \in \mathcal{T}_h\},$$

- $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{B}} \cup \mathcal{F}_h^{\mathcal{I}}$ denotes the set of all faces in the mesh $\mathcal{T}_h$.
- Trace operators

$$\{\!\{\cdot\}\!\} : \text{Average Operator} \qquad [\![\cdot]\!] : \text{Jump Operator.}$$

(Standard) Interior Penalty Method

Find $u_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{h,k}; u_{h,k}, v_{h,k}) = F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

(Standard) Interior Penalty Method

Find $u_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{h,k}; u_{h,k}, v_{h,k}) = F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

- Forms:

$$A_{h,k}(\psi; u, v) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \mu(|\nabla \psi|) \nabla u \cdot \nabla v \, d\boldsymbol{x} + \sum_{F \in \mathcal{F}_h} \int_F \sigma_{h,k} [\![u]\!] \cdot [\![v]\!] \, ds$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\mu(|\nabla \psi|) \nabla u\}\!\} \cdot [\![v]\!] + \{\!\{\mu(h_F^{-1} |[\![\psi]\!]|) \nabla v\}\!\} \cdot [\![u]\!] \right) ds,$$

$$F_{h,k}(v) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v \, d\boldsymbol{x}.$$

(Standard) Interior Penalty Method

Find $u_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{h,k}; u_{h,k}, v_{h,k}) = F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

- Interior penalty parameter:

$$\sigma_{h,k} = \gamma \frac{k_F^2}{h_F},$$

where $k_F = \max(k_{\kappa_1}, k_{\kappa_2})$ and h_F is the diameter of the face.

(Standard) Interior Penalty Method

Find $u_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{h,k}; u_{h,k}, v_{h,k}) = F_{h,k}(v_{h,k})$$

for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

- Interior penalty parameter:

$$\sigma_{h,k} = \gamma \frac{k_F^2}{h_F},$$

  where $k_F = \max(k_{\kappa_1}, k_{\kappa_2})$ and h_F is the diameter of the face.

- References:

  Bustinza & Gatica 2004, Gatica, Gonzáles & Meddahi 2004, Houston, Robson & Suli 2005, Bustinza, Cockburn & Gatica 2005, Houston, Süli & Wihler 2007, Gudi, Nataraj & Pani 2008

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

$$V(\mathcal{T}_H, \boldsymbol{K}) \subseteq V(\mathcal{T}_h, \boldsymbol{k})$$

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

$$V(\mathcal{T}_H, \boldsymbol{K}) \subseteq V(\mathcal{T}_h, \boldsymbol{k})$$

2. Compute the coarse grid approximation $u_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$ such that

$$A_{H,K}(u_{H,K}; u_{H,K}, v_{H,K}) = F_{H,K}(v_{H,K})$$

for all $v_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$.

Two-Grid Approximation

1. Construct coarse and fine FE spaces $V(\mathcal{T}_H, \boldsymbol{K})$ and $V(\mathcal{T}_h, \boldsymbol{k})$, respectively, such that

$$V(\mathcal{T}_H, \boldsymbol{K}) \subseteq V(\mathcal{T}_h, \boldsymbol{k})$$

2. Compute the coarse grid approximation $u_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$ such that

$$A_{H,K}(u_{H,K}; u_{H,K}, v_{H,K}) = F_{H,K}(v_{H,K})$$

   for all $v_{H,K} \in V(\mathcal{T}_H, \boldsymbol{K})$.

3. Determine the fine grid approximation $u_{2G} \in V(\mathcal{T}_h, \boldsymbol{k})$ such that

$$A_{h,k}(u_{H,K}; u_{2G}, v_{h,k}) = F_{h,k}(v_{h,k})$$

   for all $v_{h,k} \in V(\mathcal{T}_h, \boldsymbol{k})$.

A Priori Error Estimation

Lemma (Standard DGFEM)

The following bound holds:

$$\left\|u - u_{h,k}\right\|_{h,k}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2r_\kappa - 2}}{2k_\kappa^{s_\kappa - 3}} \|u\|_{H^{s_\kappa}(\kappa)}^2$$

with $1 \leq r_\kappa \leq \min(k_\kappa + 1, s_\kappa)$, $k_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$.

Proof.

See Houston, Robson & Süli 2005. ■

Lemma (Two-Grid Approximation)

The following bound holds:

$$\left\| u_{h,k} - u_{2G} \right\|_{h,k}^2 \leq C_2 \sum_{\kappa \in \mathcal{T}_H} \frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|u\|_{H^{S_\kappa}(\kappa)}^2$$

$$\|u - u_{2G}\|_{h,k}^2 \leq C_1 \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2r_\kappa - 2}}{2k_\kappa^{s_\kappa - 3}} \|u\|_{H^{s_\kappa}(\kappa)}^2 + C_2 \sum_{\kappa \in \mathcal{T}_H} \frac{H_\kappa^{2R_\kappa - 2}}{K_\kappa^{2S_\kappa - 3}} \|u\|_{H^{S_\kappa}(\kappa)}^2$$

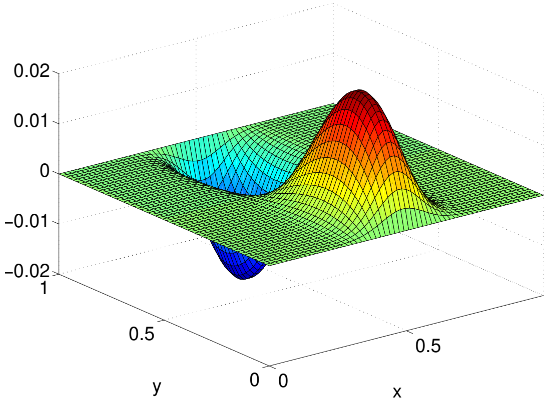
with $1 \leq r_\kappa \leq \min(k_\kappa + 1, s_\kappa)$, $k_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_h$, *and*
$1 \leq R_\kappa \leq \min(K_\kappa + 1, S_\kappa)$, $K_\kappa \geq 1$, *for* $\kappa \in \mathcal{T}_H$

Proof.

