# Weakly Terminal Objects in Quasicategories of $\mathbb{SET}$ Endofunctors

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**Abstract.** The quasicategory $\mathbb{Q}$ of all set functors (i.e. endofunctors of the category $\mathbb{SET}$ of all sets and mappings) and all natural transformations has a terminal object – the constant functor $C_1$. We construct here the terminal (or at least the smallest weakly terminal object, which is rigid) in some important subquasicategories of $\mathbb{Q}$ – in the quasicategory $\mathbb{F}$ of faithful connected set functors and all natural transformations, and in the quasicategories $\mathbb{B}^{(\kappa)}$ of all set functors and natural transformations which preserve filters of points (up to cardinality $\kappa$).

**Keywords:** weakly terminal object, set functor, rigid object

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## 1. Introduction

Endofunctors of the category $\mathbb{SET}$ of all sets and all mappings were investigated in the late sixties and early seventies under the name "set functors" in [8, 9, 10, 3, 4, 5]. After about thirty years, this field of problems has been refreshed in e.g. [2, 6, 7]. Let $\mathbb{Q}$ be the quasicategory (i.e. big category in the sense of [1], see also the last paragraph of this section) of all set functors and all natural transformations, and $\mathbb{S}$ be its subquasicategory. This paper concerns the following natural question about set functors:

> *Does there exist a set functor $W \in Ob(\mathbb{S})$, such that for every set functor $G \in Ob(\mathbb{S})$, there exists a (resp. unique) natural transformation $\mu : G \to W$, $\mu \in Mor(\mathbb{S})$?*

In other words, does the quasicategory $\mathbb{S}$ have a weakly terminal (resp. terminal) object? For $\mathbb{S} = \mathbb{Q}$, the answer is trivial – the constant functor $C_1$, which assigns a one point set to every set, is the terminal object of this quasicategory. We give the answer for the following quasicategories:

$\mathbb{F}$: The objects are all faithful connected set functors, the morphisms are all natural transformations.

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$\mathbb{B}$: (resp. $\mathbb{B}^\kappa$, where $\kappa$ is a cardinal)
The objects are all set functors, the morphisms are all natural transformations, which preserve filters of all points in the images of all sets (resp. in the images of all sets $X$ with $|X| < \kappa$).

We prove here, that there exists a terminal object $T$ in $\mathbb{B}$ (and $T^\kappa$ in $\mathbb{B}^\kappa$ for $\kappa > 2$) and a smallest weakly terminal object $W$ in $\mathbb{F}$, which is rigid. The word "smallest" means, that for every weakly terminal $V$, there exists a monotransformation $\mu : W \to V$, therefore $V$ has a subfunctor naturally equivalent to $W$; "rigid" means, that the identity is the only natural endotransformation of $W$. The quasicategory $\mathbb{F}$ has no terminal object.

The paper is organized as follows: in section 2, we recall all needed definitions and results, and we present a more detailed description of the above quasicategories. In section 3 we construct the functors $T^\kappa$, $T$ and $W$ respectively.

We close the introduction with a few notes about the set theory used.

- The term "quasicategory" is outside the scope of any classical set theory. It is used to simplify the formulations, but everything could be formulated without using it.
- An ordinal is the set of all smaller ordinals (then e.g. $0 = \emptyset$), and a cardinal is the least ordinal with its cardinality.
- We use the convention, that a mapping $f : X \to Y$ determines even the range set $Y$.

## 2. Preliminaries

Let $G$ be a set functor. If $G1 = 0$, then $G$ assigns the empty set to every set. Otherwise $G$ can be decomposed as a coproduct

$$G = \coprod_{i \in G1} G_i, \quad \text{where } G_i \text{ is } \textbf{connected}, \text{ i.e. } \quad |G_i 1| = 1$$

In this decomposition, each faithful (resp. non-faithful) component $G_i$ contains exactly one subfunctor naturally equivalent to the identity functor (resp. $C_{0,1}$ functor - this functor assigns a one point set to all nonempty sets and the empty set to the empty set). Every natural transformation $\mu : G \to G'$ sends any two points $x, y \in GX$ within the same component in $G$, to the points $\mu_X(x), \mu_X(y)$ within the same component in $G'$.

Let $f$ and $g$ be the two distinct mappings $f, g: 1 \to 2$. Then $G_i$ is faithful, iff $G_i f(x) \neq G_i g(x)$, where $x$ is the element of $G_i 1$ (see [8]).

An element $x \in GX$, which is contained in subfunctor of $G$ naturally equivalent to $C_{0,1}$ or $C_1$, is called **distinguished point** of $G$.

