

Homework 3

Deadline 03.12.2019, 11:30

1. (10 points) Let \mathcal{V} and \mathcal{W} be two varieties of groups. Define $\mathcal{V} \cdot \mathcal{W}$ to be the class of all groups \mathbf{A} containing a normal subgroup \mathbf{B} such that $\mathbf{B} \in \mathcal{V}$ and $\mathbf{A}/\mathbf{B} \in \mathcal{W}$. Prove that $\mathcal{V} \cdot \mathcal{W}$ is a variety.
2. (10 points) Let \mathbf{L}, \mathbf{M} be two bounded lattices with more than one element. By $\mathbf{L} \oplus \mathbf{M}$ we define the lattice with universe $L \dot{\cup} M$ such that every element of L lies below every element on M . Prove that $\text{HSP}(\{\mathbf{L}, \mathbf{M}\}) = \text{HSP}(\{\mathbf{L} \oplus \mathbf{M}\})$
3. (10 points) Let \mathbf{R} be a commutative ring such that for every non-zero element a also $a^n \neq 0$ holds for every $n > 1$ in \mathbf{R} (such a ring is also called *reduced*).
 - Show that for every $a \in R \setminus \{0\}$ there is a prime ideal P_a with $a \notin P_a$ (Hint: Use Zorn's lemma to show that there is a maximal ideal P_a that does not contain a, a^2, a^3, \dots)
 - Prove that \mathbf{R} is the subdirect product of integral domains.