Based on an extension of the analysis in Houston, Robson & Süli 2005 and Bi & Ginting 2011. ∎

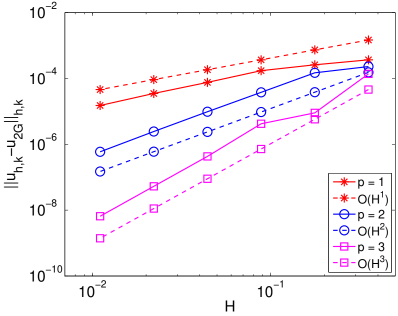
We let $\Omega = (0,1)^2$, $\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2}.$$



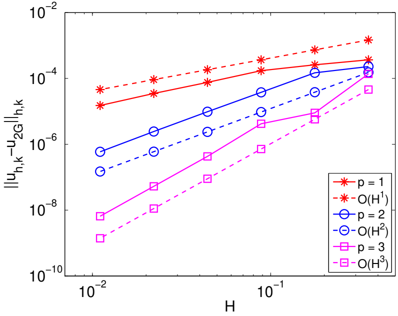
We let $\Omega = (0,1)^2$, $\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

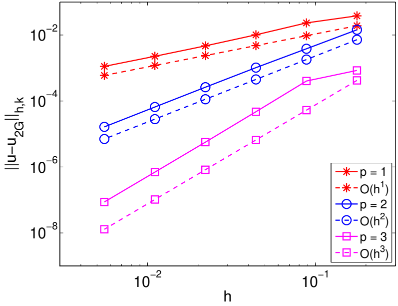
$$u(x,y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2}.$$



We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2}.$$





Lemma (Standard Quasilinear DGFEM)

The following bound holds:

$$\left\| u - u_{h,k} \right\|_{h,k}^2 \leq C_3 \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 .$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2} \left\| f + \nabla \cdot \{\mu(|\nabla u_{h,k}|)\nabla u_{h,k}\} \right\|_{L^2(\kappa)}^2 + \frac{h_\kappa}{k_\kappa} \left\| [\![\mu(|\nabla u_{h,k}|)\nabla u_{h,k}]\!] \right\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma^2 \frac{k_\kappa^3}{h_\kappa} \left\| [\![u_{h,k}]\!] \right\|_{L^2(\partial\kappa)}^2$$

Proof.

See Houston, Süli & Wihler 2008. ■

Lemma (Two-Grid Quasilinear Approximation)

The following bound holds:

$$\|u - u_{2G}\|_{h,k}^2 \leq C_4 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 \right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2} \left\| f + \nabla \cdot \{\mu(|\nabla u_{H,K}|)\nabla u_{2G}\} \right\|_{L^2(\kappa)}^2 + \frac{h_\kappa}{k_\kappa} \left\| [\![\mu(|\nabla u_{H,K}|)\nabla u_{2G}]\!] \right\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma^2 \frac{k_\kappa^3}{h_\kappa} \left\| [\![u_{2G}]\!] \right\|_{L^2(\partial\kappa)}^2$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$, as

$$\xi_\kappa^2 = \left\| (\mu(|\nabla u_{H,K}|) - \mu(|\nabla u_{2G}|))\nabla u_{2G} \right\|_{L^2(\kappa)}^2 .$$

Proof.

Split the solution $u_{2G} = u_{2G}^{\|} + u_{2G}^{\perp}$ (where $u_{2G}^{\|} \in H_0^1(\Omega)$):

$$
\begin{aligned}
C\,\|e_{2G}\|_{h,k}^2 \leq{}& \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla e_{2G}\,d\mathbf{x} \\
&+ C\sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}|[\![e_{2G}]\!]|^2\,ds \\
\leq{}& \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{H,K}|)\nabla u_{2G}\}\cdot\nabla e_{2G}^{\|}\,d\mathbf{x} \\
&- \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla u_{2G}^{\perp}\,d\mathbf{x} \\
&+ C\sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}|[\![e_{2G}]\!]|^2\,ds \\
&+ \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u_{H,K}|)\nabla u_{2G} - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla e_{2G}^{\|}\,d\mathbf{x}.
\end{aligned}
$$

Proof.

Split the solution $u_{2G} = u_{2G}^{\|} + u_{2G}^{\perp}$ (where $u_{2G}^{\|} \in H_0^1(\Omega)$):

$$C\,\|e_{2G}\|_{h,k}^2 \leq \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla e_{2G}\,d\mathbf{x}$$

$$+\,C\sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}|[\![e_{2G}]\!]|^2\,ds$$

$$\leq \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{H,K}|)\nabla u_{2G}\}\cdot\nabla e_{2G}^{\|}\,d\mathbf{x}$$

$$-\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla u_{2G}^{\perp}\,d\mathbf{x}$$

$$+\,C\sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}|[\![e_{2G}]\!]|^2\,ds$$

Using techniques similar to Houston, Süli & Wihler 2008: $\lesssim \|e_{2G}\|_{h,k}\left(\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^2\right)^{\frac{1}{2}}$

$$+\sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u_{H,K}|)\nabla u_{2G} - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla e_{2G}^{\|}\,d\mathbf{x}.$$

Proof.

