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DISERTAČNÍ PRÁCE

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Full embeddings and their modifications

Matematický ústav Univerzity Karlovy

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Prohlašuji, že jsem svou disertační práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 1.6.2006 Libor Barto

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Abstrakt:

Práce shrnuje mé příspěvky k teorii reprezentací v kategoriích. První kapitola (po úvodu) se týka množinových funktorů, t.j. endofunktorů kategorie všech množin a zobrazení. Dokážeme, že kategorie finitárních množinových funktorů a přirozenných transformací je alg-univerzální a uvedeme příklad strnulé vlastní třídy dosažitelných množinových funktorů. V další kapitole ukážeme, že kategorie variet a interpretací je rovněž alg-univerzální. Poslední kapitola se týká teorie řezů funktorů a košů konkrétních kategorií. Pro každý ordinál α zavedeme koš $\mathbb{E}_\alpha$, dokážeme, že každá esenciálně algebraická kategorie je řezem $\mathbb{E}_\alpha$, charakterizujeme malé řezy $\mathbb{E}_\alpha$ a zobecníme známé výsledky o řezech algebraického koše α.

Klíčová slova: alg-univerzální kategorie, množinový funktor, klon, řez funktoru, koše konkrétních kategorií.

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Abstract:

The present thesis summarizes my contributions to the theory of representations in categories. The first chapter (after the introduction) concerns set functors, i.e. endofunctors of the category of all sets and mappings. We prove that the category of finitary set functors and natural transformations is alg-universal and present an example of a rigid proper class of accessible set functors. In the next chapter, we show that the category of varieties and interpretations is alg-universal as well. The final chapter deals with the theory of functor slices and baskets of concrete categories. For every ordinal α we introduce a basket $\mathbb{E}_\alpha$, prove that every essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$, characterize small slices of $\mathbb{E}_\alpha$ and generalize known results about slices of the algebraic basket $\mathbb{A}$.

Keywords: alg-universal category, set functor, clone, functor slice, baskets of concrete categories.

Chapter I

Introduction

The classical results of Birkhoff [9], de Groot [11], Frucht [14] and Sabidussi [37] say that every group is isomorphic to the automorphism group of a complete distributive lattice, the autohomeomorphism group of a topological space and the automorphism group of a graph. Following Isbell's ideas [22], the concept of full embeddings (i.e. full and faithful functors – functors which are bijective on hom-sets) has been investigated and used to generalize and substantially strengthen various representations of groups as automorphism groups of given mathematical structures. It turned out that many categories (e.g. the category of graphs and graph homomorphism [21]) are even *alg-universal* – every category of universal algebras can be fully embedded into them. See Section 1 for more information about alg-universal categories and consequences of alg-universality.

Chapters II, III enrich the family of known alg-universal categories: The category of finitary set functors and natural transformations (Chapter II) and the category of varieties and interpretations (Chapter III) are both alg-universal. The results will appear in [7], [5].

The notion of full embedding can be modified in many ways, see Section 3. The final chapter deals with one such a modification – a *slice embedding* (or an *s-embedding*) between concrete categories which was introduced by J. Sichler and V. Trnková in [39]. It came out that this theory can sort many familiar concrete categories into five *baskets* $\mathbb{T}, \mathbb{P}, \mathbb{P}^{op}, \mathbb{A}, \mathbb{R}$. In Chapter IV we substantially enrich this five member collection of baskets: For every ordinal α we introduce a new basket $\mathbb{E}_\alpha$. Then we show that every essentially algebraic category of height α is a slice of (i.e. can be s-embedded into) $\mathbb{E}_\alpha$, characterize small slices of $\mathbb{E}_\alpha$ and give a common generalization of known results about slices of the algebraic basket $\mathbb{A}$. The results will appear in [6].

The chapters are independent. The preliminaries used throughout the thesis are in the next three sections of the Introduction:

1 Notation

Set theory

We work in a standard set theory with the axiom of choice for classes. At several places we use "collections larger than classes" for the sake of brevity. This can be made correct by enhancing the set theory (see [1]), but, in this thesis, everything could be formulated without any use of such monsters.

An *ordinal* is a set of all smaller ordinals and cardinal is the least ordinal with its cardinality. We write $\alpha < \beta$ in place of $\alpha \in \beta$.

Let X be a set and $\approx$ be an equivalence. $X/\approx$ denotes the factor set X modulo $\approx$. The equivalence class of an element $x \in X$ is denoted by $[x]_{\approx}$.

Category theory

To the basics we refer to [1].

The set of all morphisms in a category $\mathbf{K}$ with domain $A \in \mathrm{Obj}(\mathbf{K})$ and codomain $B \in \mathrm{Obj}(\mathbf{K})$ is denoted by $\mathbf{K}(A, B)$.

Given a faithful functor $U : \mathbf{K} \to \mathbf{H}$, $A, B \in \mathrm{Obj}(\mathbf{K})$ and $f \in \mathbf{H}(UA, UB)$ we say that f *carries* a $\mathbf{K}$-morphism from A to B provided that $f = Ug$ for a $\mathbf{K}$-morphism $g : A \to B$.

By a *concrete category* (over $\mathbf{H}$) we mean a faithful functor $U : \mathbf{K} \to \mathbf{H}$ such that

$$\mathbf{K}(A, B) \subseteq \mathbf{H}(UA, UB), \quad A, B \in \mathrm{Obj}(\mathbf{K}).$$

In this case, a $\mathbf{H}$-morphism $h : UA \to UB$ carries a $\mathbf{K}$-morphism $A \to B$ iff it *is* a $\mathbf{K}$-morphism $A \to B$. The objects UA, UB are called *underlying objects* of A, B.

We write $h \in \mathbf{H}(A, B)$, or h is a $\mathbf{H}$-morphism from A to B, in place of $h \in \mathbf{H}(UA, UB)$. Likewise, for $A \in \mathrm{Obj}(\mathbf{K})$, $H \in \mathrm{Obj}(\mathbf{H})$ we write $h \in \mathbf{H}(A, H)$ in place of $h \in \mathbf{H}(UA, H)$.

2 Full embeddings

In this section we discuss the following hierarchy of comprehensiveness: A category $\mathbf{K}$ is said to be

group-universal,	if for every group G, there exists an object $A \in \mathrm{Obj}(\mathbf{K})$ such that $\mathrm{Aut}(A)$, the automorphism group of A, is isomorphic to G;
group-universal in a stronger sense,	if for every group G, there exists $A \in \mathrm{Obj}(\mathbf{K})$ s. t. $\mathrm{End}(A)$, the endomorphism monoid of A, is a group isomorphic to G;
monoid-universal,	if for every monoid M, there exists $A \in \mathrm{Obj}(\mathbf{K})$ s. t. $\mathrm{End}(A)$ is isomorphic to G;

alg-universal,	if every category $\mathbf{Alg}(\Sigma)$ of universal algebras with a given signature Σ can be fully embedded into $\mathbf{K}$;
universal,	if every category concretizable over $\mathbf{Set}$ (i.e. a category which admits a faithful functor into $\mathbf{Set}$) can be fully embdedded into $\mathbf{K}$;
hyper-universal,	if every category can be fully embdedded into $\mathbf{K}$.

Every small category (i.e. every category with set many objects), in particular, a one object category – a monoid, can be fully embedded into some category of universal algebras (see [35]), hence every alg-universal category is monoid-universal. Alg-universality seems to be much stronger property than monoid-universality. However, no "natural" example (e.g. a variety or a quasivariety of algebras) of monoid-universal category which is not alg-universal is known. Kučera, Pultr and Hedrlín showed that the statement "every alg-universal category is universal" is equivalent to the following set-theoretical assumption (see [35]):

(M) The class of all measurable cardinals is a set.

Every universal category has a factor (morphisms are glued together in an admissible way), which is hyper-universal (see [31, 41, 45]). No "natural" example of hyper-universal category is known.

A very long list of group-universal categories is presented in the survey paper [15] and all group-universal varieties of unary algebras were characterized in [38].

The category of varieties and interpretations [47], and the category of set functors [8] are group-universal in a stronger sense. They are alg-universal as we will see in Chapters II, III.

The alg-universality seems to be the most important notion from the list above. In [21], the category $\mathbf{Rel}(2)$ of graphs and graph homomorphisms, and the category $\mathbf{Alg}(1,1)$ of algebras with two unary operations and algebra homomorphisms were shown to be alg-universal. Then a lot of varieties of universal algebras were proved to be alg-universal, e.g. the variety of $(0,1)$-lattices [19], semigroups [20], integral domains of characteristic zero [13], and many others. These older results are summarized in the monograph [35] and in the survey article [46], where also many later results are mentioned, e.g. the full characterization of alg-universal varieties of $(0,1)$-lattices [17] and of semigroups [29]. The mentioned categories are *algebraic*:

Definition 2.1. *A category is said to be* algebraic *provided that it is fully embeddable into some category of universal algebras (and hence into any alg-universal category).*

Many meanings of the term "algebraic" can be found in the literature. The present definition is used in the theory of representations in categories. Note that (M) is equivalent to "every category concretizable over $\mathbf{Set}$ is algebraic".

There are also interesting universal categories, e.g. the category of hypergraphs (Hedrlín, Kučera, see [35]), the category of topological spaces and open continuous maps [35], the category of topological semigroups and continuous homomorphisms [45]. The regular varieties of topological unary algebras, which are universal, are characterized in [27].

Recall the definition of a rigid class of objects:

Definition 2.2. *A class $\mathcal{C}$ of objects of a category* $\mathbf{K}$ *is called* rigid*, if* $\mathbf{K}(A, A) = \{\mathrm{id}_A\}$ *for every* $A \in \mathcal{C}$ *and* $\mathbf{K}(A, B) = \emptyset$ *for every* $A, B \in \mathcal{C}$*,* $A \neq B$*.*

There exists arbitrarily large rigid set of objects in any alg-universal category, since we can fully embed arbitrarily large small discrete category into it. The statement "every alg-universal category contains a rigid proper class of objects" is equivalent to the negation of Vopěnka's principle (see [25], [2]).

3 Modifications

This section contains a brief list of several modifications of full embeddings. For more information and references the book [35] and the article [46] is recommended.

