On simple Taylor algebras

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2-intersection property

for $R \subseteq A_1 \times \cdots \times A_n$, $i, j \in [n]$
let $R_{ij}$ denote the projection of $R$ onto the coordinates $(i, j)$
i.e. $R_{ij} = \{(a_i, a_j) : (a_1, \ldots, a_j) \in R\}$

Question: Which finite idempotent algebras have the 2-intersection property?
One of the motivations: CSP
2-intersection property

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**Definition**

An algebra $A$ has the 2-intersection property if

for any $n$, $R, S \leq A^n$

$\forall i, j \ R_{ij} = S_{ij} \Rightarrow R \cap S \neq \emptyset$
for \( R \subseteq A_1 \times \cdots \times A_n, \quad i, j \in [n] \)

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\[ \mathbf{A} \text{ has the 2-intersection property if} \]

\[ \text{\checkmark } \mathbf{A} \text{ has an NU (near unanimity) term operation} \quad \text{Baker, Pixley} \]

\[ \text{In fact} \quad \mathbf{R} \leq \mathbf{A} \text{ is determined by binary projections:} \]

\[ \mathbf{R} = \{ (a_1, \ldots, a_n) : \forall i, j (a_i, a_j) \in R_{ij} \} \]

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\[ \text{Example: rock-paper-scissors 3-element algebra} \]
A has the 2-intersection property if

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Non-examples and characterization

A does not have the 2-intersection property if
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\[ \textbf{A} \text{ does not have the 2-intersection property if } \]

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\> \textbf{A} \text{ is affine (\textit{\textasciitilde }essentially a module)}
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  - $R^b = \{(a_1, \ldots, a_n) : a_1 + \cdots + a_n = b\}$
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- **B** \( \in HS(A) \) is a reduct of an affine algebra
**Non-examples and characterization**

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- \( \mathbf{B} \in HS(\mathbf{A}) \) is a reduct of an affine algebra

**Recall:** No algebra in \( HS(\mathbf{A}) \) is a reduct of affine algebra

\[ \iff \quad HSP(\mathbf{A}) \text{ omits 1 and 2} \]

\[ \iff \quad \mathbf{A} \text{ is SD}(\land) \quad (= HSP(\mathbf{A}) \text{ is congruence meet semi-distributive}) \]
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- $B \in HS(A)$ is a reduct of an affine algebra

**Recall**: No algebra in $HS(A)$ is a reduct of affine algebra

$\iff HSP(A)$ omits 1 and 2

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---

**Theorem (BK, conjectured by Valeriote)**

$A$ has the 2-intersection property $\iff A$ is SD($\land$).
End of story?

Possible generalizations to Taylor algebras?
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**Recall**: \( A \) is Taylor

\[ \iff \text{No algebra in } HS(A) \text{ is a set} \]

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\[ \iff \ldots \]
The old and the new (result)

Theorem (The old)

If

- \( A \text{ is SD}(\wedge) \)

Then \( A \text{ has the 2-intersection property} \)
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Theorem (The old)

If

- \( A \) is \( \text{SD}(\wedge) \)
- \( R, S \leq A^n \)
- \( \forall i, j \quad R_{ij} = S_{ij} \)

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Theorem (The old)

If

- $A_1, \ldots, A_n$ are SD($\wedge$)
- $R, S \leq_{sd} A_1 \times \cdots \times A_n$ (sd=subdirect product)
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Theorem (The new)

If

- $A_1, \ldots, A_n$ are Taylor, simple, non-abelian
- $R, S \leq_{sd} A_1 \times \cdots \times A_n$
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Then $R \cap S \neq \emptyset$
The real result: a rectangularity theorem

**Recall:** $B \leq A$ is absorbing if $A$ has a term operation $t$ with $t(B, B, \ldots, B, A, B, B, \ldots, B) \subseteq B$
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- Similar theorem for conservative algebras $\rightarrow$ Dichotomy for conservative CSPs
Definition