Split the solution $u_{2G} = u_{2G}^{\|} + u_{2G}^{\perp}$ (where $u_{2G}^{\|} \in H_0^1(\Omega)$):

$$C\,\|e_{2G}\|_{h,k}^2 \le \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla e_{2G}\,d\mathbf{x} + C\sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}|[\![e_{2G}]\!]|^2\,ds$$

$$\le \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{H,K}|)\nabla u_{2G}\}\cdot\nabla e_{2G}^{\|}\,d\mathbf{x} - \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla u_{2G}^{\perp}\,d\mathbf{x} + C\sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}|[\![e_{2G}]\!]|^2\,ds$$

Using techniques similar to Houston, Süli & Wihler 2008:

$$\lesssim \|e_{2G}\|_{h,k}\left(\sum_{\kappa\in\mathcal{T}_h}\eta_\kappa^2\right)^{\frac{1}{2}}$$

$$+ \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \{\mu(|\nabla u_{H,K}|)\nabla u_{2G} - \mu(|\nabla u_{2G}|)\nabla u_{2G}\}\cdot\nabla e_{2G}^{\|}\,d\mathbf{x} \lesssim \|e_{2G}\|_{h,k}\left(\sum_{\kappa\in\mathcal{T}_h}\xi_\kappa^2\right)^{\frac{1}{2}}.$$

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.

The University of Nottingham
UNITED KINGDOM · CHINA · MALAYSIA

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation and two-grid solution.

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation and two-grid solution.
3. Evaluate the elemental error indicators η_κ and ξ_κ.

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation and two-grid solution.
3. Evaluate the elemental error indicators η_κ and ξ_κ.
4. Perform *h*-/*hp*-mesh refinement of the fine space based on the fine grid error indicators η_κ.

Two-Grid Adaptivity

1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation and two-grid solution.
3. Evaluate the elemental error indicators η_κ and ξ_κ.
4. Perform *h*-/*hp*-mesh refinement of the fine space based on the fine grid error indicators η_κ.
5. If $\lambda\xi_\kappa^2 \geq \eta_\kappa^2$ where $0 \leq \lambda < \infty$ is a steering parameter, then mark for refinement the coarse element $\kappa_H \in \mathcal{T}_H$ where $\kappa \subseteq \kappa_H$.
6. Perform *h*-/*hp*-refinement of the coarse space on the elements marked for refinement.

Two-Grid Adaptivity

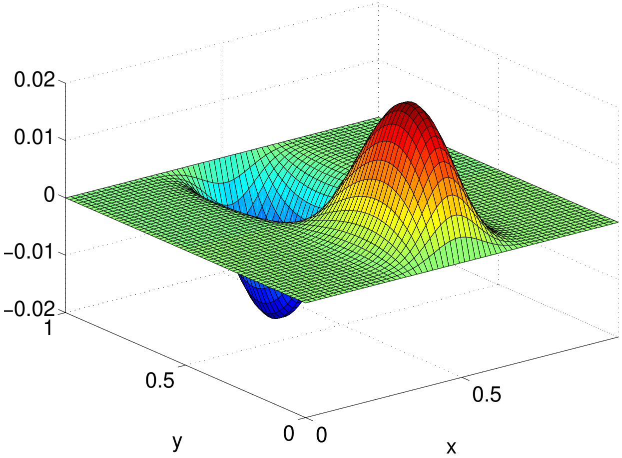
1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation and two-grid solution.
3. Evaluate the elemental error indicators η_κ and ξ_κ.
4. Perform *h*-/*hp*-mesh refinement of the fine space based on the fine grid error indicators η_κ.
5. If $\lambda\xi_\kappa^2 \geq \eta_\kappa^2$ where $0 \leq \lambda < \infty$ is a steering parameter, then mark for refinement the coarse element $\kappa_H \in \mathcal{T}_H$ where $\kappa \subseteq \kappa_H$.
6. Perform *h*-/*hp*-refinement of the coarse space on the elements marked for refinement.
7. Perform mesh smoothing to ensure the coarse space is a subset of the fine space.

Two-Grid Adaptivity

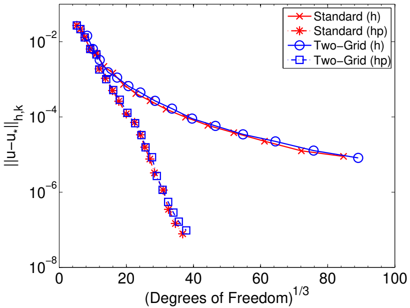
1. Construct initial coarse and fine FE spaces, ensuring that the coarse space is a subset of the fine space.
2. Compute the coarse grid approximation and two-grid solution.
3. Evaluate the elemental error indicators η_κ and ξ_κ.
4. Perform *h*-/*hp*-mesh refinement of the fine space based on the fine grid error indicators η_κ.
5. If $\lambda\xi_\kappa^2 \geq \eta_\kappa^2$ where $0 \leq \lambda < \infty$ is a steering parameter, then mark for refinement the coarse element $\kappa_H \in \mathcal{T}_H$ where $\kappa \subseteq \kappa_H$.
6. Perform *h*-/*hp*-refinement of the coarse space on the elements marked for refinement.
7. Perform mesh smoothing to ensure the coarse space is a subset of the fine space.
8. Goto 2.