Almost full embeddings

The category of topological spaces and continuous mappings isn't alg-universal, not even group-universal in a stronger sense, since every constant mapping is continuous. An *almost full embedding* is, roughly speaking, a faithful functor, which is full "up to constant mappings". Similarly we have almost alg-universality, almost universality, etc.

Examples of results of this type: the category of metrizable topological spaces and continuous mappings is almost alg-universal (Trnková, see [35]), the category of paracompact topological spaces and continuous mappings is almost universal (Koubek, see [35]).

Simultaneous representations

Definition 3.1. *Let* $U : \mathbf{K} \to \mathbf{H}$*,* $U' : \mathbf{K}' \to \mathbf{H}'$ *be concrete categories. A pair* (Φ, F) *of functors* $\Phi : \mathbf{K} \to \mathbf{K}'$*,* $F : \mathbf{H} \to \mathbf{H}'$ *is said to be a* simultaneous representation *(of U to U'), if* Φ *and* F *are full embeddings and* $U'\Phi = FU$*.*

A functor $U' : \mathbf{K}' \to \mathbf{H}'$ *is said to be* comprehensive*, if for every functor U between small categories we can find a simultaneous representation of U to U'.*

We have also simultaneous almost-representations and appropriately adapted comprehensiveness. Metric completion, β-compactification, completely regular modification, compactly generated modification, sequential modification are comprehensive (regarded as suitable functors). These are results by Trnková and Hušek, see [46].

Strong embeddings

Most of "everyday life" categories are concrete (mostly over the category **Set** of all sets and mappings) – we have the natural forgetful functor (to the category **Set**). A *strong embedding* is a full embedding which is functorial on the underlying objects:

Definition 3.2. *Let* $U : \mathbf{K} \to \mathbf{H}$*,* $U' : \mathbf{K}' \to \mathbf{H}'$ *be concrete categories. A pair* (Φ, F) *of functors* $\Phi : \mathbf{K} \to \mathbf{K}'$*,* $F : \mathbf{H} \to \mathbf{H}'$ *is said to be a* strong embedding *(of* U *to* U'*), if* Φ *is a full embedding and* $U'\Phi = FU$*.*

S-embeddings (functor slices)

If we relax the assumptions on Φ from the previous definition in a certain way, we get a definition of s-embedding (the definition is in Chapter IV). This concept was originally introduced (in [39]) to catch the complexity of additional structure needed to obtain simultaneous representations from full-embeddings. Further investigations have shown that s-embeddings can serve to compare concrete categories (especially over **Set**) according to "a manner how they choose their morphism".

Chapter II

Set functors and natural transformations

We prove that the category of finitary set endofunctors and natural transformations is alg-universal and present an example of a rigid proper class of accessible set endofunctors.

The category of accessible endofunctors of **Set** (where **Set** denotes the category of all sets and mappings) is group-universal in a stronger sense – this was proved by P. Zima and the author, see [8]. Here we are going to prove a much stronger result: The category of finitary endofunctors of **Set** is alg-universal, i.e. every category of universal algebras can be fully embedded into it. The proof is substantially easier than the proof in [8].

The basic structural properties of set functors, i.e. endofunctors of the category **Set**, were obtained in the articles [42, 43, 26, 28]. The category of all set functors and all natural transformations is not legitimate, because there are "too many" set functors and "too many" natural transformations. But it has natural legitimate subcategories – the category of κ-accessible set functors for some cardinal κ and the category of accessible set functors. See Section 2 for the definitions and preliminaries concerning set functors.

The category of finitary (ω-accessible) set functors and natural transformations is related to the category **Clone** of (abstract) clones and clone homomorphisms, or, in a different view, to the category of (finitary) varieties and interpretations. Indeed, an interpretation between varieties can be viewed as a natural transformation between their free functors, which, in some sense, preserves equations. It turned out that our main theorem is the right direction to prove alg-universality of the category **Clone**. This result is the contents of the next chapter.

Section 3 contains the proof of the main theorem of this chapter: The category of finitary set functors is alg-universal. Since the category of κ-accessible set functors is algebraic for every κ (recall that *algebraic* means here, that it can be fully embedded into some category of universal algebras), universality of this category is equivalent to the set-theoretical assumption (M) from the introduction.

In Section 4 we present an example of a rigid proper class of accessible set functors. The idea is due to V. Koubek. The following questions naturaly arise:

Open problem. *Is the category of all accessible set functors and natural transformations universal?*

Open problem. *Is the (ilegitimate) category of all set functors and natural transformations hyper-universal?*

Notation. *Let $f : X \to Y$ be a mapping.* $\mathrm{Im}(f)$ *denotes the image of f; $f(x)$ means the image of the element $x \in X$; $f[R]$ means the image of the subset $R \subseteq X$; f^{-1} is always the mapping $f^{-1} : PY \to PX$ (where PX is the set of all subsets of X), not the inverse mapping. Let F, G be set functors, $\mu : F \to G$ be a natural transformation. By μ_X we mean the component $\mu_X : FX \to GX$ of μ.*

1 Preliminaries

In this section, we recall some known facts about set functors, which will be needed in the present chapter. Their proofs can be found in [42, 26].

Every set functor F can be written as a coproduct

$$F = \coprod_{i \in F1} F_i,$$

where all components F_i are *connected*, i.e. $|F_i 1| = 1$. Each connected set functor either contains precisely one isomorphic copy of the identity functor (this is precisely when it is faithful), or contains precisely one isomorphic copy of the constant functor $\mathbb{C}_1$– the functor which assigns empty set to empty set and a one-point set to all nonempty sets. The following easy criterion will be used:

Proposition 1.1. *Let F be a connected set functor, X be a nonempty set and $x \in FX$ be an arbitrary element. Then F is faithful, iff $Ff(x) \neq Fg(x)$ for the two distinct constant mappings $f, g : X \to 2$.*

All set functors in this chapter are connected and faithful. For this reason, we formulate the next definition and propositions just for this situation. There would be some technical difficulties in the general case. The most important structural properties of a (faithful connected) set functor F are *filters* and *monoids* of elements $x \in FX$, $X \neq \emptyset$.

$$\begin{aligned} \mathrm{Flt}(x) &= \{U \subseteq X \mid (\exists u \in FU)\ Fi(u) = x,\ i : U \to X \text{ is the inclusion } \}, \\ &= \{f[U] \mid (\exists u \in FU)\ Ff(u) = x,\ f : U \to X \text{ is a mapping } \}, \\ \mathrm{Mon}(x) &= \{f : X \to X \mid Ff(x) = x\}. \end{aligned}$$

Theorem 1.2. *Let F be a faithful connected set functor, $x \in FX$, $X \neq \emptyset$. Then* $\mathrm{Flt}(x)$ *is a filter on X,* $\mathrm{Mon}(x)$ *is a submonoid of the transformation monoid on X and* $\mathrm{Flt}(x) = \{\mathrm{Im}(f) \mid f \in \mathrm{Mon}(x)\}$. *If $U \in \mathrm{Flt}(x)$ and $f \in \mathrm{Mon}(x)$, then $f[U] \in \mathrm{Flt}(x)$.*

F is said to be κ-*accessible*, if for every nonempty set X and $x \in FX$ there exists a set $U \in \mathrm{Flt}(x)$ such that $|U| < \kappa$. In other words, every element can be accessed from an element of an image of some "small" set (small means here, with cardinality less than κ). This definition agrees for a regular infinite κ with the general notion of κ-accessibility (preservation of κ-filtered colimits) from [32]. An ω-accessible functor is called *finitary*.

The category of κ-accessible (κ is a fixed cardinal) set functors and natural transformations is algebraic: A κ-accesible set functor is determined (up to natural equivalence) by its restriction $\mathbf{Card}_{<\kappa} \to \mathbf{Set}$, where $\mathbf{Card}_{<\kappa}$ is the full subcategory of $\mathbf{Set}$ generated by cardinals less than κ. Indeed, the original functor is the Kan extension of this restriction. A functor $G : \mathbf{Card}_{<\kappa} \to \mathbf{Set}$ can be viewed as a many-sorted algebra (sorts are $G\alpha, \alpha < \kappa$) with operations $Gf : G\alpha \to G\beta$ for every $f : \alpha \to \beta$, $\alpha, \beta < \kappa$. Algebra homomorphisms correspond precisely to natural transformations. It is known and easy to see that the category of S-sorted algebras is algebraic for every set S.

The next proposition is easy and often useful.

Proposition 1.3. *Let $\mu : F \to G$ be a natural transformation of faithful connected set functors, X a nonempty set, $x \in FX$. Then* $\mathrm{Flt}(x) \subseteq \mathrm{Flt}(\mu_X(x))$, $\mathrm{Mon}(x) \subseteq \mathrm{Mon}(\mu_X(x))$.

Finally, we will need the following simple observation:

Proposition 1.4. *Let $\mu : F \to G$ be a natural transformation of faithful connected set functors, X be a nonepty finite set, $x \in FX$. Let $f \in \mathrm{Mon}(x)$ for every bijection $f : X \to X$. Then* $\mathrm{Flt}(\mu_X(x)) = \{X\}$.

Proof. Due to the preceding proposition, we have $f \in \mathrm{Mon}(\mu_X(x))$ for every bijection $f : X \to X$. Suppose, we have $\emptyset \neq U \subset X$, $U \in \mathrm{Flt}(\mu_X(x))$. We can choose a sequence $f_1, \ldots, f_n : X \to X$ of bijections, such that $U \cap f_1[U] \cap \cdots \cap f_n[U] = \emptyset$. From the last part of 1.2, it follows that $f_i[U] \in \mathrm{Flt}(\mu_X(x))$. Because $\mathrm{Flt}(\mu_X(x))$ is a filter, we have $\emptyset = U \cap f_1[U] \cap \cdots \in \mathrm{Flt}(\mu_X(x))$, a contradiction. □

In the situation of this proposition, one can easily see, that $\mathrm{Flt}(x) = \{X\}$ (the same argument as in the proof) and $\mathrm{Mon}(x) = \mathrm{Mon}(\mu_X(x)) = \{f | f \text{ is a bijection}\}$ (from 1.2).

