NMAG 455 Universal Algebra 2, spring semester 2017–2018

Homework 1

Deadline 4 Apr 2018, 9:00

1.1. (10 points) Let $\mathbf{A}$ be an algebra with a central Malcev polynomial and let ring $\mathbf{R}$ and $\mathbf{R}$–module $\mathbf{M}$ be as in the proof of the fundamental theorem on Abelian algebras. Prove that every polynomial operation of $\mathbf{A}$ is a polynomial operation of $\mathbf{M}$ (this fills in one of the omitted parts of the proof of the fundamental theorem).

Recall that a *loop* is a quasigroup with a neutral element (the signature is $\cdot$ (multiplication), $/, \backslash$ (right and left division), 1 (neutral element wrt. $\cdot$). In the following problems, a *group* is a loop such that $\cdot$ is associative (and it is *commutative* if $\cdot$ is commutative).

1.2 (10 points) Prove that a loop is Abelian iff it is a commutative group.

1.3 (10 points) Consider the loop $\mathbf{L}$ with universe $\mathbb{Z}_4 \times \mathbb{Z}_2$ given by the multiplication $(a, b) \cdot (c, d) = (a + c, b + d)$ unless $b = d = 1$, and $(a, 1) \cdot (c, 1) = (a \star c, 0)$

$\star$	0	1	2	3
0	1	0	2	3
1	0	2	3	1
2	2	3	1	0
3	3	1	0	2

Consider the mapping $f : L \to \mathbb{Z}_2$ given by $(a, b) \mapsto b$, its kernel α, and the α-block N of $1 = (0, 0)$ (ie. $N = \mathbb{Z}_4 \times \{0\}$) (note that N is a subuniverse of $\mathbf{L}$, the corresponding subalgebra $\mathbf{N}$ is clearly isomorphic to the group $\mathbb{Z}_4$).

- Prove that f is a homomorphism (so $\mathbf{N}$ is a so called normal subloop, which is Abelian)
- Prove that α is not an Abelian congruence (ie. α does not centralize α modulo 0).