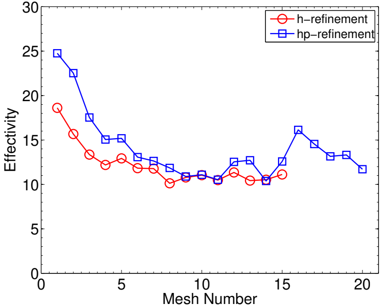
We let $\Omega = (0,1)^2, \mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1+|\nabla u|^2}$ and select f so that

$$u(x,y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2}.$$

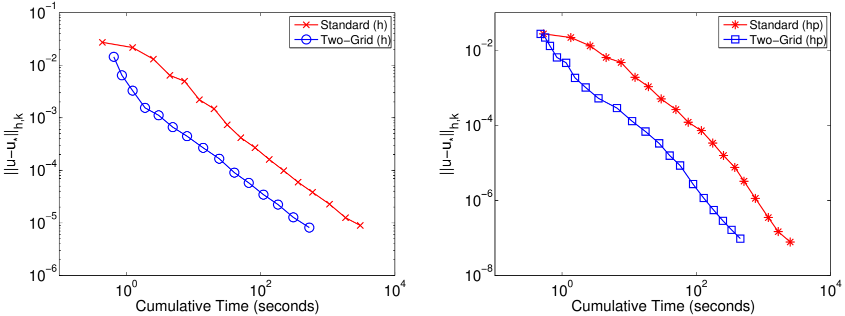


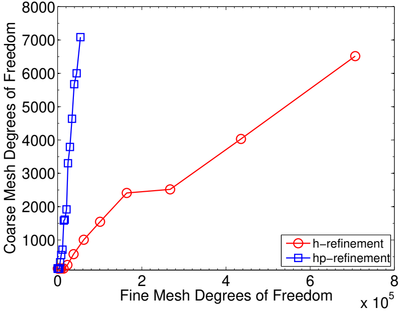
Quasilinear PDE: Smooth Solution

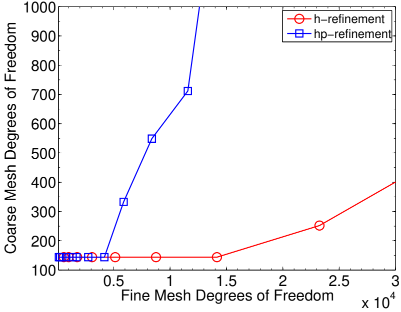




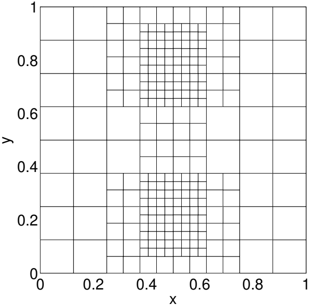
Quasilinear PDE: Smooth Solution



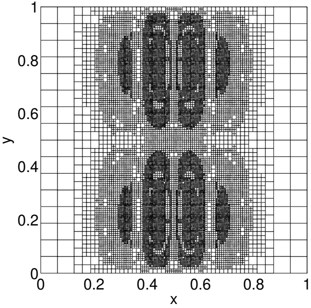

Coarse Mesh Degrees of Freedom
Fine Mesh Degrees of Freedom
x 10^5
h−refinement
hp−refinement
0 2 4 6 8
1000 2000 3000 4000 5000 6000 7000 8000


Coarse Mesh Degrees of Freedom
Fine Mesh Degrees of Freedom
x 10^4
h−refinement
hp−refinement
0.5 1 1.5 2 2.5 3
100 200 300 400 500 600 700 800 900 1000

h–Mesh after 11 adaptive refinements

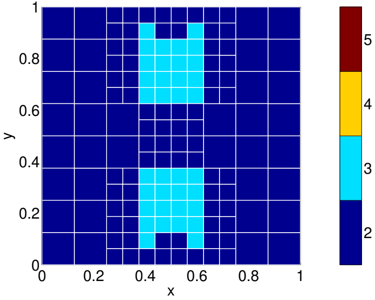


Coarse Mesh

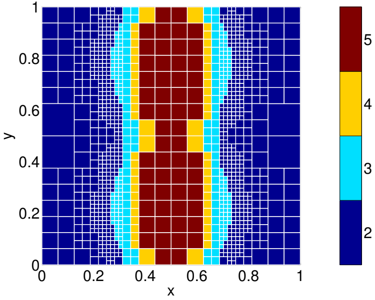


Fine Mesh

hp–Mesh after 11 adaptive refinements



Coarse Mesh



Fine Mesh

We let Ω be the Fichera corner $(-1,1)^3 \setminus [0,1)^3$,

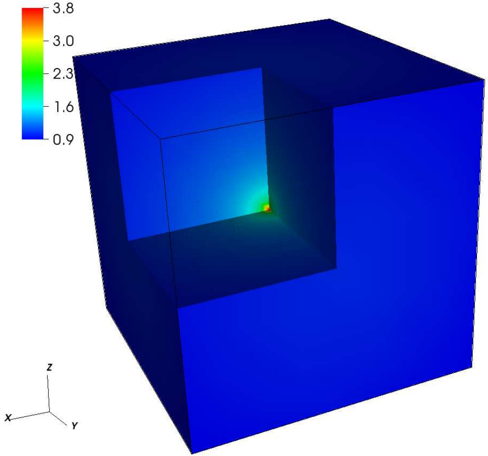
$$\mu(\boldsymbol{x}, |\nabla u|) = 2 + \frac{1}{1 + |\nabla u|^2}$$