We now define the quasicategories $\mathbb{F}$ and $\mathbb{NF}$:

$\mathbb{F}$: Quasicategory of all faithful connected set functors and all natural transformations.

$\mathbb{NF}$: Quasicategory of all non-faithful connected set functors and all natural transformations.

As mentioned above in other words, the identity functor is the initial object of $\mathbb{F}$, and $C_{0,1}$ functor is the initial object of $\mathbb{NF}$. It is easy to see, that the functor $C_1$ is the terminal object of $\mathbb{NF}$. The smallest weakly terminal object $W$ of $\mathbb{F}$ is constructed in the third section. A noticeable fact is, that $W$ is (naturally equivalent to) a subfunctor of the composition of two contravariant power set functors.

Let $P$ denote the covariant power set functor. We define a set functor $F$ by:

$$\begin{aligned} FX &:= \{\mathcal{F} \mid \mathcal{F} \text{ is filter on } X \text{ or } \mathcal{F} = PX\} \\ Ff(\mathcal{F}) &:= \{V \in Y \mid \exists U \in \mathcal{F}\ f[U] \subseteq V\}, \quad \text{for } f: X \to Y \end{aligned}$$

Thus $FX$ is the set of all nonempty families of subsets of $X$ closed under intersections and oversets. Note, that a family $\mathcal{F} \in FX$ determines the set $X$ – it is the union of $\mathcal{F}$.

Next, we define **filter** of a point $x \in GX$ (see [3]):

$$\begin{aligned} Flt^X_G(x) &:= PX, \quad \text{if x is a distinguished point, otherwise} \\ Flt^X_G(x) &:= \{U \subseteq X \mid (\exists u \in GU)\ Gi(u) = x, \\ &\qquad \text{where } i: U \to X \text{ is the inclusion}\} \end{aligned}$$

If $G$ or $X$ are clear from the context, we will write just $Flt(x)$ or $Flt_G(x)$. We have defined the filter of a distinguished point separately, because it simplifies the formulations of the following propositions and our results. But there is only a slight difference: If $x$ is distinguished, then $\{U \subseteq X \mid (\exists u \in GU)\ Gi(u) = x, i: U \to X \text{ inclusion}\} \cup \{0\} = PX$.

The following propositions contain some known facts (see [8, 3, 2]) about filters of points.

PROPOSITION 2.1. *Let $G$ be a set functor, $X$ be a set, $x \in GX$. Then*

1. *$Flt(x) \in FX$.*
2. *$0 \in Flt(x)$, iff $x$ is distinguished.*
3. *If $f : X \to Y$ is a mapping, then $Flt(Gf(x)) \supseteq Ff(Flt(x))$. Moreover, if $f$ is injective on some $R \in Flt(x)$, then $Flt(Gf(x)) = Ff(Flt(x))$.*
4. *If $f, g : X \to Y$ are mappings, such that $f$ and $g$ coincide on some $R \in Flt(x)$, then $Gf(x) = Gg(x)$.* $\square$

PROPOSITION 2.2. *Let $G, G'$ be set functors, $\mu : G \to G'$ be a natural transformation, $X$ be a set and $x \in GX$. Then $Flt^X_{G'}(\mu_X(x)) - \{0\} = \bigcup\{Flt^X_G(y) \mid \mu_X(y) = \mu_X(x)\} - \{0\}$. Every monotransformation preserves filters of points.* $\square$

Let $\mu$ be a natural transformation of set functors, $\kappa$ be a cardinal. Let ($Flt$) (resp. $Flt^\kappa$) denote the property, that $\mu$ preserves filters of all points in the image of all sets $X$ (resp. all sets $X$ with $|X| < \kappa$), formally

($Flt$): $(\forall X)\ (\forall x \in GX)\ \ Flt(\mu_X(x)) = Flt(x)$

($Flt^\kappa$): $(\forall X\ \ |X| < \kappa)\ (\forall x \in GX)\ \ Flt(\mu_X(x)) = Flt(x)$.

Note, that $\mu$ has the property ($Flt^2$), iff it sends every faithful component to a faithful component. Every natural transformation has the properties ($Flt^0$) and ($Flt^1$).

Finally, we define the quasicategories $\mathbb{B}$ and $\mathbb{B}^\kappa$ , in which the terminal objects $T$ and $T^\kappa$ (for $\kappa > 2$) will be constructed.

$\mathbb{B}$: Quasicategory of all set functors and all natural transformations $\mu$ with the property ($Flt$).

$\mathbb{B}^\kappa$: Quasicategory of all set functors and all natural transformations $\mu$ with the property ($Flt^\kappa$).

