NMAG 455 Universal Algebra 2, spring semester 2017–2018

Homework 4

Deadline 11 June 2018, 9:00

4.1. (10 points) Prove that the following are equivalent for any finite idempotent algebra $\mathbf{A}$ and any $n \geq 2$.

(i) $\mathrm{Clo}\,(\mathbf{A})$ contains an n-ary cyclic operation (i.e., t such that $t(a_1, a_2, \dots, a_n) \approx t(a_2, \dots, a_n, a_1)$.

(ii) Each $R \leq \mathbf{A}^n$ which is invariant under cyclic shift (i.e., $(a_1, \dots, a_n) \in R$ implies $(a_2, \dots, a_n, a_1) \in R$) contains a constant tuple (ie., $(a, a, \dots, a) \in R$ for some a).

(Hint: for the interesting implication, define R as the subuniverse generated by cyclic shifts of a tuple and use the assumption to get an operation that behaves like cyclic for this tuple (and its cyclic shifts). Put the operations together (inductively).)

The next two problems can be used to fill in the gaps in the proof of the general loop lemma.

4.2. (10 points) Let $\mathbf{A}$ be an idempotent finite Taylor algebra with no proper absorbing subuniverse, $R \leq_{sd} \mathbf{A}^2$, and assume that $\mathbb{G} = (A; R)$ has algebraic length 1. Prove that there exists a k such that each $a, b \in A$ are connected by a directed path of length k.

(Hint: algebraic length 1 means that each pair is connected by a fence. Define from R a linked relation so that the absorption theorem proves the conclusion.)

4.3. (10 points) Let $k \geq 2$ and $\mathbb{G} = (A; R)$ be a smooth digraph such that each $a, b \in A$ are connected by a directed path of length k. Let $B \subseteq A$ be such that $B^{+\mathbb{F}} = B$ where $\mathbb{F}$ is the $(k-1, 1)$ fence ($k-1$ edges forward and then the same number of edges backward). Finally, let C be the smooth part of $\mathbb{G}_{|B}$. Prove that $\mathbb{G}_{|C}$ is a nonempty digraph of algebraic length 1.

(Hint: To show that C is nonempty, prove that each vertex in B has an outgoing edge within B.)