Problem Set 4

Problem 1. Let $\mathbb{A} = (\{0,1,2\}; \{0\}, \{1\}, R)$ where

$$R = \{000, 110, 120, 210, 220, 101, 102, 201, 202, 011, 012, 021, 022\}.$$

(a) Find a unary polymorphism f of $\mathbb{A}$ which is not a permutation. (Hint: there is only one.)

(b) Show how to use f and the P-time algorithm for solving CSP$((\{0,1\}; \{0\}, \{1\}, \text{“}x + y = z\text{”}))$ as a “black box” to give a P-time algorithm solving CSP($\mathbb{A}$). (This is an example of a “P-time reducibility.”)

Problem 2. Let $\mathbb{A} = (A; \dots)$ be a relational structure and let f be a unary polymorphism which is not a permutation. Find a relational structure $\mathbb{B} = (B; \dots)$ such that $|B| < |A|$ and CSP($\mathbb{A}$) is P-time equivalent to CSP($\mathbb{B}$). Hints:

- First assume that $f(f(x)) = f(x)$ (for all $x \in A$). Observe that this is equivalent to $f(x) = x$ for every x in the image of f, let us denote this image by B. Consider the relational structure $(B; \dots)$ obtained by restricting all the relations in $\mathbb{A}$ to B and observe that CSP($\mathbb{A}$) and CSP($\mathbb{B}$) are P-time equivalent.
- Show that if a unary operation f which is not a permutation is in a clone, then this clone also contains an operation g which is not a permutation and satisfies $g(g(x)) = g(x)$.

Problem 3. Let $A = \{0,1\}$ and let $\mathbb{A} = (A; R, \{0\}, \{1\})$ be the relational structure from Problem 4, Problem Set 3. Let S be the 4-ary relation defined by $S = \{0,1\}^4 \setminus \{(1,1,1,0)\}$.

(a) Show that S can be equivalently defined as follows:

$$S = \{(x,y,z,w) \in A^4 : \exists u \in A \text{ such that } (x,y,u) \in R \text{ and } (u,z,w) \in R\}.$$

(b) Use part (a) to show how the P-time algorithm for CSP($\mathbb{A}$) may be used as a "black box" to give a P-time algorithm for CSP$(A; R, S, 0, 1)$.

Problem 4. Let $\mathbb{A} = (A; R_1, R_2, R_4)$ be a relational structure such that R_1 is unary, R_2 is binary and R_4 is 4-ary. Let E be the equality relation on A (i.e. $E = \{(a,b) : a = b\}$), let S be the ternary relation on A defined by

$$(x,y,z) \in S \quad \text{iff} \quad x \in R_1 \;\&\; (z,x) \in R_2 \;\&\; (y,z,y,x) \in R_4$$

and let T be the binary relation on A defined by

$$(x,y) \in T \quad \text{iff} \quad (\exists z)\ (x,y,z) \in S$$

(a) Show that CSP$(A; R_1, R_2, R_4, E)$ is P-time reducible to CSP($\mathbb{A}$)

(b) Show that CSP$(A; R_1, R_2, R_4, E, S)$ is P-time reducible to CSP($\mathbb{A}$)

(c) Show that CSP$(A; R_1, R_2, R_4, E, S, T)$ is P-time reducible to CSP($\mathbb{A}$)

Now let $\mathbb{A} = (A; R_1, \ldots, R_n)$ be any relational structure. We say that a relation W is *pp-definable* from $R_1, \ldots, R_i$ (or from $\mathbb{A}$), if it can be defined by a formula of the form

$$(x_1, x_2, \ldots, x_n) \in W$$

$$\text{iff}$$

$$(\exists x_?) \ldots (\exists x_?)(tuple\_of\_var \in R_?) \ \& \ (\cdots \in R_?) \ \& \ \ldots\ldots \ \& \ (x_? = x_?) \ \& \ \ldots \ \& \ (x_? = x_?)$$

Make sure you understand this vague definition.

(d) Show that every relation, which is pp-definable from some relations which are pp-definable from relations $R_1, \ldots, R_n$, is pp-definable from $R_1, \ldots, R_n$.

(e) Observe that $\mathrm{CSP}(A; S_1, \ldots, S_m)$ is P-time reducible to $\mathrm{CSP}(A; R_1, \ldots, R_n)$ whenever $S_1, \ldots S_m$ are relations pp-definable from $R_1, \ldots R_n$.

Problem 5. Let $\mathbb{A} = (\{0,1\}; \{0\}, \{1\}, R_1)$, where $R_1 = \{0,1\}^3 \setminus \{(0,0,0), (1,1,1)\}$ and let $\mathbb{B} = (\{0,1\}; \{0\}, \{1\}, R_{000}, R_{001}, R_{010}, \ldots, R_{111})$, where $R_{ijk} = \{0,1\}^3 \setminus \{(i,j,k)\}$

- Show that every relation in $\mathbb{A}$ is pp-definable from $\mathbb{B}$
- Show that every relation in $\mathbb{B}$ is pp-definable from $\mathbb{A}$
- Conclude that $\mathrm{CSP}(\mathbb{A})$ and $\mathrm{CSP}(\mathbb{B})$ are P-time equivalent

Problem 6. Let $\mathbb{A} = (A; R_1, \ldots, R_n)$ be a relational structure and S be a *nonempty* relation on A.

