FINITELY RELATED ALGEBRAS IN CONGRUENCE DISTRIBUTIVE VARIETIES HAVE NEAR UNANIMITY TERMS

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Abstract. We show that every finite, finitely related algebra in a congruence distributive variety has a near unanimity term operation. As a consequence we answer the near unanimity problem for relational structures: it is decidable whether a given finite set of relations on a finite set admits a compatible near unanimity operation. This consequence also implies that it is decidable whether a given finite constraint language defines a constraint satisfaction problem of bounded strict width.

1. Introduction

Since the start of the systematic study of universal algebras in the 1930’s it has been recognized that an important invariant of algebras and classes of algebras are their congruence lattices. Particularly widely studied objects are congruence distributive varieties, i.e. equationally definable classes of algebras whose congruence lattices are distributive (see section 2 for definitions).

We call an algebra in a congruence distributive variety a CD algebra. Examples of CD algebras include lattices, and, more generally, algebras which have a near unanimity term operation. These operations have also attracted a great deal of attention, not only in universal algebra, but also in graph theory and, recently, in computer science in connection with the constraint satisfaction problem (CSP), where, for instance, near unanimity operations characterize CSPs of bounded strict width [11].

Every finite algebra is, in some sense, determined by a set of relations. We call an algebra finitely related, if this set of relations can be chosen to be finite. A useful corollary of a classical result of Baker and Pixley [2] is that every algebra with a near unanimity term operation is finitely related. Our main result provides a partial converse:

Theorem 1.1. Every finite, finitely related CD algebra has a near unanimity term operation.

A special case of this theorem for algebras determined by posets was conjectured in [10] and [24]. An affirmative answer was given in [25] for bounded posets and in [19] in full generality. Another special case of the theorem, namely, for algebras determined by a reflexive undirected graph, was proved in [18]. The general version was commonly referred to as the Zádori conjecture, although it has been never stated in a journal paper, perhaps because of scant evidence.

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What made this result possible is the connection between the constraint satisfaction problem and universal algebra discovered in [15, 8]. The interaction between these areas is very fruitful in both directions: On one hand, universal algebra has brought a deeper understanding and strong results about the CSP. On the other hand, the CSP has motivated much of the recent work in universal algebra and opened new research directions. This is nicely illustrated by the main result of [3] (Theorem 5.7 in the present paper). This theorem contributed to the study of local consistency methods for the CSP (and was an important step toward the full characterization of applicability of local consistency methods given in [4]), and it also is one of the two main ingredients of the proof of our main, purely algebraic result.

We remark that none of the assumptions of Theorem 1.1 is superfluous. In [25], Zádori provides an example of an infinite, bounded poset which determines a CD algebra with no near unanimity term operation. A simple example of a finite CD algebra with no near unanimity operation is the two element set \{0, 1\} together with the implication regarded as a binary operation. Finally, the algebra determined by the complete loopless graph with 3 vertices does not have any near unanimity operation (it actually has no idempotent operations other than projections).

Of an independent interest is a corollary of the main theorem (Corollary 7.1) which gives an affirmative answer to the near unanimity problem for relational structures: It is decidable whether a given set of relations on a finite set admits a compatible near unanimity operation. This consequence is discussed in more detail in section 7.

1.1. Organization of the paper. In section 2 we recall basic notions and results about algebras and relational structures. In section 3 we show that it is enough to deal with algebras determined by at most binary relations. In section 4 we associate to such an algebra an instance of the CSP whose solutions are term operations of that algebra. The definitions and results about CSP instances which we require are stated in section 5, where we also prove the main theorem. The main new tool is only stated in this section, its proof covers section 6. Finally, in section 7 we discuss consequences and open problems.

2. Preliminaries

In this section we recall universal algebraic notions and results which will be needed throughout the paper. This material except for the notion of a Jónsson ideal is covered in any standard reference on universal algebra, for example in [9].

2.1. Algebras and varieties. An \( n \)-ary operation on a set \( A \) is a mapping \( f : A^n \to A \). In this paper we assume that all operations are finitary, i.e. \( n \) is a natural number. An operation is idempotent, if it satisfies the identity
\[
f(a, a, \ldots, a) = a,
\]
i.e., this equation holds for every \( a \in A \). An operation of arity at least 3 is called a near unanimity operation, if it satisfies the identity
\[
f(a, a, \ldots, a, b, a, a, \ldots, a) = a
\]
for every position of \( b \) in the tuple.

An algebra is a pair \( \mathbf{A} = (A, \mathcal{F}) \), where \( A \) is a set, called the universe of \( \mathbf{A} \), and \( \mathcal{F} \) is a set (possibly indexed) of operations on \( A \). We use a boldface letter to denote
an algebra and the same letter in the plain type to denote its universe. An algebra is idempotent if all of its operations are idempotent. Two algebras are similar, if their operations are indexed by the same set and corresponding operations have the same arities.

A term operation of $A$ is an operation which can be obtained from operations in $A$ using composition and the projection operations. The set of all term operations of $A$ is denoted by $\text{Clo}(A)$. Most structural properties of an algebra (like subalgebras, congruences, automorphisms, ...) depend only on the set of term operations rather than on a particular choice of the basic operations.

There are three fundamental operations on algebras of a fixed similarity type: forming subalgebras, factor algebras and products.

A subset $B$ of the universe of an algebra $A$ is called a subuniverse, if it is closed under all operations (equivalently term operations) of $A$. Given a subuniverse $B$ of $A$ we can form the algebra $B$ by restricting all the operations of $A$ to the set $B$. In this situation we say that $B$ is a subalgebra of $A$ and we write $B \leq A$ or $B \subseteq A$.

The product of algebras $A_1, \ldots, A_n$ is the algebra with the universe equal to $A_1 \times \cdots \times A_n$ and with operations computed coordinatewise. The product of $n$ copies of an algebra $A$ is denoted by $A^n$. A subalgebra (or a subuniverse) of a product of $A$ is called a subpower of $A$.

An equivalence relation $\sim$ on the universe of an algebra $A$ is a congruence, if it is a subalgebra of $A^2$. Corresponding factor algebra $A/\sim$ has, as its universe, the set of $\sim$-blocks and operations are defined using (arbitrary chosen) representatives. The set of congruences of $A$ forms a lattice, called the congruence lattice of $A$.

A variety is a class of similar algebras closed under forming subalgebras, products, factor algebras and isomorphic copies. A fundamental theorem of universal algebra, due to G. Birkhoff, states that a class of similar algebras is a variety if and only if this class can be defined via a set of identities.

2.2. Relational structures. An $n$-ary relation on a set $A$ is a subset of $A^n$ (again, $n$ is always finite in this article). A relational structure is a pair $\mathcal{A} = (A, R)$, where $A$ is the universe of $\mathcal{A}$ and $R$ is a set of relations on $A$. We use blackboard bold letters to denote relational structures.