and select f so that

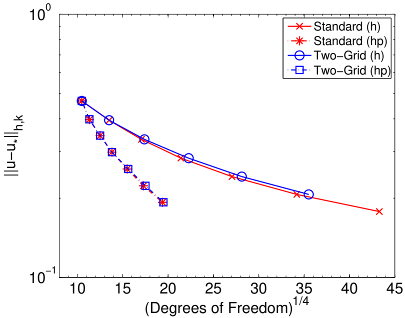
$$u(\boldsymbol{x}) = (x^2 + y^2 + z^2)^{q/2}, \quad q \in \mathbb{R};$$

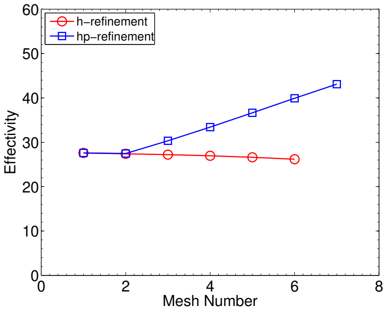
for $q > -1/2$, $u \in H^1(\Omega)$. Here, we select $q = -1/4$.

Beilina, Korotov & Křížek 2005

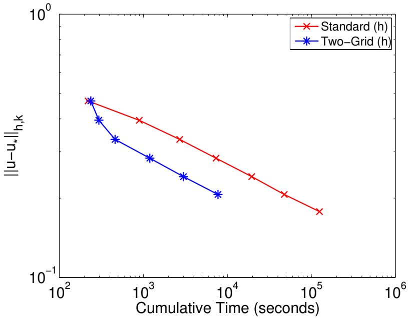


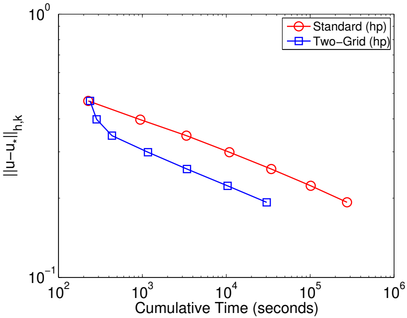
Quasilinear PDE: Singular Solution



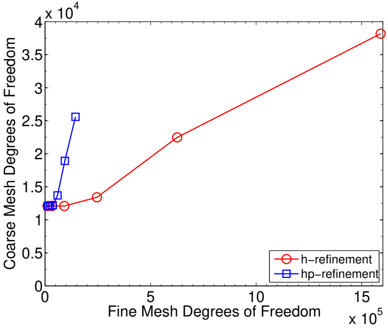


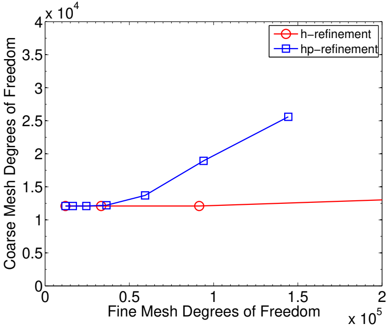
Quasilinear PDE: Singular Solution





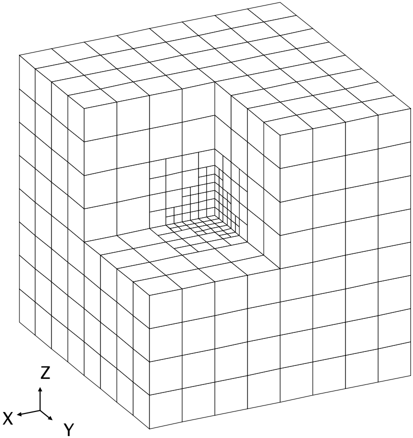
Quasilinear PDE: Singular Solution



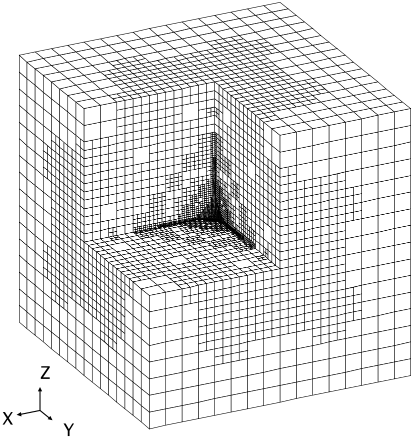


Quasilinear PDE: Singular Solution

h–Mesh after 5 adaptive refinements

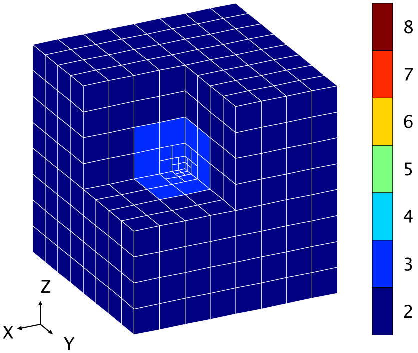


Coarse Mesh

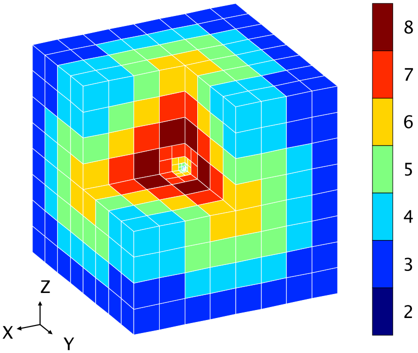


Fine Mesh

hp–Mesh after 6 adaptive refinements



Coarse Mesh



Fine Mesh

Non-Newtonian Fluid Flow

Non-Newtonian Fluid Problem

Given $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and $\boldsymbol{f} \in L^2(\Omega)^d$, find $(\boldsymbol{u}, p)$ such that