A a term operation $t$ of $A$ points to $b \in A$ if

$\exists (a_1, \ldots, a_n) \in A^n$ such that

$t(c_1, \ldots, c_n) = b$ whenever $a_i = c_i$ for all but at most one $i$
A consequence: pointing operation

<table>
<thead>
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**Known:** $A$ is $SD(\land) \iff$ every subalgebra of $A$ has a pointing term operation.
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- $R$ is irredundant
- Rectangularity theorem $\Rightarrow R = A^k$. 
Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.
A piece of proof of the rectangularity theorem

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$R_{23}, R_{13}$ full $\Rightarrow$ $S_b \leq_{sd} A_1 \times A_2$ (for each $b$)
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- $R_{23}, R_{13}$ full $\Rightarrow S_b \leq_{sd} A_1 \times A_2$ (for each $b$)
- for each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity)
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- absorption theorem $\Rightarrow$ $R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)
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- for $b \in A_3$ denote $S_b = \{(a_1, a_2) : (a_1, a_2, b) \in R\}$
- $R_{23}, R_{13}$ full $\Rightarrow$ $S_b \leq_{sd} A_1 \times A_2$ (for each $b$)
- for each $b$, $S_b$ either linked or graph of a bijection $A_1 \rightarrow A_2$ (from simplicity)

- if $S_b$ is linked for some $b$
A piece of proof of the rectangularity theorem

- Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.
- $R$ irredundant, $A_1, A_2$ simple $\Rightarrow R_{12}$ linked
- absorption theorem $\Rightarrow R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)
- View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$
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- if $S_b$ is linked for some $b$
  - $S_b = A_1 \times A_2$ (from absorption theorem)
Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.

$R$ irredundant, $A_1, A_2$ simple $\Rightarrow$ $R_{12}$ linked

absorption theorem $\Rightarrow$ $R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)

View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$

for $b \in A_3$ denote $S_b = \{(a_1, a_2) : (a_1, a_2, b) \in R\}$

$R_{23}, R_{13}$ full $\Rightarrow S_b \leq_{sd} A_1 \times A_2$ (for each $b$)

for each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity)

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$S_b = A_1 \times A_2$ (from absorption theorem)

$\bar{R}$ is linked
A piece of proof of the rectangularity theorem

- Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.
- $R$ irredundant, $A_1, A_2$ simple $\implies R_{12}$ linked
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- View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$
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- $R_{23}, R_{13}$ full $\implies S_b \leq_{sd} A_1 \times A_2$ (for each $b$)
- for each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity)

- if $S_b$ is linked for some $b$
  - $S_b = A_1 \times A_2$ (from absorption theorem)
  - $\bar{R}$ is linked
  - **Fact:** $R_{12} = A_1 \times A_2$ is absorption-free
A piece of proof of the rectangularity theorem

- Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.
- $R$ irredundant, $A_1, A_2$ simple $\Rightarrow R_{12}$ linked
- absorption theorem $\Rightarrow R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)
- View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$
- for $b \in A_3$ denote $S_b = \{(a_1, a_2) : (a_1, a_2, b) \in R\}$
- $R_{23}, R_{13}$ full $\Rightarrow S_b \leq_{sd} A_1 \times A_2$ (for each $b$)
- for each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity)

- if $S_b$ is linked for some $b$
  - $S_b = A_1 \times A_2$ (from absorption theorem)
  - $\bar{R}$ is linked
  - **Fact:** $R_{12} = A_1 \times A_2$ is absorption-free
  - absorption theorem $\Rightarrow \bar{R} = R_{12} \times A_3 = (A_1 \times A_2) \times A_3$
A piece of proof of the rectangularity theorem

- Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.