We say that an operation $f : A^n \to A$ is compatible with a relation $R \subseteq A^m$ (or, $R$ is preserved by $f$) if the tuple $(f(a_1^1, a_1^2, \ldots, a_1^n), f(a_2^1, a_2^2, \ldots, a_2^n), \ldots, f(a_m^1, a_m^2, \ldots, a_m^n))$ belongs to $R$ whenever $(a_1^1, a_2^1, \ldots, a_m^1) \in R$ for all $i \leq n$. In other words, $f$ is compatible with $R$, if $R$ is a subpower of the algebra $(A, \{f\})$.

An operation compatible with all relations of a relational structure $\mathcal{A}$ is a polymorphism of $\mathcal{A}$. The set of $n$-ary polymorphisms of $\mathcal{A}$ is denoted by $\text{Pol}_n(\mathcal{A})$ and the set of all polymorphisms of $\mathcal{A}$ is denoted by $\text{Pol}(\mathcal{A})$. This set of operations is closed under composition and contains the projection operations. On the other hand, every set of operations on a finite set closed under projections and composition can be obtained in this way:

Theorem 2.1. [6, 13] For every finite algebra $A$ there exists a relational structure $\mathcal{A}$ (with the same universe) such that $\text{Pol}(\mathcal{A}) = \text{Clo}(A)$.

An algebra is called finitely related if finitely many relations suffice to determine $\text{Clo}(A)$.
Definition 2.2. An algebra $A$ is said to be finitely related, if there exists a relational structure $A$ with finitely many relations such that $\text{Pol}(A) = \text{Clo}(A)$.

By a classic result of Baker and Pixley [2], every algebra with a near unanimity term operation is finitely related. More generally, every algebra with few subpowers is finitely related [5] (see subsection 7.3).

2.3. CD algebras.

Definition 2.3. A variety is called congruence distributive, if all algebras in it have distributive congruence lattices. A CD algebra is an algebra in a congruence distributive variety.

A theorem of Jónsson [16] characterizes CD algebras using operations satisfying certain identities.

Definition 2.4. A sequence $p_0, p_1, \ldots, p_s$ of ternary operations on a set $A$ is called a Jónsson chain, if the following identities are satisfied:

\begin{align*}
p_0(a, b, c) &= a, \\
p_1(a, b, c) &= c, \\
p_i(a, b, a) &= a \quad \text{for all } i \leq s, \\
p_i(a, a, b) &= p_{i+1}(a, a, b) \quad \text{for all even } i < s, \\
p_i(a, b, b) &= p_{i+1}(a, b, b) \quad \text{for all odd } i < s.
\end{align*}

Theorem 2.5. [16] An algebra $A$ has a Jónsson chain of term operations if and only if $A$ is a CD algebra.

Example. Every algebra with a near unanimity term operation $t$ is a CD algebra. This can be shown, for instance, by constructing a Jónsson chain:

\begin{align*}
p_0(a, b, c) &= a, \\
p_1(a, b, c) &= t(a, a, \ldots, a, b, c) \\
p_2(a, b, c) &= t(a, a, \ldots, a, c, c) \\
p_3(a, b, c) &= t(a, a, \ldots, a, b, c, c) \\
p_4(a, b, c) &= t(a, a, \ldots, a, c, c, c)
\end{align*}

\ldots

A useful notion for studying CD algebras is a Jónsson ideal:

Definition 2.6. Let $A$ be a CD algebra with Jónsson chain of term operations $p_0, p_1, \ldots, p_s$. A subuniverse $B$ of $A$ is a Jónsson ideal, if $p_i(b_1, a, b_2) \in B$ for every $a \in A, b_1, b_2 \in B$ and every $i \leq n$.

Every one element subuniverse of a CD algebra is its Jónsson ideal. Therefore, if $A$ is an idempotent CD algebra, then every singleton is a Jónsson ideal of $A$.

3. Reduction to binary structures

In this section we show that to prove the main result it is enough to consider algebras determined by binary relational structures, i.e. relational structures with at most binary relations. This will make the presentation technically easier.
Proposition 3.1. Let $\mathcal{A}$ be a relational structure whose all relations have arity less than $k$. Then there exists a binary relational structure $\check{\mathcal{A}}$ with universe $\check{\mathcal{A}} = A^k$ such that

$$\text{Pol}(\check{\mathcal{A}}) = \{ \check{f} : f \in \text{Pol}(\mathcal{A}) \},$$

where $\check{f}$ is defined (if $f$ is $n$-ary) by

$$\check{f}((a_1^1, a_2^1, \ldots, a_k^1), (a_1^2, \ldots, a_k^2), \ldots, (a_1^n, \ldots, a_k^n)) = (f(a_1^1, a_2^1, \ldots, a_k^1), f(a_2^1, \ldots, a_k^1), \ldots, f(a_k^1, \ldots, a_1^1)).$$

Proof. First we replace every relation $R$ in $\mathcal{A}$ with arity $l < k$ by the $k$-ary relation $R \times A^{k-l}$. This clearly does not change the set of polymorphisms, therefore we may assume that every relation in $\mathcal{A}$ has arity precisely $k$.

Next we introduce the relations in $\check{\mathcal{A}}$. For every $k$-ary relation $R$ (on $\mathcal{A}$) we add a binary relation $\bar{R}$ where

$$\text{Pol}(\check{\mathcal{A}}) = \{ \check{f} : f \in \text{Pol}(\mathcal{A}) \},$$

Proof of Theorem 1.1 assuming Theorem 3.2.

A $CD$ algebra, then $\mathcal{A}$ has a near unanimity polymorphism.

Proof of Theorem 1.1 assuming Theorem 3.2. Let $\mathcal{A}$ be a finite, finitely related $CD$ algebra and let $\mathcal{A}$ be a relational structure with finitely many relations (say all of them have arity less at most $k$) such that $\text{Pol}(\mathcal{A}) = \text{Clo}(\mathcal{A})$. Let $\check{\mathcal{A}}$ be the relational structure from the previous proposition. Then $\text{Pol}(\check{\mathcal{A}}) = \text{Clo}(\mathcal{A})$, since $p_0, \ldots, p_s$ is a Jónsson chain of $\mathcal{A}$ whenever $p_0, \ldots, p_s$ is a Jónsson chain of $\mathcal{A}$. By Theorem 3.2, $\check{\mathcal{A}}$ has a near unanimity polymorphism $h$. Using Proposition
3.1 again we have \( h = \bar{f} \) for some polymorphism \( f \) of \( A \), and \( f \) is clearly a near unanimity operation. \( \square \)

4. CSP Instance Associated to a Binary Relational Structure

**Definition 4.1.** An instance of the constraint satisfaction problem (CSP) is a triple \( P = (V, A, C) \) with
- \( V \) a nonempty, finite set of variables,
- \( A \) a nonempty, finite domain,
- \( C \) a finite set of constraints, where each constraint is a pair \( C = (x, R) \) with
  - \( x \) a tuple variables of length \( n \), called the scope of \( C \), and
  - \( R \) an \( n \)-ary relation on \( A \), called the constraint relation of \( C \).