## 3. Construction

We start with the construction of $T^\kappa$ for a given cardinal $\kappa > 2$. The following notation will be used:

$$\mathbb{FLT} := \bigcup\{FX \mid X \text{ is a set}\}$$
$$\mathbb{MAP}^\kappa_X := \{f \mid f : X \to Y,\ |f[X]| < \kappa\}$$

Elements of $\mathbb{MAP}^\kappa_X$ will be called **$\kappa$-small mappings**.

DEFINITION 3.1. *We say, that a mapping* $x : \mathbb{MAP}_X^\kappa \to \mathbb{FLT}$ *is a* **$\kappa$-filter structure** *on* $X$*, if for all* $\kappa$*-small mappings* $a : X \to Y$ *and* $b, b' : Y \to Z$*:*

*(A1)* $x(a) \in FY$.

*(A2)* $x(ba) \supseteq Fb(x(a))$*, if* $b$ *is injective, then* $x(ba) = Fb(x(a))$.

*(A3) If* $b, b'$ *coincide on some* $R \in x(a)$*, then* $x(ba) = x(b'a)$.

It follows from (A2), that $x$ is determined by its values on $\kappa$-small mappings to $X$ (and each value is a filter on $X$ due to (A1)), so there exist only set-many $\kappa$-filter structures on any set.

The axioms are not independent – the first part of (A2) follows from the rest of axioms: Let $R \in x(a)$, put $W := b[R] \coprod (Z - b[R]) \coprod (Z - b[R])$, let $i_1 : Z \to W$ be the embedding into the union of the first and the second component and $i_2 : Z \to W$ be the embedding into the union of the first and the third component. We have $Z \in x(ba)$ (from (A1)), $x(i_1 ba) = x(i_2 ba)$ (from (A3)), and $x(i_1 ba) = Fi_1(x(ba))$, $x(i_2 ba) = Fi_2(x(ba))$ (from the second part of (A2)). Altogether $b[R] = i_1[Z] \cap i_2[Z] \in x(i_1 ba)$, thus $b[R] \in x(ba)$.

DEFINITION 3.2. *Let* $G$ *be a set functor,* $x \in GX$*. Define* $\overline{x}$ *by*

$$\overline{x}(a) := Flt(Ga(x)), \quad \textit{for a } \kappa\textit{-small mapping } a \in \mathbb{MAP}_X^\kappa.$$

According to Proposition 2.1, $\overline{x}$ is a $\kappa$-filter structure on $X$, we say "$\kappa$-filter structure of the point $x$".

DEFINITION 3.3. *Let* $x$ *be a* $\kappa$*-filter structure on* $X$*,* $f : X \to Y$ *be a mapping. Define* $f[x]$ *by*

$$f[x](a) := x(af), \quad \textit{for a } \kappa\textit{-small mapping } a \in \mathbb{MAP}_Y^\kappa.$$

It is straightforward to verify, that $f[x]$ is a $\kappa$-filter structure on $Y$.

COROLLARY 3.4. *Let* $\mu : G \to G'$ *be a natural transformation with the property* $(Flt^\kappa)$*. Then* $\mu$ *preserves* $\kappa$*-filter structures.*

*Proof.* Let $f : X \to Y$ be a $\kappa$-small mapping, and $x \in GX$. Let $f = me$, where $e$ is ($\kappa$-small) surjective and $m$ is injective. Then

$$\begin{aligned}\overline{(\mu_X(x))}(me) &= Flt_{G'}(G'me(\mu_X(x))) = Fm(Flt_{G'}(G'e(\mu_X(x)))) = \\ &= Fm(Flt_{G'}(\mu_Y(Ge(x)))) = Fm(Flt_G(Ge(x))) = \\ &= Flt_G(Gme(x)) = \overline{x}(f),\end{aligned}$$

where the first equality follows from the definition of the $\kappa$-filter structure of $\mu_X(x)$, the second follows from Proposition 2.1, the third follows from the naturality of $\mu$, the fourth follows from the property ($Flt^\kappa$), the fifth follows from Proposition 2.1 and the sixth follows from the definition of a $\kappa$-filter structure of $x$. $\square$

Now, we define the functor $T^\kappa$:

$$\begin{aligned} T^\kappa X &:= \{x \mid x \text{ is a } \kappa\text{-filter structure on X}\} \\ T^\kappa f(x) &:= f[x], \quad \text{for } f\colon X \to Y \end{aligned}$$

COROLLARY 3.5. *$T^\kappa$ is a set functor. It has two components, one faithful and one non-faithful. A point $x \in T^\kappa X$ is distinguished, iff $0 \in x(f)$ for every $f \in \mathbb{MAP}^\kappa_X$.*

*Proof.* Strictly speaking, $T^\kappa$ is not a set functor, because each $\kappa$-filter structure is, in fact, a proper class. But the remark after the definition says, that it could be defined to be a set (the way was chosen for it is more transparent).

