NMAG 405 Universal Algebra 1, fall semester 2017–2018

Homework 4

Deadline 21 Dec 2017, 10:40

4.1. (10 points) Let $\mathbf{A} = (\{0,1\};\cdot)$ where $x \cdot y = 0$ for all $x, y \in \{0,1\}$. Let $\mathcal{V} = HSP(\mathbf{A})$. Determine a (small) equational basis of V (i.e., identities that axiomatize $\mathcal{V}$). Describe the free algebras in $\mathcal{V}$ over finite set of generators.

4.2. (10 points) Let $\mathcal{V}$ be the variety (over the signature consisting of a single binary symbol $\cdot$) defined by the identities

$$x \cdot x \approx x, \quad (x \cdot y) \cdot z \approx (z \cdot y) \cdot x \ .$$

(a) Show that every member of $\mathcal{V}$ satisfies the following identites.

$$\begin{aligned}(x \cdot y) \cdot (z \cdot w) &\approx (x \cdot z) \cdot (y \cdot w)\\ x \cdot (y \cdot z) &\approx (x \cdot y) \cdot (x \cdot z)\\ (y \cdot z) \cdot x &\approx (y \cdot x) \cdot (z \cdot x)\\ y \cdot (x \cdot y) &\approx (y \cdot x) \cdot y\\ (y \cdot x) \cdot x &\approx x \cdot y\end{aligned}$$

(b) Let $\mathcal{W}$ be the subvariety of $\mathcal{V}$ defined by the additional identity $y \cdot (x \cdot y) \approx x$. Determine $\mathbf{F}_{\mathcal{W}}(\{x, y\})$ (e.g., write out the Cayley table).

4.3. (10 points) Let $\mathbf{A}$ be the following semigroup

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

Prove that $HSP(\mathbf{A})$ is the variety of commutative semigroups satisfying $x^2 \approx x^3$.