Let \( A \) be a finite idempotent algebra. An instance of the CSP over \( A \), denoted by \( \text{CSP}(A) \), is an instance such that all constraint relations are subpowers of \( A \).

A solution to an instance \( P \) is a function \( f : V \to A \) such that, for each constraint \( C = (x, R) \in C \), the tuple \( f(x) \) belongs to \( R \).

**Remark 4.2.** The CSP is often parametrized by relational structures: an instance whose constraint relations are in a relational structure \( \mathcal{A} \) is called an instance of \( \text{CSP}((\mathcal{A}, \text{Pol}(\mathcal{A}))) \). It was proved in [15] that the computational complexity of deciding whether an instance of \( \text{CSP}(\mathcal{A}) \) has a solution is fully determined, at least when \( \mathcal{A} \) has finitely many relations, by the algebra \( \mathcal{A} = (\mathcal{A}, \text{Pol}(\mathcal{A})) \). Moreover, Bulatov, Jeavons and Krokhin proved [8] that the complexity depends only on the variety generated by \( \mathcal{A} \) (i.e., the smallest variety containing \( \mathcal{A} \)). These results are at the heart of the connection between universal algebra and the CSP mentioned in the introduction.

For simplicity we will formulate our definitions and results for a special type of CSP instances with a single binary constraint for each pair of variables, although most of the material can be generalized.

**Definition 4.3.** An instance \( P = (V, A, C) \) of the CSP is called a simple binary instance, if
- \( C = \{((x_1, x_2), R^P_{x_1,x_2}) : x_1, x_2 \in V\} \),
- \( R^P_{x_2,x_1} = R^P_{x_1,x_2}^{-1} = \{(b,a) : (a,b) \in R^P_{x_1,x_2}\} \) for every \( x_1, x_2 \in V \), and
- \( R^P_{x,x} \subseteq \{(a,a) : a \in A\} \) for every \( x \in V \).

We omit the superscript \( P \) if the instance is clear from the context.

A simple binary instance can be drawn as a \( V \)-partite graph in the following way. Each part is a copy of \( A \), one for each variable \( x \in V \) (the parts are now commonly referred to as potatoes), and elements of \( R_{x_1,x_2} \) are edges between the corresponding copies of \( A \). Solutions then correspond to cliques with \( V \) vertices (with one vertex in each part).

To every binary relational structure \( \mathcal{A} \) and natural number \( n \) we can associate, in a natural way, a simple binary instance \( P(\mathcal{A}, n) \) of \( \text{CSP}((A, \text{Pol}(\mathcal{A}))) \) whose solutions are precisely the polymorphisms of \( \mathcal{A} \).

**Definition 4.4.** Let \( \mathcal{A} \) be a binary relational structure and let \( n \geq 2 \) be a natural number. The instance \( P(\mathcal{A}, n) = (V, A, C) \) is defined by
- \( V = A^n \)
We are interested in near unanimity polymorphisms – solutions for instance, by using this reasoning for the relation $R$. The proof for a unary relation $R$ can be done, for instance, by using this reasoning for the relation $R \times A$. 

We are interested in near unanimity polymorphisms – solutions of $P(\mathbb{A}, n)$ satisfying the additional conditions $f(a, a, \ldots, a, b, a, a, \ldots, a) = a$ (for any $a, b \in A$ and any position of $b$ in the tuple). Therefore the following notion of a restriction of an instance will be useful.

**Definition 4.6.** Let $P = (V, A, C)$ be a simple binary instance of CSP and $J = \{J_x : x \in V\}$ be a family of subsets of $A$. By the restriction of $P$ to $J$ we mean the simple binary instance $P_{\mathcal{J}} = (V, A, C)$ with

$$R_{x_1, x_2}^{P_{\mathcal{J}}} = R_{x_1, x_2}^{P} \cap (J_{x_1} \times J_{x_2}).$$

for every $x_1, x_2 \in V$.

Observe that if each $J_x$ is a subuniverse of an algebra $A$, then the restriction of an instance of CSP($A$) to $J$ is an instance of CSP($A$).

To find an $n$-ary polymorphism of a binary relational structure $\mathbb{A}$, we will consider the instance $P = P(\mathbb{A}, n)$ and its restriction to the family $\mathcal{J} = \{J_x : x \in V\}$, where $J_{(a, a, \ldots, a, b, a, a, \ldots, a)} = \{a\}$ (for every $a, b \in A$ and every position of $b$ in the tuple), and $J_{(a, a, \ldots, a)} = A$ otherwise. With this choice, the set of solutions of $P_{\mathcal{J}}$ coincides with the set of $n$-ary near unanimity polymorphisms of $\mathbb{A}$. We show that this set is nonempty in two steps. First we prove that $P_{\mathcal{J}}$ contains a subinstance which is “consistent enough”, and then we apply a result from [5] which says that such instances always have a solution.

In the next section we introduce the required consistency notions.

5. Consistency notions, Proof of Theorem 3.2

5.1. (1, 2)-systems.

**Definition 5.1.** Let $P = (V, A, C)$ be a simple binary instance and let $\{R_x : x \in V\}$ be a family of nonempty subsets of $A$. We say that $P$ is a (1, 2)-system with unary projections $\{R_x : x \in V\}$, if, for any $x_1, x_2 \in V$, the projection of $R_{x_1, x_2}$ to the first coordinate is equal to $R_{x_1}$. (It follows that the projection to the second coordinate is equal to $R_{x_2}$.)

If, moreover, $A$ is an algebra and $P$ is an instance of CSP($A$) we say that $P$ is a (1, 2)-system over $A$.

\[ R_{(a_1, \ldots, a_n), (b_1, \ldots, b_n)} = \{(t(a_1, \ldots, a_n), t(b_1, \ldots, b_n)) : t \in Pol_n(A)\} \]

Note that $P(\mathbb{A}, n)$ is indeed an instance of CSP((A, Pol(A))).

**Proposition 4.5.** For every binary relational structure $\mathbb{A}$, the set of solutions of $P(\mathbb{A}, n)$ is equal to $Pol_n(\mathbb{A})$.

**Proof.** It is clear that every $n$-ary polymorphism of $\mathbb{A}$ is a solution of $P(\mathbb{A}, n)$.

Let $f$ be a solution to $P(\mathbb{A}, n)$. We have to show that every relation $R$ of $\mathbb{A}$ is preserved by $f$. But this is easy – if $R$ is binary and $n$-tuples $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n$ are such that $(a_i, b_i) \in R$ for each $1 \leq i \leq n$, then $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in R(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ (as $f$ is a solution), therefore $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) = (t(a_1, \ldots, a_n), t(b_1, \ldots, b_n))$ for some $t \in Pol_n(\mathbb{A})$. Since $t$ is a polymorphism, the right hand side is an element of $R$. The proof for a unary relation $R$ can be done, for instance, by using this reasoning for the relation $R \times A$. 