It's clear, that $T^\kappa$ preserves the compositions and the identities. On the set 1, two different $\kappa$-filter structures exist, namely $x_1$ and $x_2$, where $x_1(id_1) = \{1\}$ and $x_2(id_1) = P1$. For the two distinct mappings $f, g : 1 \to 2$, $f[x_2] = g[x_2]$ and $f[x_1] \neq g[x_1]$ (because one of the filters $f[x_1](id_2)$ and $g[x_1](id_2)$ is generated by $\{0\}$ and the other one is generated by $\{1\}$ – this is the place, where we need to assume $\kappa > 2$, otherwise $f[x_1](id_2)$ is undefined. The functor $T^2$ would be equivalent to $C_{0,1} \coprod C_1$). Therefore $T^\kappa$ has one faithful and one non-faithful component. The last statement is obvious. $\square$

PROPOSITION 3.6. *The $\kappa$-filter structure of any point $x \in T^\kappa X$ is $x$.*

*Proof.* We shall prove, that for any $\kappa$-small mapping $f\colon X \to Y$, we have $Flt(T^\kappa f(x)) = x(f)$. First, it suffices to prove $Flt(x) = x(id_X)$, where $|X| < \kappa$. (For the general case, take factorization $f = me$, where $e\colon X \to Z$ is $\kappa$-small surjective and $m$ is injective. Then

$$\begin{aligned} Flt(T^\kappa f(x)) &= Flt(T^\kappa me(x)) = Fm(Flt(T^\kappa e(x))) \\ &= Fm(T^\kappa e(x)(id_Z)) = Fm(x(e)) = x(me) = x(f), \end{aligned}$$

where the second equality is implied by 2.1, the third one shall be proved, the fourth one follows from the definition of $T^\kappa$, the fifth one is again implied by (A2). )

Next, we prove that $Flt(x) \subseteq x(id_X)$. Let $R \in Flt(x)$. If $R = 0$, then $x$ is distinguished (2.1), hence $0 \in x(id_X)$ due to Corollary 3.5.

Otherwise, there exists $r \in T^\kappa R$ such that $T^\kappa i(r) = x$, $i$ is the inclusion $i\colon R \to X$. Thus $x(id_X) = (T^\kappa i(r))(id_X) = r(i) = Fi(r(id_R)) \ni R$.

Finally, let $R \in x(id_X)$. If $R = 0$, then $0 \in x(f)$ for every $f \in \mathbb{MAP}^\kappa_X$ (use the first part of (A2)), and $x$ is distinguished (Corollary 3.5), hence $R = 0 \in Flt(x)$. Otherwise, let $i$ denote the inclusion $i\colon R \to X$. Take $e$, such that $ei = id_R$, and for a $\kappa$-small mapping $g \in \mathbb{MAP}^\kappa_R$ define $r(g) := x(ge)$. It is routine to verify that $r \in T^\kappa R$. For arbitrary $\kappa$-small mapping $h \in \mathbb{MAP}^\kappa_X$, we have $T^\kappa i(r)(h) = r(hi) = x(hie) = x(h)$, because the last two mappings coincide on $R$, therefore we can use (A3). Thus $T^\kappa i(r) = x$, hence $R \in Flt(x)$. □

THEOREM 3.7. *$T^\kappa$ is the terminal object of $\mathbb{B}^\kappa$.*

*Proof.* Let $G$ be a set functor. We shall prove, that there exists exactly one natural transformation $\mu\colon G \to T^\kappa$ with the property $(Flt^\kappa)$. Define $\mu$ by $\mu(x) := \overline{x}$. For every point $x \in GX$, every mapping $f\colon X \to Y$, and every $\kappa$-small mapping $a\colon Y \to Z$, we have

$$(\mu_Y(Gf(x)))(a) = \overline{Gf(x)}(a) = Flt(Ga(Gf(x)) = Flt(Gaf(x)),$$

and

$$(T^\kappa f(\mu_X(x)))(a) = f[\overline{x}](a) = \overline{x}(af) = Flt(Gaf(x))$$

so $\mu$ is a natural transformation. The uniqueness follows from Corollary 3.4 and Proposition 3.6. □

The functor $T^\kappa$ is rigid and it is the smallest weakly terminal object of $\mathbb{B}^\kappa$ (the terminal object in every (quasi)category has these properties).