$$-\nabla \cdot \{\mu(\boldsymbol{x}, |\underline{e}(\boldsymbol{u})|)\underline{e}(\boldsymbol{u})\} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega,$$
$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega,$$
$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma,$$

where $\underline{e}(\boldsymbol{u})$ is the *symmetric $d \times d$ strain tensor* defined by

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Assumption

1. *$\mu \in C(\bar{\Omega} \times [0, \infty))$ and*
2. *there exists positive constants m_μ and M_μ such that*

$$M_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s), \quad t \geq s \geq 0, \quad \boldsymbol{x} \in \bar{\Omega}.$$

- *hp*-DG finite element space:

$$\boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_\kappa \in \mathcal{S}_{k_\kappa}(\kappa)^d, \forall \kappa \in \mathcal{T}_h\},$$
$$Q(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L_0^2(\Omega) : q|_\kappa \in \mathcal{S}_{k_\kappa - 1}(\kappa), \forall \kappa \in \mathcal{T}_h\}.$$

- Jump operator: $\underline{[\![\boldsymbol{v}]\!]} = \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-$

- *hp*-DG finite element space:

$$\boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) = \{\boldsymbol{v} \in L^2(\Omega)^d : \boldsymbol{v}|_\kappa \in \mathcal{S}_{k_\kappa}(\kappa)^d, \forall \kappa \in \mathcal{T}_h\},$$
$$Q(\mathcal{T}_h, \boldsymbol{k}) = \{q \in L^2_0(\Omega) : q|_\kappa \in \mathcal{S}_{k_\kappa - 1}(\kappa), \forall \kappa \in \mathcal{T}_h\}.$$

- Jump operator: $\underline{[\![\boldsymbol{v}]\!]} = \boldsymbol{v}^+ \otimes \boldsymbol{n}^+ + \boldsymbol{v}^- \otimes \boldsymbol{n}^-$

(Standard) Interior Penalty Method

Find $(\boldsymbol{u}_{h,k}, p_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) \times Q(\mathcal{T}_h, \boldsymbol{k})$ such that

$$\begin{aligned} A_{h,k}(\boldsymbol{u}_{h,k}; \boldsymbol{u}_{h,k}, \boldsymbol{v}_{h,k}) + B_{h,k}(\boldsymbol{v}_{h,k}, p_{h,k}) &= F_{h,k}(\boldsymbol{v}_{h,k}) \\ -B_{h,k}(\boldsymbol{u}_{h,k}, q_{h,k}) &= 0 \end{aligned}$$

for all $(\boldsymbol{v}_{h,k}, q_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) \times Q(\mathcal{T}_h, \boldsymbol{k})$.

$$A_{h,k}(\psi;\boldsymbol{u},\boldsymbol{v}) = \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mu(|\underline{e}(\psi)|)\underline{e}(\boldsymbol{u}) : \underline{e}(\boldsymbol{v})\,d\boldsymbol{x}$$
$$- \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mu(|\underline{e}(\psi)|)\underline{e}(\boldsymbol{u})\}\!\} : \underline{[\![\boldsymbol{v}]\!]}\,ds$$
$$+\,\theta \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\mu(h_F^{-1}|\underline{[\![\psi]\!]}|)\underline{e}(\boldsymbol{v})\}\!\} : \underline{[\![\boldsymbol{u}]\!]}\,ds$$
$$+ \sum_{F\in\mathcal{F}_h}\int_F \sigma_{h,k}\underline{[\![\boldsymbol{u}]\!]} : \underline{[\![\boldsymbol{v}]\!]}\,ds,$$

$$B_{h,k}(\boldsymbol{v}.q) = -\sum_{\kappa\in\mathcal{T}_h}\int_\kappa q\nabla\cdot\boldsymbol{v}\,d\boldsymbol{x} + \sum_{F\in\mathcal{F}_h}\int_F \{\!\{q\}\!\}[\![\boldsymbol{v}]\!]\,ds,$$

$$F_{h,k}(v) = \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x}.$$

Two-Grid Approximation

1. Construct $\boldsymbol{V}(\mathcal{T}_H, \boldsymbol{K})$, $Q(\mathcal{T}_H, \boldsymbol{K})$, $\boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k})$ and $Q(\mathcal{T}_h, \boldsymbol{k})$ such that

$$\boldsymbol{V}(\mathcal{T}_H, \boldsymbol{K}) \subseteq \boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) \quad \text{and} \quad Q(\mathcal{T}_H, \boldsymbol{K}) \subseteq Q(\mathcal{T}_h, \boldsymbol{k})$$