We show that this set is equal to $Pol_n(\mathbb{A})$. 

\[ R_{(a_1, \ldots, a_n), (b_1, \ldots, b_n)} = \{(t(a_1, \ldots, a_n), t(b_1, \ldots, b_n)) : t \in Pol_n(\mathbb{A})\} \]
Observe that if $P$ is a $(1, 2)$-system over $A$, then each $R_x$ is a subuniverse $A$ (since the set $R_x$ is equal to the projection of $R_{x,x}$ to the first coordinate). In this case we denote the subalgebra of $A$ with universe $R_x$ by $R_x$.

When a simple binary instance $P$ is drawn as a multipartite graph (see the note after Definition 4.3), then $P$ is a $(1, 2)$-system if and only if, for every pair $x_1, x_2$ of variables, every vertex $a \in R_{x_1}$ is adjacent to at least one vertex from $R_{x_2}$ and to no vertex outside $R_{x_2}$ (in particular, vertices outside the sets $R_x$ are isolated).

Whether an instance has a restriction which is a (nonempty) $(1, 2)$-system can be decided using trees:

**Definition 5.2.** Let $P = (V, A, C)$ be a simple binary instance. A $P$-tree $T$ is a tree (i.e., an undirected connected graph without loops or cycles) whose vertices are labeled by variables in $V$. The vertex set of $T$ is denoted by $\text{vert}(T)$ and the label of a vertex $v \in \text{vert}(T)$ by $\text{lbl}(v)$.

A realization of a $P$-tree $T$ in $P$ is a mapping $r : \text{vert}(T) \to A$ such that $(r(v_1), r(v_2)) \in R_{\text{lbl}(v_1), \text{lbl}(v_2)}$ whenever $v_1, v_2$ are adjacent vertices of $T$. For a vertex $v$ of $T$ we put

$$T[v] = \{r(v) : r \text{ is a realization of } T \text{ in } P\}$$

If $P$ is a $(1, 2)$-system with unary projections $\{R_x : x \in V\}$ then every $P$-tree is clearly realizable. Moreover, for every $P$-tree $T$ and every vertex $v$ of $T$, we have $T[v] = R_{\text{lbl}(v)}$.

The following proposition provides a converse to this observation.

**Proposition 5.3.** Let $P = (V, A, C)$ be a simple binary instance over an algebra $A$. If every $P$-tree is realizable in $P$ then, for every $x \in V$, the set

$$R_x = \bigcap_{T \text{ is a } P\text{-tree}} T[v]$$

is nonempty and $P_{\{R_x : x \in V\}}$ is a $(1, 2)$-system over $A$.

**Proof.** Since $A$ is a finite set, each $R_x$ can be obtained by intersecting the sets $T[v]$ for only finitely many $P$-trees $T$. Moreover, there exists a single tree $T_x$ with vertex $x$ labeled by $x$ such that $R_x = T_x[x]$: We take the disjoint union of the finite collection of trees and identify the vertices $v$ to a single vertex. It follows that $R_x$ is nonempty for every $x \in V$.

Next we prove that $P_{\{R_x : x \in V\}}$ is a $(1, 2)$-system. It is enough to show that for every $x_1, x_2 \in V$ and every $a_1 \in R_{x_1}$, there exists $a_2 \in R_{x_2}$ such that $(a_1, a_2) \in R_{x_1, x_2}$. Consider the $P$-tree $T$ constructed from $T_{x_2}$ by adding a vertex $v_1$ adjacent to $v_2$ with label $x_1$. This $P$-tree has a realization $r$ such that $r(v_1) = a_1$ (since $R_{x_1} \subseteq T[v_1]$). Now we can put $a_2 = r(v_2)$, because $(r(v_1), r(v_2)) \in R_{x_1, x_2}$ and $r(v_2) \in T[v_2] \subseteq T_{x_2}$. Finally, we have to show that $P_{\{R_x : x \in V\}}$ is an instance of CSP($A$). It is clearly enough to prove that $R_x (= T_x[x])$ is a subuniverse of $A$ for every $x \in V$. But this is a straightforward consequence of the definitions: for any $P$-tree $T$, any operation $t$ of $A$ (say, $k$-ary) and any $k$-tuple of realizations $r_1, \ldots, r_k$ of $T$ in $P$, the mapping $r$ defined by $r(v) = t(r_1(v), \ldots, r_k(v))$ is a realization of $T$ in $P$ (as $R_{x_1, x_2}$ is a subuniverse of $A^2$ for every $x_1, x_2 \in V$).

**Remark 5.4.** The family $R = \{R_x : x \in V\}$ from the previous proposition is actually the largest family such that $P_{\{R\}}$ is a $(1, 2)$-system. Also observe that if some $P$-tree is not realizable, then no such a family exists.
5.2. (2, 3)-systems. A (2, 3)-system is a (1, 2)-system such that every edge extends to a triangle:

**Definition 5.5.** A (1, 2)-system \( P = (V, A, C) \) is called a (2, 3)-system if for every \( x_1, x_2, x_3 \in V \) and every \( (a_1, a_2) \in R_{x_1,x_2} \) there exists \( a_3 \in A \) such that \( (a_1, a_3) \in R_{x_2,x_3} \) and \( (a_2, a_3) \in R_{x_2,x_3} \).

Examples of (2, 3)-system include the instances \( P(\mathcal{A}, n) \) introduced in Section 4. Indeed, \( P(\mathcal{A}, n) \) is a (2, 3)-system with unary projections \( \{R_x : x \in V\} \), where

\[
R_{(a_1, \ldots, a_n)} = \{t(a_1, \ldots, a_n) : t \in \text{Pol}_n(\mathcal{A})\}.
\]

The following theorem is the main new ingredient for the proof of the Zádori conjecture. Its proof spans Section 6.

**Theorem 5.6.** Let \( P = (V, A, C) \) be a (2, 3)-system with unary projections \( \{R_x : x \in V\} \) over a CD algebra \( \mathcal{A} \) and let \( \mathcal{J} = \{J_x : x \in V\} \) be a family of (nonempty) subsets of \( A \) such that each \( J_x \) is a Jonsson ideal of \( R_x \). If all \( P \)-trees with at most \( 4^{h(A)} \) vertices are realizable in \( P_{|\mathcal{J}} \), then all \( P \)-trees are realizable in \( P_{|\mathcal{J}} \).

The core result of [3] states that every (2, 3)-system over a CD algebra has a solution (Theorem 5.2. in [3]). We will need a refinement proved (although not explicitly stated) in the same article.

**Theorem 5.7.** Let \( P = (V, A, C) \) be a (2, 3)-system with unary projections \( \{R_x : x \in V\} \) over a CD algebra \( \mathcal{A} \) and let \( \mathcal{J} = \{J_x : x \in V\} \) be a family of (nonempty) subsets of \( A \) such that each \( J_x \) is a Jonsson ideal of \( R_x \). If \( P_{|\mathcal{J}} \) is a (1, 2)-system, then \( P_{|\mathcal{J}} \) has a solution.