2 Main theorem

Theorem 2.1. *The category* $\mathbf{SetFunc_{fin}}$ *of finitary set functors and natural transformations is algebraic and alg-universal.*

Remark 2.2. *In fact, we will prove a stronger result: The category of 7-accessible connected faithful set functors is alg-universal.*

We are going to construct a full embedding $\Phi : \mathbf{Alg}(1,1) \to \mathbf{SetFunc_{fin}}$. This is enough, since the category $\mathbf{Alg}(1,1)$ is alg-universal and $\mathbf{SetFunc_{fin}}$ is algebraic (see Section 1).

Let $\mathcal{A} = (A, \alpha, \beta) \in \mathbf{Alg}(1, 1)$ be an algebra with two unary operations. For every $a \in A$, we now define a mapping

$$s_{\mathcal{A},a} : P6 \to A \cup \{o, j\}.$$

The union is assumed to be disjoint. For $R \subseteq 6$, let

$$s_{\mathcal{A},a}(R) = \begin{cases} o & \text{if } R = 0, \\ a & \text{if } |R| = 1 \text{ or } |R| = 5, \\ \alpha(a) & \text{if } |R| = 2 \text{ or } |R| = 4, \\ \beta(a) & \text{if } |R| = 3, \\ j & \text{if } R = 6. \end{cases}$$

Observe that the mappings $s_{\mathcal{A},a_1}$ and $s_{\mathcal{A},a_2}$ are distinct for distinct $a_1, a_2 \in A$. For a set X and a mapping $f : X \to Y$, we put

$$\begin{aligned} \mathbb{A}X &= \{s_{\mathcal{A},a}g^{-1} : PX \to A \cup \{o, j\} \,|\, a \in A,\ g : 6 \to X \text{ is a map}\}, \\ \mathbb{A}f(s_{\mathcal{A},a}g^{-1}) &= s_{\mathcal{A},a}g^{-1}f^{-1}. \end{aligned}$$

$\mathbb{A}$ is a set functor: For every $f_1 : X \to Y, f_2 : Y \to Z$, we have

$$\begin{aligned} \mathbb{A}\mathrm{id}_X(s_{\mathcal{A},a}g^{-1}) &= s_{\mathcal{A},a}g^{-1}\mathrm{id}_X^{-1} = s_{\mathcal{A},a}g^{-1}, \\ \mathbb{A}f_1(\mathbb{A}f_2(s_{\mathcal{A},a}g^{-1})) &= s_{\mathcal{A},a}g^{-1}f_2^{-1}f_1^{-1} = s_{\mathcal{A},a}g^{-1}(f_1 f_2)^{-1} = \\ &= \mathbb{A}f_1 f_2(s_{\mathcal{A},a}g^{-1})). \end{aligned}$$

Let $R \subseteq X$. Let $\chi_{R,X} : X \to 2$ denote the characteristic mapping of R, i.e. $\chi_{R,X}(x) = 1$, iff $x \in R$.

Claim 1. *The functor $\mathbb{A}$ is faithful, connected and 7-accessible.*

Proof. 7-accessibility is clear – every element can be accessed from some $s_{\mathcal{A},a} \in \mathbb{A}6$.

Connectedness: The elements of $\mathbb{A}1$ are of the form $s_{\mathcal{A},a}f^{-1}$, where $f : 6 \to 1$ is the unique mapping. But $s_{\mathcal{A},a}f^{-1}(0) = s_{\mathcal{A},a}(0) = o$ and $s_{\mathcal{A},a}f^{-1}(1) = s_{\mathcal{A},a}(6) = j$, hence $s_{\mathcal{A},a}f^{-1}$ doesn't depend on a – $|\mathbb{A}1| = 1$.

Faithfulness: We will use Proposition 1.1. Take arbitrary $s = s_{\mathcal{A},a} \in \mathbb{A}6$. Then $s\chi_{0,6}^{-1}$ and $s\chi_{6,6}^{-1}$ differs on $\{0\}$:

$$\begin{aligned} s\chi_{0,6}^{-1}(\{0\}) &= s(6) = j, \\ s\chi_{6,6}^{-1}(\{0\}) &= s(0) = o. \end{aligned}$$

$\square$

Given two algebras $\mathcal{A} = (A, \alpha, \beta)$, $\mathcal{B} = (B, \gamma, \delta)$ and a homomorphism $h : \mathcal{A} \to \mathcal{B}$, we define a natural transformation $\mu^h : \mathbb{A} \to \mathbb{B}$ as follows

$$\mu^h_X(s_{\mathcal{A},a}g^{-1}) = s_{\mathcal{B},h(a)}g^{-1}.$$

Claim 2. *The definition is correct.*

Proof. We must check that if $s_{\mathcal{A},a_1}g_1^{-1} = s_{\mathcal{A},a_2}g_2^{-1}$, then $s_{\mathcal{B},h(a_1)}g_1^{-1} = s_{\mathcal{B},h(a_2)}g_2^{-1}$. For $R \subseteq X$, we have

$$\begin{aligned} s_{\mathcal{B},h(a_1)}g_1^{-1}(R) &= \begin{cases} o & |g_1^{-1}(R)| = 0 \\ h(a_1) & |g_1^{-1}(R)| = 1,5 \\ \gamma(h(a_1)) & |g_1^{-1}(R)| = 2,4 \\ \delta(h(a_1)) & |g_1^{-1}(R)| = 3 \\ j & |g_1^{-1}(R)| = 6 \end{cases} = \\ &= \begin{cases} o & |g_1^{-1}(R)| = 0 \\ h(a_1) & |g_1^{-1}(R)| = 1,5 \\ h(\alpha(a_1)) & |g_1^{-1}(R)| = 2,4 \\ h(\beta(a_1)) & |g_1^{-1}(R)| = 3 \\ j & |g_1^{-1}(R)| = 6 \end{cases} = \overline{h}(s_{\mathcal{A},a_1}g_1^{-1}(R)), \end{aligned}$$

where $\overline{h} : A \cup \{o, j\} \to B \cup \{o, j\}$ coincides with h on A and is identical on $\{o, j\}$. The same computation gives $s_{\mathcal{B},h(a_2)}g_2^{-1}(R) = \overline{h}(s_{\mathcal{A},a_2}g_2^{-1}(R))$.
Since $s_{\mathcal{A},a_1}g_1^{-1}(R) = s_{\mathcal{A},a_2}g_2^{-1}(R)$, we are done. □

Claim 3. *μ is natural.*

Proof. Let $s_{\mathcal{A},a}g^{-1} \in \mathbb{A}X$, $f : X \to Y$ be arbitrary. Then

$$\mathbb{B}f(\mu_X^h(s_{\mathcal{A},a}g^{-1})) = \mathbb{B}f(s_{\mathcal{B},h(a)}g^{-1}) = s_{\mathcal{B},h(a)}g^{-1}f^{-1},$$

$$\mu_Y^h(\mathbb{A}f(s_{\mathcal{A},a}g^{-1})) = \mu_Y^h(s_{\mathcal{B},a}g^{-1}f^{-1}) = s_{\mathcal{B},h(a)}g^{-1}f^{-1}.$$

□

The functor $\Phi : \mathbf{Alg}(1,1) \to \mathbf{SetFunc_{fin}}$ given by

$$\Phi(\mathcal{A}) = \mathbb{A}, \quad \Phi(h) = \mu^h$$

is the searched full and faithful functor:

Claim 4. *Φ is a faithful functor.*

Proof. It is clear, that Φ preserves the composition and identities.

Faithfulness: Take distinct homomorphisms $h, h' : A \to B$ and then, an element $a \in A$, for which $h(a) \neq h'(a)$. Then $\mu_6^h(s_{\mathcal{A},a}) = s_{\mathcal{B},h(a)} \neq s_{\mathcal{B},h'(a)} = \mu_6^{h'}(s_{\mathcal{A},a})$ from the note after the definition of the mappings $s_{\dots}$. □

Let $\mathcal{A} = (A, \alpha, \beta)$, $\mathcal{B} = (B, \gamma, \delta)$ be algebras. Let $\mu : \mathbb{A} \to \mathbb{B}$ be a natural transformation. We will check that $\mu = \mu^h$ for some homomorphism $h : \mathcal{A} \to \mathcal{B}$ proving the fullness of Φ.

Claim 5. *Let $g : 6 \to 6$, $b \in B$. Then* $\mathrm{Im}(g) \in \mathrm{Flt}(s_{\mathcal{B},b}g^{-1})$.

Proof. Take the factorization $g = ih$, where $i : \mathrm{Im}(g) \to 6$ is the inclusion. Then clearly $Fi(s_{\mathcal{B},b}h^{-1}) = s_{\mathcal{B},b}g^{-1}$. □

Claim 6. *Let $g : 6 \to 6$ be a bijection, then $g \in \mathrm{Mon}(s_{\mathcal{A},a})$.*

Proof. We should check that $s_{\mathcal{A},a}(R) = s_{\mathcal{A},a}g^{-1}(R)$ $(= s_{\mathcal{A},a}(g^{-1}(R)))$ for every $R \subseteq 6$. This is true, since $|g^{-1}(R)| = |R|$ and the value of $s_{\mathcal{A},a}$ on some subset $S \subseteq 6$ depends only on the cardinality of S. $\square$

From these two claims, it follows that the only elements $s \in \mathbb{B}6$ with $\mathrm{Flt}(s) = \{6\}$ are the elements $s_{\mathcal{B},b}$ $(b \in B)$. Combining this with Proposition 1.4, we obtain $\mathrm{Flt}(\mu_6(s_{\mathcal{A},a})) = \{6\}$, hence

$$\mu_6(s_{\mathcal{A},a}) = s_{\mathcal{B},h(a)}$$

for some $h(a) \in B$. Now we aim to show, that this $h : A \to B$ is a homomorphism of the algebras $\mathcal{A}$, $\mathcal{B}$.

Let $d_{\mathcal{A},a} : P2 \to A \cup \{o, j\}$ be the following mapping ($R \subseteq 2$):

$$d_{\mathcal{A},a}(R) = \begin{cases} o & \text{if } R = 0, \\ a & \text{if } R = \{0\} \text{ or } R = \{1\}, \\ j & \text{if } R = 2. \end{cases}$$

Claim 7. *Let $a \in A$, $R \subseteq 6$. Then*

$$\begin{aligned} d_{\mathcal{A},a} &= \mathbb{A}\chi_{R,6}(s_{\mathcal{A},a}), \ \textit{if } |R| = 1, \\ d_{\mathcal{A},\alpha(a)} &= \mathbb{A}\chi_{R,6}(s_{\mathcal{A},a}), \ \textit{if } |R| = 2, \\ d_{\mathcal{A},\beta(a)} &= \mathbb{A}\chi_{R,6}(s_{\mathcal{A},a}), \ \textit{if } |R| = 3. \end{aligned}$$

In particular $d_{\mathcal{A},a} \in \mathbb{A}2$.