Two-Grid Approximation

1. Construct $\boldsymbol{V}(\mathcal{T}_H, \boldsymbol{K})$, $Q(\mathcal{T}_H, \boldsymbol{K})$, $\boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k})$ and $Q(\mathcal{T}_h, \boldsymbol{k})$ such that

$$\boldsymbol{V}(\mathcal{T}_H, \boldsymbol{K}) \subseteq \boldsymbol{V}(\mathcal{T}_h, \boldsymbol{k}) \quad \text{and} \quad Q(\mathcal{T}_H, \boldsymbol{K}) \subseteq Q(\mathcal{T}_h, \boldsymbol{k})$$

2. Compute $(\boldsymbol{u}_{H,K}, p_{H,K}) \in \boldsymbol{V}(\mathcal{T}_H, \boldsymbol{K}) \times Q(\mathcal{T}_H, \boldsymbol{K})$ such that

$$\begin{aligned} A_{H,K}(\boldsymbol{u}_{H,K}; \boldsymbol{u}_{H,K}, \boldsymbol{v}_{H,K}) + B_{H,K}(\boldsymbol{v}_{H,K}, p_{H,K}) &= F_{H,K}(\boldsymbol{v}_{H,K}), \\ -B_{H,K}(\boldsymbol{u}_{H,K}, q_{H,K}) &= 0 \end{aligned}$$

for all $(\boldsymbol{v}_{H,K}, q_{H,K}) \in \boldsymbol{V}(\mathcal{T}_H, \boldsymbol{K}) \times Q(\mathcal{T}_H, \boldsymbol{K})$.

Two-Grid Approximation

1. Construct $\boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K})$, $Q(\mathcal{T}_H,\boldsymbol{K})$, $\boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k})$ and $Q(\mathcal{T}_h,\boldsymbol{k})$ such that
$$\boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K}) \subseteq \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \quad \text{and} \quad Q(\mathcal{T}_H,\boldsymbol{K}) \subseteq Q(\mathcal{T}_h,\boldsymbol{k})$$
2. Compute $(\boldsymbol{u}_{H,K}, p_{H,K}) \in \boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K}) \times Q(\mathcal{T}_H,\boldsymbol{K})$ such that
$$\begin{aligned} A_{H,K}(\boldsymbol{u}_{H,K};\boldsymbol{u}_{H,K},\boldsymbol{v}_{H,K}) + B_{H,K}(\boldsymbol{v}_{H,K},p_{H,K}) &= F_{H,K}(\boldsymbol{v}_{H,K}),\\ -B_{H,K}(\boldsymbol{u}_{H,K},q_{H,K}) &= 0 \end{aligned}$$
for all $(\boldsymbol{v}_{H,K}, q_{H,K}) \in \boldsymbol{V}(\mathcal{T}_H,\boldsymbol{K}) \times Q(\mathcal{T}_H,\boldsymbol{K})$.
3. Determine $(\boldsymbol{u}_{2G}, p_{2G}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$ such that
$$\begin{aligned} A_{h,k}(\boldsymbol{u}_{H,K};\boldsymbol{u}_{2G},\boldsymbol{v}_{h,k}) + B_{h,k}(\boldsymbol{v}_{h,k},p_{2G}) &= F_{h,k}(\boldsymbol{v}_{h,k}),\\ -B_{H,K}(\boldsymbol{u}_{2G},q_{h,k}) &= 0 \end{aligned}$$
for all $(\boldsymbol{v}_{h,k}, q_{h,k}) \in \boldsymbol{V}(\mathcal{T}_h,\boldsymbol{k}) \times Q(\mathcal{T}_h,\boldsymbol{k})$.

Lemma (Standard Non-Newtonian DGFEM)

The following bound holds:

$$\left\|(\boldsymbol{u}-\boldsymbol{u}_{h,k}, p-p_{h,k})\right\|_{DG}^2 \leq C_5 \sum_{\kappa\in\mathcal{T}_h} \eta_\kappa^2 .$$

Here the local error indicators η_κ *are defined, for all* $\kappa \in \mathcal{T}_h$ *as*

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2}\left\|\boldsymbol{f} + \nabla\cdot\{\mu(|\underline{e}(\boldsymbol{u}_{h,k})|)\underline{e}(\boldsymbol{u}_{h,k})\} - \nabla p\right\|_{L^2(\kappa)}^2 + \left\|\nabla\cdot\boldsymbol{u}_{h,k}\right\|_{L^2(\kappa)}^2$$
$$+ \frac{h_\kappa}{k_\kappa}\left\|[\![p_{h,k}]\!] - [\![\mu(|\underline{e}(\boldsymbol{u}_{h,k})|)\underline{e}(\boldsymbol{u}_{h,k})]\!]\right\|_{L^2(\partial\kappa\setminus\Gamma)}^2 + \gamma^2\frac{k_\kappa^3}{h_\kappa}\left\|\underline{[\![\boldsymbol{u}_{h,k}]\!]}\right\|_{L^2(\partial\kappa)}^2$$

Proof.