**Remark 5.8.** The method used to prove Theorem 5.2. in [3] was the following. If \( \mathcal{J} \) satisfies the assumptions (such families are called absorbing system in [3]) and some of the sets \( J_x \) is more than one element, then it is possible (Lemma 6.9. in [4]) to find a family \( \mathcal{J}' = \{J_x' : x \in V\} \) such that \( \mathcal{J}' \) satisfies the same conditions, \( J_x' \subseteq J_x \) and at least one of these inclusions is proper. In this way we eventually get a solution to \( P_{|\mathcal{J}} \).

More recently, the result that every (2, 3)-system over a CD algebra \( \mathcal{A} \) has a solution was generalized in two directions. First, a weaker consistency notion than (2, 3)-system is enough to guarantee a solution. It suffices to assume that the instance is so called Prague strategy (see [4]). A more “modern” proof of Theorem 5.7 would be to show that \( P_{|\mathcal{J}} \) is a Prague strategy (which is not hard).

A weaker condition can also be put on the algebra. It is enough to assume that \( \mathcal{A} \) lies in a meet semi-distributive variety (actually, for an idempotent finite algebra \( \mathcal{A} \), the statement “every (2, 3)-system (or Prague strategy) over \( \mathcal{A} \) has a solution” is equivalent to “\( \mathcal{A} \) lies in a meet semi-distributive variety” [4]).

5.3. Proof of the Zádori conjecture. We are ready to proof the main theorem. As discussed in Section 3 it is enough to prove Theorem 3.2.

**Theorem.** If \( A \) is a finite binary relational structure such that \((A, \text{Pol}(A))\) is a CD algebra, then \( A \) has a near unanimity polymorphism.

**Proof.** Let \( p_0, \ldots, p_s \) be a Jónsson chain of (term) operations of the algebra \((A, \text{Pol}(A))\). Let \( A \) be the algebra with universe \( A \) whose operations are idempotent polymorphisms of \( A \). Since \( p_i \)'s are idempotent, \( p_0, \ldots, p_s \) is a Jónsson chain for the algebra \( A \).
Let \( n \) be a natural number greater than \( 4^{8^{(|A|)}} \) and let \( P = P(\mathbb{A}, n) \). It was observed above that \( P \) is a \((2,3)\)-system over \( A \) with unary projections \( \{ R_x : x \in V \} \).

Let \( J = \{ J_x : x \in V \} \), where \( J_{(a,a,\ldots,a,b,a,a,\ldots,a)} = \{ \alpha \} \) (for every \( a, b \in A \) and every position of \( b \) in the tuple), and \( J_{(a_1,\ldots,a_n)} = A \) otherwise. Since \( A \) is idempotent, each \( J_x \) is a Jónsson ideal of \( R_x \). As discussed in Section 4 the solutions to the instance \( Q = P_{|J} \) are \( n \)-ary near unanimity polymorphisms of \( A \), therefore it is enough to show that \( Q \) has a solution.

First we observe that every \( P \)-tree \( T \) with at most \( n - 1 \) vertices is realizable in \( Q \). Indeed, the variables are \( n \)-tuples and \( T \) has less than \( n \) vertices, therefore there exists a natural number \( i \) (with \( 1 \leq i \leq n \)) such that \( b \) is not on the \( i \)-th position of any tuple \( x = (a, a, \ldots, a, b, a, a, \ldots) \), \( a \neq b \) which is a label of a vertex of \( T \). Then the mapping assigning \( a_i \) to a vertex of label \( (a_1, \ldots, a_n) \) is a realization of \( T \) in \( Q \).

By Theorem 5.6 every tree is realizable in \( Q \).

Proposition 5.3 (applied to the simple binary instance \( Q \)) gives us a system \( J' = \{ J'_x : x \in V \} \) such that \( Q_{|J'} \) is a \((1,2)\)-system with unary projections \( J' \).

For every \( x \in V \), \( J'_x \) is a Jónsson ideal of \( R_x \). Indeed, in the proof of Proposition 5.3 we have shown that \( J'_x = T^2_x[v_x] \) for certain tree \( T_x \) and its vertex \( x \). If \( a_1, a_2 \in J'_x \) and \( b \in R_x \), then there exists a realization \( r_1 \) (resp. \( r_2 \)) of \( T_x \) in \( Q \) such that \( r_1(v_x) = a_1 \) (resp. \( r_2(v_x) = a_2 \)) and, since \( P \) is a \((1,2)\)-system, there exists a realization \( r_3 \) of \( T_x \) in \( P \) such that \( r_3(v_x) = b \). Now we apply the Jónsson term operation \( p_i \) to \( r_1, r_2, r_3 \) (in the same way as in the last paragraph of the proof of Proposition 5.3) and we get a realization \( r \) of \( T_x \) in \( P \) such that \( r(v_x) = p_i(a_1, b, a_2) \).

From the assumption that \( J_x \) is a Jónsson ideal of \( R_x \), (for every \( x' \in V \)) it follows that \( r \) is a realization in \( Q \). Therefore \( p_i(a_1, b, a_2) \in T'^2_x[v_x] = J'_x \).

Finally, \( P \) is a \((2,3)\)-system, \( J' \) is formed by Jónsson ideals of appropriate \( R_x \)'s and \( P_{|J'} \) (= \( Q_{|J'} \)) is a \((1,2)\)-system, thus, by Theorem 5.7, \( P_{|J'} \) has a solution, which is of course also a solution to \( P_{|J} \). \( \square \)

6. Proof of Theorem 5.6

The entire section is devoted to the proof of Theorem 5.6.

**Theorem.** Let \( P = (V, A, C) \) be a \((2,3)\)-system with unary projections \( \{ R_x : x \in V \} \) over a CD algebra \( A \) and let \( J = \{ J_x : x \in V \} \) be a family of subsets of \( A \) such that each \( J_x \) is a Jónsson ideal of \( R_x \). If all \( P \)-trees with at most \( 4^{8^{(|A|)}} \) vertices are realizable in \( P_{|J} \), then all \( P \)-trees are realizable in \( P_{|J} \).

We argue by contradiction – we take a tree which is not realizable in \( P_{|J} \) and we eventually obtain a configuration (a tuple \( (B, L, U, E, F, a, b) \) which will contradict the following auxiliary result. In this lemma we look at the binary relations \( E, F \) on \( B \) as digraphs.

**Lemma 6.1.** Let \( B \) be a finite CD algebra and let \( U, L \subseteq B \), \( E, F \leq B^2 \), \( a, b \in B \) be such that

- \( E \) is a Jónsson ideal of \( F \),
- \( U \) is disjoint from \( L \),
- \( a \in U, b \in L, (a, b) \in F \),
- the digraph \( E \cap U^2 \) has no sources (that is, for all \( c \in U \) there exists \( d \in U \) such that \( (d, c) \in E \)), and
-
• the digraph $E \cap L^2$ has no sinks (that is, for all $c \in L$ there exists $d \in L$ such that $(c, d) \in E$).