Proof. This is an easy calculation. $\square$

Of course, a similar claim holds for b, γ, δ and the functor $\mathbb{B}$.

Claim 8. *Let $a \in A$. Then $\mu_2(d_{\mathcal{A},a}) = d_{\mathcal{B},h(a)}$.*

Proof. We use the naturality of μ for $\chi_{R,6} : 6 \to 2$, where $|R| = 1$, and the preceding claim.

$$\begin{gathered} \mathbb{B}\chi_{R,6}(\mu_6(s_{\mathcal{A},a})) = \mathbb{B}\chi_{R,6}(s_{\mathcal{B},h(a)}) = d_{\mathcal{B},h(a)} = \\ = \mu_2(\mathbb{A}\chi_{R,6}(s_{\mathcal{A},a})) = \mu_2(d_{\mathcal{A},a}). \end{gathered}$$

$\square$

Claim 9. *Let $a \in A$. Then $h(\alpha(a)) = \gamma(h(a))$.*

Proof. We use the naturality of μ for $\chi_{R,6} : 6 \to 2$, where $|R| = 2$, and the last two claims.

$$\begin{gathered} \mathbb{B}\chi_{R,6}(\mu_6(s_{\mathcal{A},a})) = \mathbb{B}\chi_{R,6}(s_{\mathcal{B},h(a)})) = d_{\mathcal{B},\gamma(h(a))} = \\ = \mu_2(\mathbb{A}\chi_{R,6}(s_{\mathcal{A},a})) = \mu_2(d_{\mathcal{A},\alpha(a)}) = d_{\mathcal{B},h(\alpha(a))}. \end{gathered}$$

Because the mappings $d_{\mathcal{B},b}, d_{\mathcal{B},b'}$ are distinct for distinct $b, b' \in B$, we have $\gamma(h(a)) = h(\alpha(a))$. $\square$

Claim 10. *Let $a \in A$. Then $h(\beta(a)) = \delta(h(a))$.*

Proof. The proof is similar to the previous – use a subset $R \subseteq 6$ such that $|R| = 3$. $\square$

We have proved, that h is a homomorphism. To conclude the proof, we must observe:

Claim 11. $\mu = \mu^h$.

Proof. Let $g : 6 \to X$ be an arbitrary mapping, $a \in A$. From the naturality of μ, we have

$$\mathbb{B}g(\mu_6(s_{\mathcal{A},a})) = \mathbb{B}g(s_{\mathcal{B},h(a)}) = s_{\mathcal{B},h(a)}g^{-1} =$$
$$= \mu_X(\mathbb{A}g(s_{\mathcal{A},a})) = \mu_X(s_{\mathcal{A},a}g^{-1}).$$

$\square$

3 Rigid proper class of accessible set functors

Let $\mathcal{F}$ be a filter on a set X and $f : X \to Y$ be a mapping. By an f-image of $\mathcal{F}$ is meant the following filter on Y:

$$\begin{aligned} f(\mathcal{F}) &= \{S \subseteq Y \,|\, f[R] \subseteq S \text{ for some } R \in \mathcal{F}\} = \\ &= \{f^{-1}(R) \subseteq Y \,|\, R \in \mathcal{F}\}. \end{aligned}$$

It is known and easy to see that the filter functor $\mathbb{F}$ defined by

$$\begin{aligned} \mathbb{F}X &= \{\mathcal{F} \,|\, \mathcal{F} \text{ is a filter on } X\} && \text{for a set } X, \\ \mathbb{F}f(\mathcal{F}) &= f(\mathcal{F}) && \text{for a mapping } f : X \to Y \end{aligned}$$

is a faithful connected set functor. In this functor $\mathrm{Flt}(\mathcal{F}) = \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$.

For an infinite cardinal κ, we put

$$\mathcal{F}_\kappa = \{R \subseteq \kappa \,|\, |\kappa - R| < \kappa\}.$$

It is easy to see that $\mathcal{F}_\kappa$ is a filter on κ.

Let $\mathcal{C}$ be a nonempty class of regular cardinals. For a set X and a mapping $f : X \to Y$ we define

$$\begin{aligned} \mathbb{C}X &= \{g(\mathcal{F}_\kappa) \,|\, \kappa \in \mathcal{C},\ g : \kappa \to X\}, \\ \mathbb{C}f(g(\mathcal{F}_\kappa)) &= fg(\mathcal{F}_\kappa). \end{aligned}$$

$\mathbb{C}$ is a subfunctor of the filter functor $\mathbb{F}$. Hence it is faithful and connected and $\mathrm{Flt}(\mathcal{F}) = \mathcal{F}$ for every $\mathcal{F} \in \mathbb{C}X$. It is λ-accessible for every cardinal λ greater than all $\kappa \in \mathbb{C}$.

Theorem 3.1. *Let $\mathcal{C}, \mathcal{D}$ be nonempty classes of regular cardinals. Then there exists a natural transformation $\mathbb{C} \to \mathbb{D}$, iff $\mathcal{C} \subseteq \mathcal{D}$. In this case, it is unique.*

Proof. First, we describe the filters $f(\mathcal{F}_\kappa)$ for a regular cardinal κ and $f : \kappa \to X$. Let $U \subseteq V \subseteq X$. Let $\mathcal{F}_{U,V,X,\kappa}$ be the following filter on X:

$$\mathcal{F}_{U,V,X,\kappa} = \{R \subseteq X \,|\, U \subseteq R,\ |V - R| < \kappa\}.$$

Note that

- If $U, U' \subseteq X$, $U \neq U'$, then $\mathcal{F}_{U,V,X,\kappa} \neq \mathcal{F}_{U',V',X,\lambda}$ for every V, V', where $U \subseteq V \subseteq X$, $U' \subseteq V' \subseteq X$, and κ, λ are regular cardinals.
- Let $V, V' \subseteq \kappa$, $|V| = \lambda$. Then $\mathcal{F}_{0,V,\kappa,\lambda} = \mathcal{F}_{0,V',\kappa,\lambda}$ iff the symmetric difference $(V - V') \cup (V' - V)$ has cardinality less than λ.

Claim 1. *Let κ be a regular cardinal, $f : \kappa \to X$ be a mapping. Let $U = \{x \,|\, |f^{-1}(\{x\})| = \kappa\}$, $V = f[\kappa]$. Then $f(\mathcal{F}_\kappa) = \mathcal{F}_{U,V,X,\kappa}$. If $U = 0$ then $|V| = \kappa$.*

Proof. The inclusion "$\subseteq$". Let $R \in f(\mathcal{F}_\kappa)$, so $|\kappa - f^{-1}(R)| < \kappa$. If $x \in U$ and $x \notin R$, then $|\kappa - f^{-1}(R)| \geq |\kappa - f^{-1}(X - \{x\})| = |f^{-1}(\{x\})| = \kappa$, a contradiction, hence $U \subseteq R$. Since moreover $|f[\kappa] - R| \leq |\kappa - f^{-1}(R)| < \kappa$, we have $R \in \mathcal{F}_{U,V,X,\kappa}$.

The inclusion "$\supseteq$". Let $R \in \mathcal{F}_{U,V,X,\kappa}$, so $U \subseteq R$, $|V - R| < \kappa$. Since $\kappa - f^{-1}(R) = \cup_{x \in V-R} f^{-1}(\{x\})$, we have $|\kappa - f^{-1}(R)| < \kappa$ (the right hand side is a union of less then κ sets, each of cardinality less than κ, κ is regular). Thus $R \in f(\mathcal{F}_\kappa)$.

The last statement is obvious. □

Now, let $\mu : \mathbb{C} \to \mathbb{D}$ be a natural transformation.

Claim 2. *Let $\kappa \in \mathcal{C}$. Then $\kappa \in \mathcal{D}$ and $\mu_\kappa(\mathcal{F}_\kappa) = \mathcal{F}_\kappa$.*

Proof. Let $\lambda \in \mathcal{D}$, $f : \lambda \to \kappa$, $U \subseteq V \subseteq \kappa$ be such that $\mu_\kappa(\mathcal{F}_\kappa) = f(\mathcal{F}_\lambda) = \mathcal{F}_{U,V,\kappa,\lambda}$.

Every bijection $\kappa \to \kappa$ is in the monoid of $\mathcal{F}_\kappa \in \mathbb{C}\kappa$. According to 1.4, every bijection is in the monoid of $\mathcal{F}_{U,V,\kappa,\lambda}$. It is obvious that $b(\mathcal{F}_{U,V,\kappa,\lambda}) = \mathcal{F}_{b[U],b[V],\kappa,\lambda}$, thus $b[U] = U$ for every bijection $b : \kappa \to \kappa$ (see the note above), hence either $U = 0$ or $U = \kappa$.

Suppose $U = \kappa$. Let $x \in \kappa$ be arbitrary. The set $X - \{x\}$ is in the filter of $\mathcal{F}_\kappa$, but it isn't in the filter of $\mathcal{F}_{\kappa,\kappa,\kappa,\lambda}$. This contradicts 1.4 (recall that $\mathrm{Flt}(\mathcal{F}) = \mathcal{F}$).