- $R$ irredundant, $A_1, A_2$ simple $\Rightarrow R_{12}$ linked

- Absorption theorem $\Rightarrow R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)

- View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$

- For $b \in A_3$ denote $S_b = \{(a_1, a_2) : (a_1, a_2, b) \in R\}$

- $R_{23}, R_{13}$ full $\Rightarrow S_b \leq_{sd} A_1 \times A_2$ (for each $b$)

- For each $b$, $S_b$ either linked or graph of a bijection $A_1 \rightarrow A_2$ (from simplicity)

- If $S_b$ is a graph of a bijection for every $b$
A piece of proof of the rectangularity theorem

- Assume \( R \leq_{sd} A_1 \times A_2 \times A_3 \) is irredundant and each \( A_i \) is simple, absorption-free, non-abelian.
- \( R \) irredundant, \( A_1, A_2 \) simple \( \Rightarrow \) \( R_{12} \) linked
- Absorption theorem \( \Rightarrow \) \( R_{12} = A_1 \times A_2 \) (similarly for \( R_{23}, R_{13} \))
- View \( R \) as \( \bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3 \)
- For \( b \in A_3 \) denote \( S_b = \{(a_1, a_2) : (a_1, a_2, b) \in R\} \)
- \( R_{23}, R_{13} \) full \( \Rightarrow \) \( S_b \leq_{sd} A_1 \times A_2 \) (for each \( b \))
- For each \( b \), \( S_b \) either linked or graph of a bijection \( A_1 \rightarrow A_2 \) (from simplicity)

- If \( S_b \) is a graph of a bijection for every \( b \)
  - \( \bar{R} \) is a graph of surjective mapping \( R_{12} \rightarrow A_3 \) (from simplicity)
A piece of proof of the rectangularity theorem

- Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.
- $R$ irredundant, $A_1, A_2$ simple $\Rightarrow R_{12}$ linked
- absorption theorem $\Rightarrow R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)
- View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$
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- $R_{23}, R_{13}$ full $\Rightarrow S_b \leq_{sd} A_1 \times A_2$ (for each $b$)
- for each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity)

- if $S_b$ is a graph of a bijection for every $b$
  - $\bar{R}$ is a graph of surjective mapping $R_{12} \to A_3$ (from simplicity)
  - $\Rightarrow \{S_b : b \in A_3\}$ is a partition of $A_1 \times A_2$ that defines a congruence $\alpha$ on $A_1 \times A_2$
Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.

$R$ irredundant, $A_1, A_2$ simple $\Rightarrow$ $R_{12}$ linked.

Absorption theorem $\Rightarrow$ $R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$).

View $R$ as $\bar{R} \leq_{sd} R_{12} \times A_3 = (A_1 \times A_2) \times A_3$.

For $b \in A_3$ denote $S_b = \{(a_1, a_2) : (a_1, a_2, b) \in R\}$.

$R_{23}, R_{13}$ full $\Rightarrow$ $S_b \leq_{sd} A_1 \times A_2$ (for each $b$).

For each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity).

If $S_b$ is a graph of a bijection for every $b$:

- $\bar{R}$ is a graph of surjective mapping $R_{12} \to A_3$ (from simplicity).
- $\Rightarrow \{S_b : b \in A_3\}$ is a partition of $A_1 \times A_2$ that defines a congruence $\alpha$ on $A_1 \times A_2$.
- Using $\alpha$ we can find a congruence $\beta$ on $A_1^2$ whose one block is the diagonal.
A piece of proof of the rectangularity theorem

- Assume $R \leq_{sd} A_1 \times A_2 \times A_3$ is irredundant and each $A_i$ is simple, absorption-free, non-abelian.
- $R$ irredundant, $A_1, A_2$ simple $\Rightarrow R_{12}$ linked
- absorption theorem $\Rightarrow R_{12} = A_1 \times A_2$ (similarly for $R_{23}, R_{13}$)
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- for each $b$, $S_b$ either linked or graph of a bijection $A_1 \to A_2$ (from simplicity)
- if $S_b$ is a graph of a bijection for every $b$
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  - $\Rightarrow \{S_b : b \in A_3\}$ is a partition of $A_1 \times A_2$ that defines a congruence $\alpha$ on $A_1 \times A_2$
  - Using $\alpha$ we can find a congruence $\beta$ on $A_1^2$ whose one block is the diagonal
  - $\Rightarrow A_1$ (and $A_2$) is abelian.
Thank you!