Then there exist $c \in U$ and $d \in B \setminus U$ such that $(c, d) \in E$.

Proof. We take a counterexample to the lemma and fix a Jónsson chain $p_0, p_1, \ldots, p_s$ of term operations of $B$. We may assume that $B$ is idempotent, otherwise we can replace $B$ by the algebra $(B, \{p_0, p_1, \ldots, p_s\})$.

Let us quickly sketch the proof on a smallest choice which does not satisfy the conclusion: $B = \{1, 2\}, U = \{1\}, L = \{2\}, E = \{(1, 1), (2, 2)\}, F = \{(1, 1), (2, 2), (1, 2)\}$, $a = 1$, $b = 2$. The first step of the proof is to transform our counterexample to a form closer to this simplest one, then we prove that $E$ must at least contain the edge $(2, 1)$ and finally we show that in this case we would have a directed path from $1$ to $2$ in the digraph $E$.

Since $E \cap L^2$ has no sources we can find a sequence $a = a_1, a_2, \ldots$ of elements in $U$ such that $(a_{i+1}, a_i) \in E$ for all $i$. As $U$ is finite there are positive numbers $k$ and $l$ such that $a_k = a_{k+l}$. Similarly we find a sequence $b = b_1, b_2, \ldots$ of elements in $L$ such that $(b_i, b_{i+1}) \in E$ and positive numbers $k'$ and $l'$ such that $b_{k'} = b_{k'+l'}$.

Let $m$ be a natural number greater than or equal to $k + k' - 1$ and divisible by $l$ and $l'$, let

$$E' = F \circ \cdots \circ F, \quad F' = F \circ \cdots \circ F, \quad U' = U, \quad B' = B, \quad a' = a_k,$$

where $\circ$ denotes the composition of relations defined by

$$S \circ S' = \{(s, s'') : 3s' \in B, (s, s') \in S, (s', s'') \in S'\},$$

and let $b' \in \{b_{k'}, b_{k'+1}, \ldots, b_{k'+l'}\}$ be an element such that there exists a directed path in the digraph $F$ from $a'$ to $b'$ of length $m$ (we can take the element of appropriate distance from $a'$ on the path $a' = a_k, a_k, \ldots, a_1, b_1, b_2, \ldots, b_{k'}, b_{k'+1}, \ldots, b_{k'+l'} = b_{k'}, b_{k'+1}, \ldots$).

These new sets $B', E', F', U'$ and elements $a', b'$ have the following properties.

• $B'$ is a CD algebra, $E' \leq F' \leq B'^2$, $E'$ is a Jónsson ideal of $F'$. This is straightforward. (That $E', F'$ are subalgebras follows from a more general fact that any relation positively primitively defined from subpowers is a subpower, but it is easy to check the claims directly.)

• $a' \in U', b \in B' \setminus U'$, $(a', a') \in E'$, $(b', b') \in E'$, $(a', b') \in F'$. We have chosen $b'$ so that there exists a directed path in $F$ of length $m$ between them, thus $(a', b') \in F'$. Since $a'$ (resp. $b'$) is in closed path of length $l$ (resp. $l'$) and this length divides $m$, it follows that $(a', a'), (b', b') \in E'$.

• There do not exist $c \in U', d \in B' \setminus U'$ such that $(c, d) \in E'$. Otherwise there is a path in $E$ from $c$ to $d$ in $E$, which is impossible as there is no edge from $U'$ to $B' \setminus U'$.

We will show that even such a configuration (consisting of $B', U', E', F', a', b'$) is impossible. Let us assume that $B', U', E', F', a', b'$ satisfy these three conditions and $|B'|$ is the smallest possible.

The minimality assumption has some useful consequences.

• $(c, c) \in E'$ for any $c \in B'$. Otherwise the following choice would form a smaller counterexample: $B'' = \{c : (c, c) \in E'\}$, $U'' = U' \cap B''$, $E'' = E' \cap B''^2$, $F'' = F' \cap B''^2$, $a'' = a'$, $b'' = b'$. That $B''$ is a subuniverse of
Lemma 6.3. There exists a $T$ of a non-realizable tree to a tree whose every vertex has degree 1 or 3 and which has $T$.

To obtain a configuration contradicting the previous lemma we will first transform $T$ to a tree whose every vertex has degree 1 or 3 and which has $T$.

Now we consider the sequence $a' = p_1(a', a', b'), p_1(a', b', b') = p_2(a', b', b'), p_2(a', a', b') = p_3(a', a', b'), \ldots$

where $s'$ is odd and $s' = s - 1$ if $s$ is even.

As $(a', a'), (b', b') \in E'$, $(a', b') \in E'$ and $E'$ is a Jónsson ideal of $E'$, it follows that the first pair of elements of this sequence is in $E'$. Similarly, the second pair is in $E'^{-1}$, the third pair in $E'$, and so on. Thus we have a “fence” in $E'$ from $a'$ to $b'$ and, since we are assuming that there are no $c \in U' \cap B' \cap U'$ such that $(c, d) \in E'$, there must exist elements $c \in U'$ and $d \in E' \setminus U'$ such that $(d, c) \in E'$.

We have $(c, c), (d, d), (d, c) \in E'$ and $(c, d) \in E'$. It follows that $c = p_1(c, c, d), p_1(c, d, d) = p_2(c, d, d), p_2(c, d, d) = p_3(c, c, d), \ldots, d$

is a sequence where all the pairs are in $E'$. This contradicts the assumption that there is no element in $U'$ which is $E'$-related to an element outside $U'$.$\Box$

To obtain a configuration contradicting the previous lemma we will first transform a non-realizable tree to a tree whose every vertex has degree 1 or 3 and which has no realization in $P$ with leaves realized in $P_{\mathcal{J}}$. We require the following definition.

**Definition 6.2.** Let $T$ be a $P$-tree and $S$ a subset of vertices of $T$. A realization $r$ of $T$ in $P$ is called an $S$-realization, if $r(v) \in J_{\mathcal{J}}(v)$ for every $v \in S$.

For a vertex $v$ of $T$ we define $T_S[v] = \{r(v) : r$ is an $S$-realization of $T$ in $P\}$

The set $S$ from the definition will often be the set of all leaves of $T$, which we denote by $\text{leaves}(T)$.

**Lemma 6.3.** There exists a $P$-tree $T$ such that

- the degree of any vertex of $T$ is 1 or 3,
- $T$ has no leaves($T$)-realization,
- $T$ has a $S$-realization for every proper subset $S$ of leaves($S$).

