Problem Set 1

Problem 1.

(a) Find all operations on a given set \( A \) which are polymorphisms of every relational structure with universe \( A \).

(b) Assume that binary operations \( \circ, \star \) are polymorphisms of a relational structure \( A = (A; \ldots) \). Let \( f \) be a ternary operation on \( A \) defined by \( f(x, y, z) = (x \circ (y \star z)) \star x \). Is it always the case that \( f \) is a polymorphism of \( A \)? Can you generalize this?

(c) Find all relations on a fixed set \( A \) which are compatible with all operations on the set \( A \).

Problem 2. Find all polymorphisms of the following relational structures:

(a) \( \{\{0,1\};\{0\},\{1\},R_1\} \), where \( R_1 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\} \)

(b) \( \{\{0,1,2\};\{0\},\{1\},R_2\} \), where \( R_2 = \{0,1,2\}^2 \setminus \{(0,0),(1,1),(2,2)\} \) (the inequality relation)

(c) \( \{\{0,1,2\};R_2\} \) (the same \( R_2 \) as above)

Problem 3. Let us call the polymorphisms from Problem 1, item (a) trivial.

Consider the relational structure:

(a) \( A = (\{0,1\};\{0\},\{1\},S) \), where \( S = \{0,1\}^3 \setminus \{(1,1,0)\} \)

(b) \( A = (\{0,1\};\{0\},\{1\},R_1,R_2,\ldots,R_n) \), where \( R_1,\ldots,R_n \) is a list of all binary relations on \( \{0,1\} \) (what is the value of \( n \), by the way?)

(c) \( A = (\{0,1\};\{0\},\{1\},R) \), where \( R = \{(0,0,1),(0,1,0),(0,1,0),(1,1,1)\} \).

Does \( A \) admit any nontrivial polymorphism?

Is the answer different when we remove the unary relations \( \{0\},\{1\} \) from \( A \)?

Problem 4. Describe all relations compatible with the operation \( f(x, y, z) = x + y + z \), where + is the addition modulo 2 on the set \( \{0,1\} \).

Problem 5. Let us say that an operation \( f \) on the set \( \{0,1\} \) is a min operation, if \( f \) can be written in the form \( f(x_1, \ldots, x_n) = \min\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \) for some natural number \( m \) and some \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \).

Moreover, let us say that \( f \) is a max-min operation, if \( f(x_1, \ldots, x_n) = \max\{g_1(x_1, \ldots, x_n), g_2(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\} \), where \( g_1, \ldots, g_k \) are min operations.

(a) Show that the set of all min-max-max-min-max-min-max operations (defined in the obvious way) coincide with the set of all max-min operations.

(b) Find a relational structure \( A = (\{0,1\},\ldots) \) such that polymorphisms of \( A \) are precisely the max-min operations.

Problem 6. Let us call the polymorphisms from Problem 1, item (a) trivial (as in Problem 3).

Find the smallest digraph \( A \) such that \( A \) has only trivial polymorphisms. Where “smallest” is

(a) with respect to the number of vertices,

(b) with respect to the number of edges.

Problem 7. Find an infinite sequence \( A_1, A_2, \ldots \) of relational structures with universe \( \{0,1\} \) such that, for all \( i > j \), every polymorphism of \( A_i \) is a polymorphism of \( A_j \), and there exists a polymorphism of \( A_j \) which is not a polymorphism of \( A_i \).