See C., Houston, Süli & Wihler (In Preparation). ∎

Lemma (Two-Grid Non-Newtonian Approximation)

The following bound holds:

$$\|(\boldsymbol{u} - \boldsymbol{u}_{2G}, p - p_{2G})\|_{DG}^2 \le C_6 \sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \xi_\kappa^2 \right).$$

Here the local error indicators η_κ are defined, for all $\kappa \in \mathcal{T}_h$ as

$$\eta_\kappa^2 = \frac{h_\kappa^2}{k_\kappa^2} \left\| \boldsymbol{f} + \nabla \cdot \{\mu(|\underline{e}(\boldsymbol{u}_{H,K})|)\underline{e}(\boldsymbol{u}_{2G})\} - \nabla p \right\|_{L^2(\kappa)}^2 + \|\nabla \cdot \boldsymbol{u}_{2G}\|_{L^2(\kappa)}^2$$
$$+ \frac{h_\kappa}{k_\kappa} \left\| [\![p_{2G}]\!] - [\![\mu(|\underline{e}(\boldsymbol{u}_{H,K})|)\underline{e}(\boldsymbol{u}_{2G})]\!] \right\|_{L^2(\partial\kappa \setminus \Gamma)}^2 + \gamma^2 \frac{k_\kappa^3}{h_\kappa} \left\| \underline{[\![\boldsymbol{u}_{2G}]\!]} \right\|_{L^2(\partial\kappa)}^2$$

and the local two-grid error indicators are defined, for all $\kappa \in \mathcal{T}_h$ as

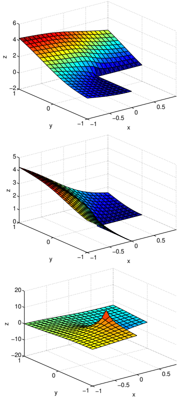
$$\xi_\kappa^2 = \left\| (\mu(|\nabla \boldsymbol{u}_{H,K}|) - \mu(|\nabla \boldsymbol{u}_{2G}|)) \nabla \boldsymbol{u}_{2G} \right\|_{L^2(\kappa)}^2.$$

Non-Newtonian Fluid: Singular Solution

Let $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$, $\mu = 1 + e^{-|\underline{e}(\boldsymbol{u})|}$ and select $\boldsymbol{f}$ so that

$$\boldsymbol{u}(x,y) = r^{\lambda} \begin{pmatrix} (1+\lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi) \\ \sin(\varphi)\Psi'(\varphi) - (1+\lambda)\cos(\varphi)\Psi(\varphi) \end{pmatrix},$$

$$p(x,y) = -r^{\lambda-1}\left\{(1+\lambda)^2\Psi'(\varphi) + \Psi'''(\varphi)\right\}/(1-\lambda),$$



where (r,φ) denotes polar coordinates,

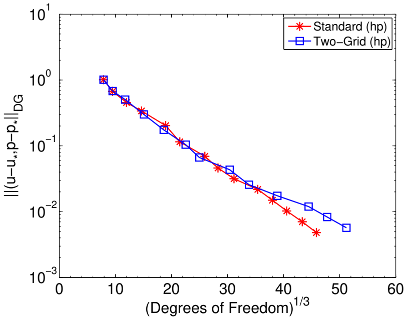
$$\Psi(\varphi) = \frac{\sin((1+\lambda)\varphi)\cos(\lambda\omega)}{1+\lambda} - \cos((1+\lambda)\varphi) - \frac{\sin((1-\lambda)\varphi)\cos(\lambda\omega)}{1-\lambda} + \cos((1-\lambda)\varphi),$$

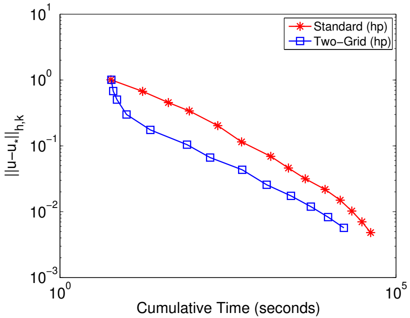
and $\omega = \frac{3\pi}{2}$. Here, the exponent λ is the smallest positive solution of

$$\sin(\lambda\omega) + \lambda\sin(\omega) = 0;$$

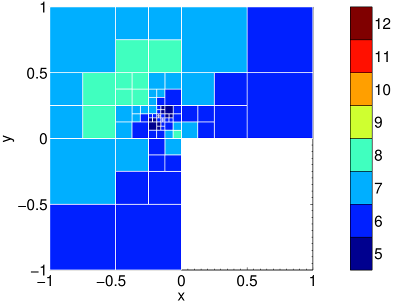
thereby, $\lambda \approx 0.54448373678$. Note that $\boldsymbol{u} \notin H^2(\Omega)^2$ and $p \notin H^1(\Omega)$.

Non-Newtonian Fluid: Singular Solution

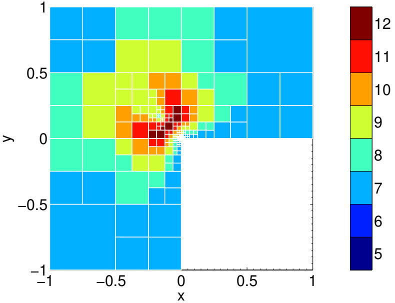




hp–Mesh after 11 adaptive refinements



Coarse Mesh



Fine Mesh

Summary and Future Work

- Summary:
  - Two-grid h–/hp–DGFEMs proposed.
  - *A priori* and *a posteriori* error analysis undertaken.

Summary and Future Work

- Summary:
  - Two-grid h–/hp–DGFEMs proposed.
  - *A priori* and *a posteriori* error analysis undertaken.

- Future Work:
  - Non-Newtonian fluid *a priori* error analysis.
  - Compressible flows.
  - Two-grid h-/hp-adaptive algorithms developed to control the discretization error in both the coarse and fine grid solutions.