**Proof.** We start with a $P$-tree $T$ which is not realizable in $P_{\mathcal{J}}$. To every inner vertex (i.e. a vertex of degree greater than one) we add an adjacent vertex with the same label. Since $R_{\mathcal{J}}$ is a subset of the equality relation, any realization maps the new leaf to the same element of $A$ as the inner vertex. It follows that the new tree is not leaves($P$)-realizable.

In a similar way we can modify the tree so that all the vertices have degree at most 3. If a vertex $v$ has degree at least 4, we can split it into two adjacent vertices $v_1, v_2$ with the same label in such a way that $v_1$ is adjacent to 2 of the original neighbors of $v$ and $v_2$ is adjacent to the remaining neighbors. Clearly, $v_1$
and $v_2$ have smaller degree than $v$, therefore we can repeat this splitting procedure until we obtain a tree whose every vertex has degree at most 3 and which is not leaves($P$)-realizable.

Let $T$ be such a tree with minimal number of vertices.

Now we show that $T$ has no vertex of degree 2. Suppose otherwise, that is, there is a vertex $v$ with precisely two neighbors $v_1, v_2$. The tree $T'$ obtained by removing the vertex $v$ and adding the edge $v_1 - v_2$ is smaller than $T$, therefore $T'$ has a leaves($T$)-realization $r$. As $(r(v_1), r(v_2)) \in R_{\text{lbl}}(v_1), \text{lbl}(v_2)$ and $P$ is a $(2,3)$-system, there exists $a \in A$ such that $(r(v_1), a) \in R_{\text{lbl}}(v_1), \text{lbl}(v)$ and $(r(v_2), a) \in R_{\text{lbl}}(v_2), \text{lbl}(v)$ (This is the only place in this section where we use the assumption that $P$ is a $(2,3)$-system. For the rest it would suffice to assume that $P$ is a $(1,2)$-system.) It follows that the extension of the mapping $r$ by $r(v) = a$ is a leaves($T$)-realization of $T$, a contradiction.

It remains to show that $T$ is $S$-realizable for every proper subset $S$ of leaves($S$). But this is easy: If we remove a leaf outside $S$ the remaining tree is $S$-realizable (from the minimality of $T$) and this realization can be extended to an $S$-realization of $T$ as $P$ is a $(1,2)$-system. 

For the remainder of the proof we fix a $P$-tree $T$ with the properties stated in the previous lemma.

**Lemma 6.4.** $T$ contains a path of length at least $2 \cdot 8^{|A|}$.

**Proof.** It can be easily computed that a tree, which has all vertices of degree at most 3 and which does not contain any path with more than $k$ vertices, has size at most $2^k$ (this is a crude estimate, one computes that the most accurate estimate is $3 \cdot 2^{k/2} - 2$ for even $k$ and $2^{(k+3)/2} - 2$ for odd $k > 1$).

Since $T$ has more than $4^{|A|}$ vertices by our assumption (smaller $P$-trees are even realizable in $P_{(2)}$), the claim follows. 

We fix a subpath $v_1, v_2, \ldots, v_m$ of $T$ where $m \geq 2 \cdot 8^{|A|}$. Let $S_i, i = 1, 2, \ldots, m$, denote the leaves of $T$ whose shortest path to $v_i$ does not contain neither $v_{i-1}$ nor $v_{i+1}$. (For $v_1$ only the vertex $v_2$ is considered. If $v_1$ is a leaf then $S_1 = \{v_1\}$. Similarly for $v_m$.) In other words, we straighten the line $v_1, \ldots, v_m$ and shake the tree. Then $S_i$ are the leaves below $v_i$.

The next lemma will enable us to find the sought after configuration.

**Lemma 6.5.** There exist natural numbers $k, l$ such that

- $1 \leq k, l \leq m, k \leq l + 2$,
- $T_{S_k}[v_k] = T_{S_l}[v_1]$,  
- $T_{S_1 \cup S_2 \cup \ldots \cup S_k}[v_k] = T_{S_1 \cup S_2 \cup \ldots \cup S_l}[v_1] \neq \emptyset$, and
- $T_{S_k \cup S_{k+1} \cup \ldots \cup S_m}[v_k] = T_{S_l \cup S_{l+1} \cup \ldots \cup S_m}[v_1] \neq \emptyset$.

**Proof.** There is at least $m/2 - 1 \geq 8^{|A|} - 1$ even numbers less than $m$. For each such number $i$ we consider the triple

$$(T_{S_i}[v_1], T_{S_1 \cup \ldots \cup S_i}[v_1], T_{S_1 \cup \ldots \cup S_m}[v_1])$$

of subsets of $A$ (note that these subsets are nonempty by the third item of Lemma 6.3). There are less than $(2^{|A|} - 1)^3 < 8^{|A|} - 2$ possible triples, therefore, by pigeonhole principle, there exist distinct $k, l$ with the same associated triples and the lemma follows.
Again, the estimates we used are very rough. For instance, the second and third sets in the triple are disjoint subsets of the first subset. This significantly reduces the number of possibilities, etc.

Let

\[ Q_1 = S_1 \cup S_2 \cup \cdots \cup S_k, \quad Q_2 = S_k \cup \cdots \cup S_l, \quad Q_3 = S_l \cup \cdots \cup S_m. \]

Now we define

\[ B = T_{S_k}[v_k] = T_{S_l}[v_l], \]
\[ L = T_{Q_2 \cup Q_3}[v_k] = T_{Q_1}[v_l], \]
\[ U = T_{Q_1}[v_l] = T_{Q_2 \cup Q_3}[v_k], \]
\[ E = \{(r(v_k), r(v_l)) : r \text{ is a } Q_2\text{-realization of } T \}, \]
\[ F = \{(r(v_k), r(v_l)) : r \text{ is a } (S_k \cup S_l)\text{-realization of } T \}, \]
\[(a, b) = (r(v_k), r(v_l)) \text{ for a chosen } (Q_1 \cup Q_3)\text{-realization } r \text{ of } T.\]

Since \( k \leq l - 2 \) and \( S_{k+1} \neq \emptyset \) (by the first item of Lemma 6.3), \( Q_1 \cup Q_3 \) is a proper subset of leaves\( (T) \), therefore \( T \) has a \((Q_1 \cup Q_3)\)-realization by the third item of Lemma 6.3, and the definition of \( a \) and \( b \) makes sense. This choice satisfies all the assumptions of Lemma 6.1:

- \( B \) is a subuniverse of \( A \). It follows directly from the definitions (see the last paragraph of the proof of Proposition 5.3). Let \( \mathbf{B} \) be the subalgebra of \( A \) with universe \( B \).
- \( E, F \leq B, E \) is a Jónsson ideal of \( F \). This is also straightforward. That \( E \) is a Jónsson ideal of \( F \) follows from the assumption that \( J_x \) is a Jónsson ideal of \( R_x \) for every \( x \in Q_2 \).
- \( U \) and \( L \) are disjoint. Suppose \( c \in U \cap L \). Since \( U = T_{Q_1}[v_k] \), there exists a \( Q_1\)-realization \( r_1 \) of \( T \) such that \( r_1(v_k) = c \). Similarly, since \( L = T_{Q_2 \cup Q_3}[v_k] \), there exists a \((Q_2 \cup Q_3)\)-realization \( r_2 \) of \( T \) such that \( r_2(v_k) = c \). The realizations \( r_1 \) and \( r_2 \) can be joined in the following way: we put \( r(v) = r_1(v) \) for vertices \( v \) whose shortest path to \( v_k \) does not contain \( v_{k+1} \), and \( r(v) = r_2(v) \) for the other vertices. Now \( r \) is a \((Q_1 \cup Q_2 \cup Q_3)\)-realization of \( T \). But \( Q_1 \cup Q_2 \cup Q_3 \) is the set of all leaves of \( T \), a contradiction (see the second item of Lemma 6.3).
- \( a \in U, b \in L, (a, b) \in F \). The element \( a \) is defined as \( r(v_k) \) for a \((Q_1 \cup Q_3)\)-realization \( r \) of \( T \). Since \( Q_1 \subseteq Q_1 \cup Q_3 \) we have \( T_{Q_1 \cup Q_3}[v_k] \subseteq T_{Q_1}[v_k] = U \), therefore \( a = r(v_k) \in U \). Similarly, \( b \in L \) follows from \( b = r(v_l) \), \( Q_3 \subseteq Q_1 \cup Q_3 \) and \( L = T_{Q_1}[v_l] \), and \((a, b) \in F \) follows from \( S_k \cup S_l \subseteq Q_1 \cup Q_3 \).
- \( E \cap U^2 \) has no sources, \( E \cap L^2 \) has no sinks. Let \( c \) be an arbitrary element of \( U \). Since \( U = T_{Q_1 \cup Q_3}[v_k] \) there exists a \((Q_1 \cup Q_2)\)-realization \( r \) of \( T \) such that \( r(v_l) = c \). But \( r \) is also a \((Q_1 \cup Q_3)\)-realization of \( T \), hence \((r(v_k), r(v_l)) \in U \). The element \( d = r(v_k) \) lies in \( T_{Q_1 \cup Q_3}[v_k] \subseteq T_{Q_2}[v_k] = U \). We can analogically show that \( E \cap L^2 \) has no sinks: any \( c \in L \) is equal to \( r(v_k) \) for a \((Q_2 \cup Q_3)\)-realization \( r \) of \( T \), and \( r(v_l) \in T_{Q_3}[v_l] = L \).
- There do not exist \( c \in U \) and \( d \in B \setminus U \) such that \((c, d) \in E \). If \( c \in U = T_{Q_1}[v_k] \) and \((c, d) \in E \), then there exists a \( Q_1\)-realization \( r_1 \) of \( T \) and a \( Q_2\)-realization \( r_2 \) of \( T \) such that \( r_1(v_k) = c = r_2(v_k) \) and \( r_2(v_l) = d \). When we join \( r_1 \) and \( r_2 \) in the same way as in the proof that \( U \) and \( L \) are disjoint we
get a \((Q_1 \cup Q_2)\)-realization \(r\) of \(T\) such that \(r(v_1) = d\). But \(U = T_{Q_1 \cup Q_2}[v_1]\), thus \(d \in U\).

The last property contradicts Lemma 6.1 which concludes the proof of Theorem 5.6.

7. Conclusion

7.1. Decidability of near unanimity for relational structures. As a corollary of the main theorem we obtain an affirmative answer to the near unanimity problem for relations:

**Corollary 7.1.** It is decidable whether a finite relational structure with finitely many relations has a near unanimity polymorphism.

**Proof.** It is enough to decide whether the given relational structure has a Jónsson chain of polymorphisms. This can be decided as follows. We first compute the set \(P\) of all ternary idempotent polymorphisms satisfying \(p(a, b, a) = a\) and then compute the graph whose vertices are idempotent binary operations and edges are \(f - g\), if there exists \(p_1, p_2\) such that

\[
f(a, b) = p_1(a, a, b), p_1(a, b, b) = p_2(a, b, b) \text{ and } g(a, b) = p_2(a, a, b)
\]

Jónsson chain exists if and only if \(\pi_1\) is connected to \(\pi_2\), where \(\pi_i\) is the binary projection to the \(i\)-th coordinate. \(\square\)

It was shown in [22] that the corresponding decision problem for algebras (that is, does a given finite algebra with finitely many operations have a near unanimity term operation?) is decidable. This was a surprising development after undecidability results about closely related questions [23, 21].

The naïve algorithm described in the proof of Corollary 7.1 runs in exponential time:

**Open Problem 7.2.** Determine the computational complexity of deciding whether a finite relational structure with finitely many relations has a near unanimity polymorphism.

There exist polynomial time algorithms for finite posets [17] and for finite reflexive undirected graphs [18].

The complexity of the same problem for algebras is also unknown. There exists a polynomial time algorithm for deciding whether a finite idempotent algebra (with finitely many operations) is a CD algebra, and the same problem without assuming idempotency is exponential time complete [12].

7.2. Arities. Our proof gives some upper bound on the minimal arity of a near unanimity polymorphism, namely, a binary relational structure \(A\) either has a near unanimity polymorphism of arity \(4^{|A|} + 1\) or has none. For a relational structure whose relations have maximum arity \(k\) the upper bound is \(4^{|A| k} + 1\). We have used quite rough estimates in a couple of places, however the present proof most likely cannot provide a better upper bound than doubly exponential.

For finite algebras with finitely many operations the upper bound also exists, but is tremendously large and is not even computed in [22].

Therefore we ask the following.
Open Problem 7.3. Give a better upper bound for the minimal arity of a near unanimity polymorphism (resp. term operation) for relational structures with finitely many relations (resp. finite algebras).

7.3. Valeriote conjecture. The most important open problem related to this work is the Valeriote conjecture (also known as the Edinburgh conjecture [7]):

Conjecture 7.4. Every finite, finitely related algebra in a congruence modular variety has few subpowers.

Congruence modularity is a widely studied generalization of congruence distributivity. An algebra $A$ has few subpowers, if the logarithm of the number of subalgebras of $A^n$ is bounded by a polynomial in $n$. This property was defined and its importance in the CSP demonstrated in [14, 5]. Examples of algebras with few subpowers include algebras with a Maltsev operation (e.g. groups, rings, modules) and algebras with a near unanimity operation. It is known [5, 20] that every finite CD algebra with few subpowers has a near unanimity term. Therefore a positive solution to the Valeriote conjecture would imply the main result of this paper. It would also have deep consequences in the complexity of constraints.

A converse to the conjecture generalizing the Baker-Pixley result [2] was proved recently. E. Aichinger, P. Mayr and R. McKenzie [1] have shown that every finite algebra with few subpowers is finitely related.

References


[23] Ralph McKenzie. Is the presence of a nu-term a decidable property of a finite algebra?


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