Now, we have $U = 0$, thus $\lambda = |V|$ (see the last statement in the previous claim). If $|\kappa - V| = \kappa$, we can find a bijection such that the symmetric difference $(V - b[V]) \cup (b[V] - V)$ has cardinality κ, hence $\mathcal{F}_{0,V,\kappa,\lambda} \neq \mathcal{F}_{0,b[V],\kappa,\lambda}$ (see the note above again), a contradiction. Hence $\lambda = \kappa$ and $|\kappa - V| < \kappa$. Then $\mathcal{F}_{0,V,\kappa,\kappa} = \mathcal{F}_{0,\kappa,\kappa,\kappa} = \mathcal{F}_\kappa$. □

We now know that $\mathcal{C} \subseteq \mathcal{D}$ and $\mu_\kappa(\mathcal{F}_\kappa) = \mathcal{F}_\kappa$. From the naturality of μ, it follows that for every $\kappa \in \mathcal{C}$, set X and mapping $f : \kappa \to X$

$$\mu_X(f(\mathcal{F}_\kappa)) = f(\mu_\kappa(\mathcal{F}_\kappa)) = f(\mathcal{F}_\kappa).$$

Thus the transformation μ is uniqely determined - it is the inclusion. □

Let $\mathtt{E}$ be a conglomerate (i.e. collection of classes in the sense of [1]) of pairwise incomparable classes of regular cardinals. From the last theorem, it follows that $\{\mathbb{C} \mid \mathcal{C} \in \mathtt{E}\}$ is a rigid conglomerate of set functors. Putting $\mathtt{E} = \{\{\kappa\} \mid \kappa \text{ is a regular cardinal}\}$, we obtain:

Corollary 3.2. *There exists a rigid proper class of accessible set functors.*

Chapter III

Varieties and interpretations

We prove that the category of varieties and interpretations, or in other words, the category of abstract clones and clone homomorphisms, is alg-universal.

The lattice $\mathcal{L}$ of interpretability types of varieties (of finitary mono-sorted universal algebras) was first introduced and investigated in [34]. Then an issue [16] of Memoirs of the AMS was devoted to the study of $\mathcal{L}$. One of the many open problems formulated there, whether the breadth of this lattice is uncountable, was solved in [47]. The authors proved there (among other) that every poset can be embedded into $\mathcal{L}$ and that the existence of proper class antichain is equivalent to the negation of Vopěnka's principle (see [25]).

In fact, they investigated the category **Clone** of all abstract clones and all their homomorphisms and then used the well-known fact that $\mathcal{L}$ can be obtained by forming a partially ordered class from the category **Clone** in a standard way (we introduce a quasiordering on objects – $A \leq B$ iff $\mathbf{Clone}(A, B) \neq \emptyset$ and then make a partial ordering from $\leq$). They constructed a semifull embedding from the category of semigroups to **Clone**, i.e. a functor $\Phi : \mathbf{Smg} \to \mathbf{Clone}$ such that $\mathbf{Smg}(A, B) \neq \emptyset$ precisely when $\mathbf{Clone}(\Phi A, \Phi B) \neq \emptyset$, for every $A, B \in \mathrm{Obj}(\mathbf{Smg})$. The mentioned results are consequences of the fact that the category of semigroups is alg-universal.

In the same article, the authors also proved that every group is isomorphic to the endomorphism monoid of some clone A, i.e. the category of clones is group-universal in a stronger sense.

Here we prove a substantial strengthening of both results by answering the open problem formulated there – the category **Clone** is alg-universal. Moreover, the clones will be idempotent. Let us use an alternative formulation (see the next paragraph).

> The category of idempotent varieties and interpretations is alg-universal.

There are several ways how to view a variety: class of algebras, equational theory, finitary monad over **Set** or clone (the last two describe variety up to term equivalence). Clone homomorphisms then correspond to concrete functors (going in the opposite direction), interpretations and monad homomorphisms respectively. We recall these well-known facts in Section 2

Some basic notions and results from the theory of rewriting systems, which we will need for the proof, are recalled in Section 3.

Section 4 contains the proof of the main theorem. To enhance readability, several facts are formulated there, their proofs are in Sections 5,6,7.

1 Auxiliary alg-universal category

To prove that a certain category is alg-universal, it suffices to fully embed any alg-universal category into it. The following auxiliary category will be used to prove the main result.

Definition 1.1. $\mathbf{Alg}_*(1,1)$ *is the full subcategory of* $\mathbf{Alg}(1,1)$ *consisting of algebras* (A,α,β) *such that* $a,\alpha(a),\beta(a)$ *are pairwise distinct for every* $a\in A$.

Proposition 1.2. $\mathbf{Alg}_*(1,1)$ *is alg-universal.*

Proof. We will construct a full embedding $\Phi:\mathbf{Alg}(1,1)\to\mathbf{Alg}(1,1)$ such that for every $\mathcal{A}=(A,\alpha,\beta)\in\mathbf{Alg}(1,1)$, the algebra $\Phi(\mathcal{A})=(\overline{A},\overline{\alpha},\overline{\beta})$ will satisfy $\overline{\alpha}(\overline{a})\neq\overline{a}$, $\overline{\alpha}(\overline{a})\neq\overline{\beta}(\overline{a})$ for all $\overline{a}\in\overline{A}$. Moreover, if $\alpha(a)\neq a$ for every $a\in A$, then $\overline{\beta}(\overline{a})\neq\overline{a}$ for every $\overline{a}\in\overline{A}$. Therefore $\Phi\Phi$ will be a full embedding $\mathbf{Alg}(1,1)\to\mathbf{Alg}_*(1,1)$.

For an algebra $\mathcal{A}=(A,\alpha,\beta)$, let $\Phi(\mathcal{A})=(\overline{A},\overline{\alpha},\overline{\beta})$ be as follows:

$$\begin{aligned}
\overline{A} &= 3\cup A\times 2,\\
\overline{\alpha}(0) &= 1,\quad \overline{\alpha}(1)=0,\quad \overline{\alpha}(2)=1,\\
\overline{\alpha}(a,0) &= 2,\\
\overline{\alpha}(a,1) &= (\beta(a),0),\\
\overline{\beta}(0) &= 2,\quad \overline{\beta}(1)=2,\quad \overline{\beta}(2)=0,\\
\overline{\beta}(a,0) &= (a,1),\\
\overline{\beta}(a,1) &= (\alpha(a),1).
\end{aligned}$$

For a homomorphisms $f:(A,\alpha,\beta)\to(B,\gamma,\delta)$, let

$$\Phi(f)=\overline{f}=id_3\cup f\times 2.$$

It is easy to see, that Φ is a faithful functor and that $(\overline{A},\overline{\alpha},\overline{\beta})$ has all required properties. It remains to prove that Φ is full. So, let $g:(\overline{A},\overline{\alpha},\overline{\beta})\to(\overline{B},\overline{\gamma},\overline{\delta})$ be a homomorphism. We have to prove that $g=\overline{f}$ for some homomorphism $f:A\to B$.

1. Observe that $\overline{\alpha}(\overline{\alpha}(0))=0$ and the only elements $\overline{b}\in\overline{B}$ for which $\overline{\gamma}(\overline{\gamma}(\overline{b}))=\overline{b}$ are $0,1$. Hence $g(0)\in\{0,1\}$, since g is a homomorphism.

2. Suppose $g(0)=1$. Then $g(1)=0$ (because $g(1)=g(\overline{\alpha}(0))=\overline{\gamma}(g(0))=0$), $g(2)=2$ (because $g(2)=g(\overline{\beta}(1))=\overline{\delta}(g(1))=2$. But $0=g(\overline{\alpha}(2))=\overline{\gamma}(g(2))=1$, a contradiction.

3. We have $g(0)=0$, thus $g(1)=1$ (because $g(1)=g(\overline{\alpha}(0))=\overline{\gamma}(g(0))=1$) and $g(2)=2$ (because $g(2)=g(\overline{\beta}(0))=\overline{\delta}(g(0))=2$)).

4. For every $a \in M$, we have $2 = g(2) = g(\overline{\alpha}(a,0)) = \overline{\gamma}(g(a,0))$. The only elements of $\overline{B}$ which are sent to 2 by $\overline{\gamma}$ are the elements $(b,0)$. Therefore $g(a,0) = (f(a),0)$ for some mapping $f : M \to N$. Moreover $g(a,1) = g(\overline{\beta}(a,0)) = \overline{\delta}(g(a,0)) = \overline{\delta}(f(a),0) = (f(a),1)$.

5. Now, we have $g = \overline{f}$. It remains to prove that f is a homomorphism: $(f(\beta(a)),0) = g(\overline{\alpha}(a,1)) = \overline{\gamma}(g(a,1)) = \overline{\gamma}(f(a),1) = (\delta(f(a)),0)$, and
$(f(\alpha(a)),1) = g(\overline{\beta}(a,1)) = \overline{\delta}(g(a,1)) = \overline{\gamma}(f(a),1) = (\gamma(f(a)),1)$. This concludes the proof. □

2 Varieties, interpretations

The basic notions such as universal algebras, varieties, terms, etc. are used in the standard way, see e. g. [18], [33]. We recall several notions to fix the notation.

A (finitary, mono-sorted) *signature* is a set Σ of operational symbols together with a mapping arity : $\Sigma \to \omega$. To avoid some technical difficulties, we assume that there is no nullary operation in any signature. All signatures in this chapter have this property.

Let $\mathbb{V}$ be a (mono-sorted) variety of a (finitary) signature Σ. Let X be a fixed countably infinite set. In this chapter, we assume that $\{x, y, x_0, \dots, x_{18}\} \subset X$. The absolutely free algebra on X in the signature Σ (the algebra of terms in the operational symbols in Σ over the set X) will be denoted by $\mathrm{Term}(\Sigma)$. An endomorphism of $\mathrm{Term}(\Sigma)$ is called a *substitution*, it is determined by values on variables.

The *equational theory* of $\mathbb{V}$, i.e. the fully invariant congruence of $\mathrm{Term}(\Sigma)$ determined by $\mathbb{V}$, will be denoted $\approx_{\mathbb{V}}$. The congruence $\approx_{\mathbb{V}}$ is often given by its generating set – *base*.

$\mathbb{V}$ is said to be *idempotent*, if $\sigma(x, \dots, x) \approx_{\mathbb{V}} x$ for all $\sigma \in \Sigma$ or, equivalently, for all $\sigma \in \mathrm{Term}(\Sigma)$.