(a) Show that if S is pp-definable from $\mathbb{A}$ then every polymorphism of $\mathbb{A}$ is a polymorphism of S.

(b)** Show that if every polymorphism of $\mathbb{A}$ is a polymorphism of S, then S is pp-definable from $\mathbb{A}$. Hints:

- Let $S = \{(a_{11}, \ldots, a_{1m}), \ldots, (a_{n1}, \ldots, a_{nm})\}$ and let F denote the set of n-ary polymorphisms of $\mathbb{A}$
- Observe that F can be viewed as an A^n-ary relation on A (i.e. coordinates of the relation are indexed by n-tuples of elements of A)
- Show that F is pp-definable from $\mathbb{A}$
- Consider relation T defined by from F by existentially quantifying over all the coordinates different from $(a_{11}, \ldots, a_{n1}), \ldots, (a_{1m}, \ldots, a_{nm})$.
- Show that $S = T$

Problem 7. Using the result of Problem 4 (and another problem from different problem set) show that

(a) Every relation on $\{0,1,2\}$ is pp-definable from $(\{0,1,2\}; \{0\}, \{1\}, \neq)$

(b) Every relation on $\{0,1\}$ is pp-definable from $(\{0,1\}; \{0\}, \{1\}, \{0,1\}^3 \setminus \{(0,0,0), (1,1,1)\}$

(c) Characterize relations on A pp-definable from the empty set of relations. Compare with Problem 1.(c) in Problem Set 1.

Problem 8. Let $\mathbb{A} = (\{0, 1, 2\}; R)$, where $R = \{(a, b) : a \neq b\}$.

(a) Explain why solving CSP($\mathbb{A}$) is essentially the same as finding 3-coloring of a given graph.

(b) Let $\mathbb{B} = (\{0, 1, 2\}; \{0\}, \{1\}, \{2\}, R)$. Show that CSP($\mathbb{B}$) is P-time reducible to CSP($\mathbb{A}$).

Now let $\mathbb{A} = \{\{1, 2, \dots, n\}; R_1, \dots, R_n\}$ be any relational structure whose every unary polymorphism is a permutation. (Such a structure is called a *core*.)

(c) Show that the relation n-ary relation

$$P = \{(f(1), f(2), \dots, f(n)) : f \text{ is a unary polymorphism of } \mathbb{A}\}$$

is pp-definable from $\mathbb{A}$. (Hint: Problem 4). Use this fact to prove that CSP$((1, 2, \dots, n); R_1, \dots, R_n, P)$ is P-time reducible to CSP($\mathbb{A}$).

(d) Show that CSP$(\{1, 2, \dots, n\}; R_1, \dots, R_n, \{1\}, \{2\}, \dots, \{n\}\})$ is P-time reducible to CSP($\mathbb{A}$).

Problem 9. Let $\mathbb{A} = (\{0, 1\}; R_1, \dots, R_n)$ be a relational structure such that the operation $\wedge$ is a polymorphism. Find a P-time algorithm for solving CSP($\mathbb{A}$). (Hint: Consider first the case that $R_1, \dots, R_n$ are at most binary. To do that, start with finding all binary relations compatible with $\wedge$.)

Problem 10. Let $\mathbb{A} = (\{0, 1\}; R_1, \dots, R_n)$ be a relational structure such that the majority operation is a polymorphism. Show that CSP($\mathbb{A}$) can be solved in P-time. Hints:

- Show that the CSP of any relational structure on $\{0, 1\}$ containing at most binary relation is solvable in P-time. (See Problem 7 in Problem Set 3.)
- Let $R \subseteq \{0, 1\}^n$. For $1 \leq i, j \leq n$, the projection of R to coordinates i, j is defined by

  $$R|_{i,j} = \{(c, d) : \exists (a_1, \dots, a_n) \in R \quad a_i = c, a_j = d\}$$

  Prove that if R is compatible with the majority operation, then R is determined by projections to pair of coordinates in the following sense. For any tuple $(a_1, \dots, a_n) \in \{0, 1\}^n$,

  $$(a_1, \dots, a_n) \in R$$

  iff

  $$(a_i, a_j) \in R|_{i,j} \text{ for every } 1 \leq i, j \leq n$$

- Conclude that every R_i is pp-definable from (at most) binary relations
- Finish the proof using previous problems.

Problem 11. Show that for any relational structure $\mathbb{A} = \{(0, 1); \dots\}$ the following dichotomy holds:

Either CSP($\mathbb{B}$) is P-time reducible to CSP($\mathbb{A}$) for every relational structure $\mathbb{B}$ on the set $\{0, 1\}$,
or
CSP($\mathbb{A}$) is P-time solvable. (Hint: Combine several problems from several problem sets.)

Problem 12. ****** Show similar dichotomy for relational structures on arbitrary finite set.