The construction of the terminal object $T$ of the quasicategory $\mathbb{B}$ is similar – just delete all $\kappa$'s above (for example $\mathbb{MAP}_X$ is the class of all mappings; we have filter structures instead of $\kappa$-filter structures, etc.. $T$ is still a set functor – a filter structure on $X$ is determined by values on mappings to $X$, therefore there are only set-many filter structures on $X$).

Note, that $T^\alpha X = T^\beta X = TX$ for cardinals $\alpha, \beta$ and a set $X$ such that $|X| < \alpha, \beta$ (this follows from the definition of these functors), and that there exists a (unique) monotransformation $\mu^{\alpha\beta}\colon T^\alpha \to T^\beta$ (and $\mu^\alpha\colon T^\alpha \to T$) with the property $Flt^\alpha$ for cardinals $\alpha < \beta$ (this follows from 3.7). Moreover, the $X$-th component of this monotransformation is the identity for every set $X$ such that $|X| < \alpha$.

It remains to construct the smallest weakly terminal object $W$ of the quasicategory $\mathbb{F}$. In fact, this functor was already constructed: $W$ is the faithful component of $T^3$.

THEOREM 3.8. *$W$ is the smallest weakly terminal object in $\mathbb{F}$, $W$ is rigid.*

*Proof.* Let $G$ be a faithful connected set functor, and $\mu : G \to T^3$ be the natural transformation with the property $(Flt^3)$ (it exists due to Theorem 3.7). Since $\mu$ preserves the filter of the point $x \in G1$, it sends $G$ to the faithful component $W$ of $T^3$. Hence, $\mu$ is indeed a transformation $\mu : G \to W$.

Next, we show that $W$ is rigid. Let $\mu : W \to W$ be a natural transformation. Let $x$ be the point $x \in W1$ and $f, g$ be the two mappings $f, g : 1 \to 2$, $f(0) = 0$, $g(0) = 1$. There are three 2-filter structures in $W2$. Namely $a = Wf(x)$, $b = Wg(x)$, $c$, here $a(id_2)$ (resp. $b(id_2)$) is generated by $\{0\}$ (resp. $\{1\}$), $c(id_2) = \{\{0, 1\}\}$. The naturality of $\mu$ gives $\mu_2(a) = a$ (proof: $\mu_2(a) = \mu_2(Wf(x)) = Wf(\mu_1(x)) = Wf(x) = a$) and $\mu_2(b) = b$ (analogically). If $\mu_2(c) = a$, we obtain a contradiction using the naturality of $\mu$ for the mapping $h : 2 \to 2$, $h(0) = 1$, $h(1) = 0$ ($a = \mu_2(Wh(c)) = Wh(\mu_2(c)) = b$, a contradiction). Symmetrically, it is not possible, that $\mu_2(c) = b$. Hence $\mu$ has the property $(Flt^3)$, so it is the identity due to Corollary 3.4 and Proposition 3.6.

Since $W$ is rigid, standard argumentation shows, that every natural transformation from $W$ to a weakly terminal object is a coretraction, thus $W$ is the smallest weakly terminal object. $\square$

Note, that $W$ is also the smallest weakly terminal object in the quasicategory of set functors with all components faithful; and $W \coprod C_1$ is the smallest weakly terminal object in $\mathbb{B}^2$.

$W$ is not terminal: Consider for example the square functor $Q_2$ ($Q_2 X = X \times X$, $Q_2 f = f \times f$). The projections are natural transformations from $Q_2$ to the identity functor, so by composing them with the monotransformation from the identity functor to $W$, we obtain two distinct natural transformations from $Q_2$ to $W$ (there is still another natural transformation — the one with property $(Flt^3)$). Therefore $\mathbb{F}$ has no terminal object.

$W$ is naturally equivalent to the following subfunctor $W'$ of the composition of two contravariant power set functors:

$$\begin{aligned} W' &:= \{\mathcal{X} \mid 0 \notin \mathcal{X},\ (\forall R \subseteq X)\ (R \in \mathcal{X} \text{ or } X - R \in \mathcal{X})\} \\ W'f(\mathcal{X}) &:= \{S \mid f^{-1}(S) \in \mathcal{X}\}, \quad \text{for } f : X \to Y \end{aligned}$$

The (unique) natural equivalence $\mu : W \to W'$ is:

$$\mu_X(x) = \{R \mid 1 \in x(\chi_{R,X})\},$$

where $\chi_{R,X} : X \to 2$ is the characteristic mapping of the subset $R \subseteq X$. Let us omit the proof, that $\mu$ is a natural equivalence. It is technical and without any deep idea.

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