An (abstract) *clone*, in its algebraic definition, is an ω-sorted algebra
(C_n, S^n_m, e^n_i) with underlying sets C_n for $n \in \omega$, constants $e^n_i \in C_n$ for $i < n \in \omega$ and heterogeneous operations $S^n_m : C_n \times (C_m)^n \to C_m$, where the following identities hold:

(i) $S^n_k(u; S^m_k(v_1; w_1, \dots, w_m), \dots, S^m_k(v_n; w_1, \dots, w_m)) =$
$= S^m_k(S^n_m(u; v_1, \dots, v_n); w_1, \dots, w_m)$,

(ii) $S^n_n(u; e^n_0, \dots, e^n_{n-1}) = u$,

(iii) $S^n_m(e^n_i; v_0, \dots, v_{n-1}) = v_i$

for any $m, n, k \in \omega$, $u \in C_n$, $v_1, \dots, v_n \in C_m$ and $w_1, \dots, w_m \in C_k$. *Clone homomorphism* $f : (C_n, S^n_m, e^n_i) \to (C'_n, S'^n_m, e'^n_i)$ is a homomorphism of this heterogeneous algebras – a family of mappings $f = \{f_n : C_n \to C'_n \mid n \in \omega\}$ respecting the operations.

From the variety $\mathbb{V}$ we can form its clone $\mathrm{Clone}(\mathbb{V})$ by putting C_n to be the free algebra on the set $\{e^n_0, \dots, e^n_{n-1}\}$ and $S^n_m(u; v_0, \dots, v_{n-1})$ to be the image of u under the homomorphism $C_n \to C_m$ which takes each e^n_i to v_i. Conversely, every clone is a clone of "many" varieties which have the same variety of termal operations (see [40]).

Let $\mathbb{V}, \mathbb{W}$ be varieties of signatures Σ, Γ respectively. By an *interpretation* of $\mathbb{V}$ in $\mathbb{W}$, we mean a mapping $\nu : \mathrm{Term}(\Sigma) \to \mathrm{Term}(\Gamma)$ such that

(i) $\nu(x) = x$ for every $x \in X$. If $t \in \mathrm{Term}(\Sigma)$ is a term over $Y \subseteq X$, then $\nu(t)$ is a term over Y.

(ii) ν preserves substitutions, i.e. $\nu(t(s_0, \ldots, s_n)) = \nu(t)(\nu(s_0), \ldots, \nu(s_n))$ if the left hand side is defined.

(iii) ν preserves equations, i.e. if $s \approx_{\mathbb{V}} t$, then $\nu(s) \approx_{\mathbb{W}} \nu(t)$.

We identify ν and ν', if $\nu(s) \approx_{\mathbb{W}} \nu'(s)$ for all $s \in \mathrm{Term}(\Sigma)$. More precisely, an interpretation should be defined as a mapping $\nu : \mathrm{Term}(\Sigma) \to \mathrm{Term}(\Gamma)/ \approx_{\mathbb{W}}$.

It is clear that ν is determined by values on the terms $\sigma(x_0, \ldots, x_n)$, $\sigma \in \Sigma$ and that in (iii) it suffices to consider only equations from some base of $\approx_{\mathbb{V}}$.

An interpretation $\nu : \mathrm{Term}(\Sigma) \to \mathrm{Term}(\Gamma)$ determines a clone homomorphism $\mathrm{Clone}(\mathbb{V}) \to \mathrm{Clone}(\mathbb{W})$ and vice versa, see [40].

We can also form a concrete functor (i.e. a functor which commutes with the forgetful functors) $\mathbb{W} \to \mathbb{V}$ from an interpretation in a natural way, and vice versa, see [12].

Finally, interpretations between varieties precisely correspond to monad homomorphisms between their monads. For these notions and related results, we refer to [4].

Altogether, the following categories are equivalent.

(i) The category of varieties and interpretations.

(ii) The dual of the category of varieties and concrete functors.

(iii) The category of abstract clones and clone homomorphisms.

(iv) The category of finitary monads over **Set** and monad homomorphisms.

Remark 2.1. *Strictly speaking, (i) and (ii) are not correct formulations, because a variety is a* class *of algebras. But this can be obviously avoided.*

3 Terms, rewriting systems

Here we recall some notions and results about terms and term rewriting systems, see [3] for their proofs.

Let Σ be a signature.

A term t over X (in the signature Σ) can be viewed as a labeled tree, where *leaves* are labeled by elements of X, nodes are labeled by elements of $\sigma \in \Sigma$ and every node labeled by σ has $\mathrm{arity}(\sigma)$ sons.

A *height* $\mathrm{ht}(t)$ of a term t has its obvious meaning, we should just mention that height of a variable is 0.

By an *address* we mean a finite (possible empty) sequence of natural numbers 0,1,.... The concatenation of addresses R, S is denoted by $R\hat{\ }S$. By a subterm of t at the address R, we mean the term $t[R]$ defined inductively by

1. $t[\emptyset] = t$.
2. If $R = S\hat{\ }i$, $t[S] = \sigma(t_0, t_1, \ldots, t_n)$ and $i \leq n$, then $t[R] = t_i$; otherwise $t[R]$ is undefined.

If $t[R]$ is defined, we say that R is a *valid* address in t. We say that s is a *subterm* of t and write $s \subseteq t$, if $s = t[R]$ for some valid address R.

An *equation* (E) (called also *rewriting rule* in some situations) is a pair of terms $(E) = (u, v)$ often written in the form $(E) = u \approx v$.

We say that a term s can be rewritten in one step to t using (E) and write $s \xrightarrow{(E)}_1 t$, if there exists a valid address A in s and a substitution f such that $s[A] = f(u)$ and t is obtained by replacing the subterm $f(u)$ by $f(v)$ at A. We can also say that (E) can be applied to s at the address A and t is the result of the application.

Let $\mathcal{S}$ be a set of equations (called also *rewriting system*) and $\approx$ denote the equational theory it generates. We write $s \xrightarrow{\mathcal{S}}_n t$, if $s = r_0 \xrightarrow{(S_1)}_1 r_1 \ldots \xrightarrow{(S_n)}_1 r_n = t$ for some $(S_i) \in \mathcal{S}$, and write $s \xrightarrow{\mathcal{S}} t$ (and say that s can be rewritten to t), if $s \xrightarrow{\mathcal{S}}_n t$ for some n. A term t is called *reduced*, if no rewriting rule from $\mathcal{S}$ can be applied to t.

It is known and easy to see that $s \approx t$, iff there exists a sequence $r_0, \ldots, r_n$ of terms such that $s = r_0 \xrightarrow{\mathcal{S}}_1 r_1 \xleftarrow{\mathcal{S}}_1 r_2 \longrightarrow \ldots r_n = t$. Such a sequence is called a *derivation* of $s \approx t$.

$\mathcal{S}$ is said to be *finitely terminating*, if every sequence of the form $t_0 \xrightarrow{\mathcal{S}}_1 t_1 \xrightarrow{\mathcal{S}}_1 t_2 \ldots$ is finite. It is said to be *confluent* (resp. *locally confluent*), if for arbitrary terms t, s_0, s_1 such that $t \xrightarrow{\mathcal{S}} s_0, s_1$ (resp. $t \xrightarrow{\mathcal{S}}_1 s_0, s_1$), there exists a term r such that $s_0, s_1 \xrightarrow{\mathcal{S}} r$. If S is finitely terminating and locally confluent, then it is confluent. In this situation, every term s can be rewritten to a unique reduced term $\mathrm{Red}_{\mathcal{S}}(s)$ called *reduced form* of s. Moreover $s \approx t$ iff $\mathrm{Red}_{\mathcal{S}}(s) = \mathrm{Red}_{\mathcal{S}}(t)$.

To verify that $\mathcal{S}$ is locally confluent it is enough to consider *critical overlaps* (see [3], pp 134-141). It is such a situation, when we have a term t, two rules $(E_0), (E_1) \in \mathcal{S}$ and a substitution f such that (E_0) can be applied to $f(t)$ at $\emptyset$ and (E_1) can be applied to $f(t)$ at A, where A is a valid address of t and not an address of some leaf.

By a *reduced height* of a term s is meant the height of the reduced form of s.

4 Main theorem

Theorem 4.1. *The category* **IdempVar** *of idempotent varieties and interpretations is alg-universal.*

Remark 4.2. *It is easy to see that* **IdempVar** *is algebraic (see [47], for example).*

As mentioned, we'll construct a full embedding $\Phi : \mathbf{Alg}_*(1,1) \to \mathbf{IdempVar}$. This is sufficient due to 1.2.

For an algebra $\mathcal{A} = (A, \alpha, \beta) \in \mathbf{Alg}_*(1,1)$, let $\Sigma_{\mathcal{A}}$ be the signature consisting of 19-ary operational symbols c_a, $a \in A$ and binary operational symbols d_a, $a \in A$. Let $\mathbb{A}$ be the variety which equational theory is based by

(C) $c_a(x_0, x_1, \ldots, x_{18}) \approx c_a(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(18)})$ for every permutation σ on 19,

$(D1)$ $c_a(x, 18y) \approx d_a(x, y)$,

$(D3)$ $c_a(3x, 16y) \approx d_{\alpha(a)}(x, y)$,

$(D7)$ $c_a(7x, 12y) \approx d_{\beta(a)}(x, y)$,

$(E0)$ $d_a(d_a(x, y), y) \approx d_a(x, y)$,

$(E1)$ $d_a(x, d_a(x, y)) \approx d_a(x, y)$,

(I) $d_a(x, x) \approx x$.

Each row is to be understood as a set of equations, for example (C) says that for every $a \in A$ and every permutation on 19, we have the equation $c_a(x_0, x_1, \ldots, x_{18}) \approx c_a(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(18)})$. In $(D1), (D3), (D7)$ we use the following abbreviation: $c_a(3x, 16y)$ denotes any term of the form $c_a(W)$, where there are 3 occurrences of x and 16 occurrences of y in W, for example the term $c_a(y, y, x, y, x, y, x, y, y, \ldots, y)$.

For a homomorphisms $f : (A, \alpha, \beta) \to (B, \gamma, \delta)$ we define an interpretation $\nu_f : \mathrm{Term}(\Sigma_{\mathcal{A}}) \to \mathrm{Term}(\Sigma_{\mathcal{B}})$ of $\mathbb{A}$ in $\mathbb{B}$ by

$$\nu_f(d_a(x, y)) = d_{f(a)}(x, y), \quad \nu_f(c_a(x_0, \ldots, x_{18})) = c_{f(a)}(x_0, \ldots, x_{18}).$$

The functor $\Phi : \mathbf{Alg}_*(1,1) \to \mathbf{IdempVar}$ defined $\Phi(\mathcal{A}) = \mathbb{A}$ on objects and $\Phi(f) = \nu_f$ on morphisms is the seeked full and faithful functor.

We postpone the proof of the following facts after the proof of the theorem.

Fact 1. The equations $(D1), (D3), (D7), (E0), (E1), (I)$ form a finitely terminating confluent rewriting system. For any terms s, t in $\Sigma_{\mathcal{A}}$, we have $s \approx_{\mathbb{A}} t$ iff $\mathrm{Red}(s) \sim \mathrm{Red}(t)$, where $\sim$ is the equational theory based by (C) and $\mathrm{Red}(s)$ is the reduced form of s in the equational theory based by $(D1)$, $(D3)$, $(D7)$, $(E0)$, $(E1)$, (I).

From now on by "reduced, reduced height, ...", we mean reduced, reduced height with respect to the above rewriting system. It is clear that if $t \sim s$ and t is reduced, then s is also reduced.

Fact 2. Let t be a term over $\{x, y\}$ in $\Sigma_{\mathcal{A}}$ such that $t(t(x, y), y) \approx_{\mathbb{A}} t(x, y)$, $t(x, t(x, y)) \approx_{\mathbb{A}} t(x, y)$. Then t is of reduced height at most 1.

Fact 3. Let $\mathcal{P} = \{1, 3, 7, 12, 16, 18\}$, $P \subseteq \{x_0, x_1, \ldots, x_{18}\}$, $|P| \in \mathcal{P}$. The substitution g_P sending all variables in P to x and all other variables to y is called *permissible substitution.* Let t be a term over $\{x_0, \ldots, x_{18}\}$ in $\Sigma_{\mathcal{A}}$ such that $g_P(t)$

is of reduced height at most 1 for every permissible substitution g_P. Then the reduced height of t is at most 1.

First, observe that Φ is a correctly defined faithful functor. For better readability, we write $\nu(d_a)$ instead of $\nu(d_a(x,y))$, $\nu(c_a)$ instead of $\nu(c_a(x_0,\ldots,x_{18}))$, and so on.

1. For every $\mathcal{A}$, $\mathbb{A}$ is idempotent: The operations d_a are idempotent (I) and c_a are idempotent because of the equations $(D1)$ and (I), for instance.

2. Φ preserves the composition and the identities: This is clear.

3. Φ is faithful: From Fact 1 it follows that for distinct $b, b' \in B$ the terms $d_b(x,y), d_{b'}(x,y)$ are inequivalent in $\mathbb{B}$.

4. ν_f is an interpretation: The equations $(C), (D1), (E0), (E1)$ and (I) are readily preserved. Preservation of $(D3)$ follows from the fact that f is a homomorphism: $\nu_f(c_a)(3x,16y) = c_{f(a)}(3x,16y) \approx_{\mathbb{B}} d_{\gamma(f(a))}(x,y) = d_{f(\alpha(a))}(x,y) = \nu_f(d_{\alpha(a)})(x,y)$. The proof for $(D7)$ is similar.

It remains to prove that Φ is full. In other words, we have to prove that every interpretation ν of $\mathbb{A}$ in $\mathbb{B}$ is of the form $\nu = \nu_f$ for some homomorphism $f : \mathcal{A} \to \mathcal{B}$. So, let $\nu : \mathrm{Term}(\Sigma_{\mathcal{A}}) \to \mathrm{Term}(\Sigma_{\mathcal{B}})$ be an interpretation.

1. Let $a \in A$. Put $t = \nu(d_a)$. The equations $(E0), (E1)$ are satisfied in $\mathbb{A}$, hence $t(t(x,y),y) \approx_{\mathbb{B}} t(x,y) \approx_{\mathbb{B}} t(x,t(x,y))$. Therefore t is of reduced height at most 1 due to Fact 2.

2. Let g_P be a permissible substitution. We have $g_P(c_a(x_0,\ldots,x_{18})) \approx d_{a'}(x,y)$ in $\mathbb{A}$ for some $a' \in A$ (see the equations $(D1), (D3), (D7)$). Hence $g_P(\nu(c_a)) \approx_{\mathbb{B}} \nu(d_{a'})$. We know from the preceding step that the right hand side is a term of reduced height at most 1. From Fact 3 it follows that $\nu(c_a)$ is of reduced height at most 1.

3. The term $c_a(x_0,\ldots,x_{18})$ is commutative in $\mathbb{A}$ (in the sense of (C)). Therefore the term $\nu(c_a)$ is commutative in $\mathbb{B}$. It is clear (see Fact 1 again) that the only commutative terms in $\mathbb{B}$ of height 1 are the terms $c_b(x_0,\ldots,x_{18})$. Thus $\nu(c_a) = c_{f(a)}(x_0,\ldots,x_{18})$ for some $f(a) \in B$.

4. Since $c_a(x,18y) \approx_{\mathbb{A}} d_a(x,y)$, we have

$$d_{f(a)}(x,y) \approx_{\mathbb{B}} c_{f(a)}(x,18y) = \nu(c_a)(x,18y) \approx_{\mathbb{B}} \nu(d_a).$$

Hence $\nu(d_a) = d_{f(a)}(x,y)$.

5. We have proved, that $\nu = \nu_f$. The last thing is to prove that f is a homomorphism. We have $c_a(3x,16y) \approx_{\mathbb{A}} d_{\alpha(a)}$, hence $\nu(c_a)(3x,16y) \approx_{\mathbb{B}} \nu(d_{\alpha(a)})$. The left hand side equals $c_{f(a)}(3x,16y) \approx_{\mathbb{B}} d_{\gamma(f(a))}(x,y)$. The right hand side equals $d_{f(\alpha(a))}(x,y)$. Using Fact 1 we obtain $\gamma(f(a)) = f(\alpha(a))$.

6. Analogically as in the previous step, using the equation $c_a(7x,12y) \approx_{\mathbb{A}} d_{\beta(a)}(x,y)$, we get $\delta(f(a)) = f(\beta(a))$ and the proof is complete.

5 Fact 1

Fact 1, first part. *The equations $(D1), (D3), (D7), (E0), (E1), (I)$ form a finitely terminating confluent rewriting system.*

Proof. The system is finitely terminating, since each rewriting rule decreases the number of occurrences either of c_a or d_a. To prove its local confluency, it is enough to consider the critical overlaps (see Section 3). In our system, we have to consider the following cases:

1. We can apply two different rules (Di), (Dj) $(i, j \in \{1, 3, 7\})$ at the address $\emptyset$. Consider the case $(D1), (D3)$, the other possibilities are analogical. All terms $t[i]$, $i \in 19$ are equal, say, to a term t_0. We have

$$c_a(t_0, 18t_0) \xrightarrow{(D1)} d_a(t_0, t_0) \xrightarrow{(I)} t_0,$$

$$c_a(3t_0, 16t_0) \xrightarrow{(D3)} d_{\alpha(m)}(t_0, t_0) \xrightarrow{(I)} t_0.$$

2. We can apply the rule (Ei) $(i \in 2)$ at the address $\emptyset$ and the rule (Ej) $(j \in 2)$ at the address j. First, let $i = j = 0$. We can apply $(E0)$ at 0, thus $t[0\hat{}0] = d_a(t_0, t_1)$ and $t[0\hat{}1] = t_1$ for some terms t_0, t_1. We can apply $(E0)$ at $\emptyset$, hence $t[1] = t_1$. Therefore $t = d_a(d_a(d_a(t_0, t_1), t_1), t_1)$. But the application of both rules gives the same result:

$$d_a(d_a(d_a(t_0, t_1), t_1), t_1) \xrightarrow{(E0,1)} d_a(d_a(t_0, t_1), t_1).$$

Next, let $i = 0, j = 1$. We can apply $(E1)$ at 0, hence $t[0\hat{}1] = d_a(t_0, t_1)$ and $t[0\hat{}0] = t_0$. We can apply $(E0)$ at $\emptyset$, hence $t[1] = d_a(t_0, t_1)$. Thus $t = d_a(d_a(t_0, d_a(t_0, t_1)), d_a(t_0, t_1))$. We have

$$d_a(d_a(t_0, d_a(t_0, t_1)), d_a(t_0, t_1)) \xrightarrow{(E0)} d_a(t_0, d_a(t_0, t_1)) \xrightarrow{(E1)} d_a(t_0, t_1),$$

$$d_a(d_a(t_0, d_a(t_0, t_1)), d_a(t_0, t_1)) \xrightarrow{(E1)} d_a(d_a(t_0, t_1), d_a(t_0, t_1)) \xrightarrow{(I)} d_a(t_0, t_1).$$

The two cases $i = 1, j = 0, 1$ are symmetric.

3. We can apply (Ei), $i \in 2$ at $\emptyset$ and (I) at i. In this case $t = d_a(d_a(t_0, t_0), t_0)$ or $t = d_a(t_0, d_a(t_0, t_0))$ which can be rewritten to t_0. □

Recall that the reduced form of a term t in this rewriting system is denoted by $\mathrm{Red}(t)$.

Fact 1, second part. *Let s, t be terms. Then $s \approx t$ in $\mathbb{A}$ if and only if $\mathrm{Red}(s) \sim \mathrm{Red}(t)$, where $\sim$ is the equational theory based by (C).*

Proof. Only the "only if" part is nontrivial. Let $s \approx t$ in $\mathbb{A}$.

Let $\mathcal{S} = \{(Ei), (Dj), (I), i \in \{1, 3, 7\}, j \in 2\}$ and $\equiv$ denote the equational theory generated by $\mathcal{S}$. Let p_0, p_1, p_2 be terms. Observe that if $p_0 \xleftrightarrow{(C)}_1 p_1 \xleftrightarrow{\mathcal{S}}_1 p_2$, then also $p_0 \xleftrightarrow{\mathcal{S}}_1 p_3 \xleftrightarrow{(C)}_1 p_2$ for some term p_3. Hence a derivation of $s \approx t$ can be rearranged to obtain a derivation of $s \equiv s_0 \sim t$, where s_0 is a term. From the previous lemma and Section 3, we know that $s \xrightarrow{\mathcal{S}} \mathrm{Red}(s) \xleftarrow{\mathcal{S}} s_0 \sim t$. After further rearrangement we get $s \xrightarrow{\mathcal{S}} \mathrm{Red}(s) \sim s_1 \xleftarrow{\mathcal{S}} t$ for some term s_1. Clearly, every term $\sim$-equivalent to a reduced term is reduced, thus $s_1 = \mathrm{Red}(t)$. □

6 Fact 2

All terms in this section will be over $\{x, y\}$ in the signature $\Sigma_{\mathcal{A}}$.

Fact 2. *Let t be a term such that $t(x, t(x, y)) \approx_{\mathbb{A}} t(x, y)$, $t(t(x, y), y) \approx_{\mathbb{A}} t(x, y)$. Then t is of reduced height at most 1.*

Proof. Striving for a contradiction, suppose that t is a reduced term with $\mathrm{ht}(t) > 1$ satisfying the equations. Since $\mathbb{A}$ is idempotent, t contains both variables x, y.

Let f_x denote the substitution sending x to t and y to y. Symmetrically, let f_y denote the substitution sending x to x and y to t. The equation $t(x, t(x, y)) \approx_{\mathbb{A}} t(x, y)$ means $f_y(t) \approx_{\mathbb{A}} t$. The equation $t(t(x, y), y) \approx_{\mathbb{A}} t(x, y)$ means $f_x(t) \approx_{\mathbb{A}} t$.

Lemma 6.1. *Let s_1, s_2 be terms and $f_x(s_1) = f_x(s_2)$. Then $s_1 = s_2$.*

Proof. Assume $f_x(s_1) = f_x(s_2)$ (the second case is symmetric). Assume $\mathrm{ht}(s_1) \leq \mathrm{ht}(s_2)$. We proceed by induction on $\mathrm{ht}(s_1)$. First, let $s_1 = y$. Then $f_x(s_1) = y$ and clearly $s_2 = y$. Next, suppose $s_1 = x$, $s_2 \neq x$. Then $f_x(s_1) = t$. If s_2 doesn't contain x then clearly $f_x(s_1) \neq f_x(s_2)$. If s_2 contains x, then $\mathrm{ht}(s_2) > \mathrm{ht}(t)$.

The induction step is trivial. $\square$

Lemma 6.2. *Let s_1, s_2 be terms, $s_2 \subseteq t$, $f_x(s_1) = s_2$. Then $s_1 = y$.*

Proof. Evident. $\square$

Lemma 6.3. *Let t be a reduced term. If $f_x(t)$ is not reduced, then $t = d_a(s, y)$ or $t = d_a(y, s)$, where $a \in A$ and s is a term. If $f_y(t)$ is not reduced, then $t = d_a(s, x)$ or $t = d_a(x, s)$, where $a \in A$ and s is a term.*

Proof. We prove only the first part, the second part being symmetric.

Suppose that we can apply a rewriting rule to $f_x(t)$ at an address R. Since t is reduced, R is a valid address of t and R is not an address of a leaf of t.

We can not apply $(D1), (D3), (D7), (I)$ at R: The term t is reduced, so, if one of these rules can be applied to $f_x(t)$, we have $f_x(t[R\hat{\ }i]) = f_x(t[R\hat{\ }j]))$ for some $i, j \in 19$ such that $t[R\hat{\ }i] \neq t[R\hat{\ }j]$, which contradicts 6.1.

Suppose, we can apply $(E0)$ at R, hence $t[R] = d_a(t_0, t_1)$. If $t_0 \neq x$, we have $t_0 = d_a(t_2, t_3)$ (because $(E0)$ can be applied to $f_x(t)$ at R), and $t_1 \neq t_3$ (because t is reduced). But $f_x(t_1) = f_x(t_3)$ (again, because we can apply $(E0)$ to $f_x(t)$ at R), which contradicts 6.1. So $t_0 = x$. Then $t = d_a(s_0, s_1)$ and $s_1 = f_x(t_1)$. By 6.2 $t_1 = y$, hence $s_1 = y$. Together $t = d_a(s_0, y)$.

Suppose, we can apply $(E1)$ at R, hence $t[R] = d_a(t_0, t_1)$. As in the last paragraph $t_1 = x$, $t = d_a(s_0, s_1)$ and $s_0 = f_x(t_0)$. Hence $t_0 = y$ and $s_0 = y$. $\square$

The last lemma contradicts our assumption $\mathrm{ht}(t) > 1$. $\square$

7 Fact 3

This is the most technical part of the proof. The longest part is an examination of terms of height 2 over $\{x_0, \dots, x_{18}\}$. For those readers who don't want to read the whole proof, we'd like to explain the following:

- **Why 19-ary operations, why 1,3,7?** We will need the properties of those numbers stated in 7.1 and 7.2 several times, for example 7.4.E.4.
- **Why $\mathbf{Alg}_*(1,1)$ instead of $\mathbf{Alg}(1,1)$?** We will use the property of algebras in $\mathbf{Alg}_*(1,1)$ in the proof of 7.4.E.5.

In this section all terms are in the signature $\Sigma_{\mathcal{A}}$.

Recall that

$$\mathcal{P} = \{1, 3, 7, 12, 16, 18\}.$$

Let $P \subseteq \{x_0, x_1, \dots, x_{18}\}, |P| \in \mathcal{P}$. The substitution sending all variables in P to x and all other variables to y is called permissible substitution.

Fact 3 can be formulated as follows:

Fact 3. *Let t be a reduced term over $\{x_0, \dots, x_{18}\}$ of height at least 2. Then there exists a permissible substitution g_P such that $g_P(t)$ is of reduced height at least 2.*

Proof. We proceed by induction on $\mathrm{ht}(t)$ starting from $\mathrm{ht}(t) = 2$. First, we prove the induction step. Let $t_0 = t[i], i \in 19$ be of height at least $\mathrm{ht}(t) - 1 \geq 2$ (it is reduced, since t is). From the induction hypotheses, we can find a permissible substitution g_P such that $\mathrm{ht}(g_P(t_0)) \geq 2$. Put $s_0 = \mathrm{Red}(g_P(t_0))$ and let s be the term obtained by taking the term $g_P(t)$ and applying all possible rewriting rules, but not at the root. We have $\mathrm{ht}(s) \geq 3$.

If $s = d_a(s_0, s_1)$ or $s = d_a(s_1, s_0)$, the only possible rules, which can be applied, are $(E0), (E1), (I)$. In each of these cases $\mathrm{Red}(s) = s_i$, where $\mathrm{ht}(s_i) \geq \mathrm{ht}(s_{1-i})$. Hence $\mathrm{ht}(s) \geq 2$.

if $s = c_a(\dots, s_0, \dots)$, the only possible rules are $(D1), (D3), (D7)$. After applying one of these rules, we obtain a term of the form from the last paragraph, and again $\mathrm{ht}(s) \geq 2$.

It remains to prove the first step. So, we assume $\mathrm{ht}(t) = 2$ and shall find a permissible substitution g_P such that $g_P(t)$ is of reduced height 2.

The following properties of $\mathcal{P}$ will be needed.

Lemma 7.1. *If $i, j \in \mathcal{P}$, then $i + j \notin \mathcal{P}$.*

Proof. Simple computation. □

Lemma 7.2. *If $i, j, k, l \in \mathcal{P}$ and $19 > i + j = k + l$, then $\{i, j\} = \{k, l\}$.*

Proof. Simple computation. □

Let

$$\begin{aligned} S &= \{t[i] \mid i \in 19 \text{ is a valid address of } t\}, \\ g_P(S) &= \{\mathrm{Red}(g_P(s)) \mid s \in S\}. \end{aligned}$$

Lemma 7.3. *If g_P is a permissible substitution such that $g_P(S)$ satisfies one the following conditions (R1-3), then* $\mathrm{ht}(g_P(t)) = 2$.

(R1) $g_P(S)$ contains two different terms of height 1.

(R2) $g_P(S)$ contains two different terms, one of which is of the form $c_a(...)$.

(R3) $g_P(S)$ contains three pairwise different terms.

Proof. Clear. $\square$

Lemma 7.4. *If one of the following set H of terms is contained in S, then there exists a permissible substitution such that $g_P(S)$ satisfies one of the conditions (R1-3).*

(A) $\{d_a(x_i, x_j), d_{a'}(x_k, x_l)\}$, *if $i \neq k$, $j \neq l$ or $a \neq a'$.*

(B) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), x_i\}$, *if there exists $j \in 19$ such that $e_j \neq 1$.*

(C) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), d_{a'}(x_i, x_j)\}$.

(D) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), c_{a'}(e_0x_0, e_1x_1, \ldots, e_{18}x_{18})\}$, *if $a \neq a'$.*

(E) $\{c_a(e_0x_0, e_1x_1, \ldots, e_{18}x_{18}), c_{a'}(f_0x_0, f_1x_1, \ldots, f_{18}x_{18})\}$, *if there exists $i \in 19$ such that $e_i \neq f_i$.*

(F) $\{d_a(x_i, x_j), x_k, x_l\}$, $k \neq l$.

(G) $\{c_a(e_0x_0, \ldots, e_{18}x_{18}), x_i, x_j\}$, $i \neq j$.

(H) $\{d_a(x_i, x_j), d_{a'}(x_k, x_l), x_m\}$, *if $i \neq k$ or $j \neq l$.*

Proof. The proof is shown in the table. For example row A.1 reads as follows: If $a \neq a'$, let $P = ...$ We have $g_P(H) = ...$ and this satisfies the condition (R1) from the last lemma. The letters i, j, k, l, m denote elements of 19. In rows A.3,F.3,H.4, o is an arbitrary element of 19 distinct from i, j, k, l. In B.2, $p \in 19$ is such that $e_p = 0$. In B.2 we need Lemma 7.1 to know that the term $c_a((e_k + e_l)x, \ldots y)$ is reduced. Note also that, for example, $i \neq j$, $k \neq l$ in the case (A), because H is a set of reduced terms; $e_k + e_l < 19$ in B.2 for the same reason, etc.

Case	Assumption	$P =$	$g_P(H) =$
A.1	$a \neq a'$, $i = k$	$\{x_i\}$	$\{d_a(x,y), d_{a'}(x,y)\}$ (R1)
A.2	$a \neq a'$, $j = l$	$\{x_j\}$	$\{d_a(y,x), d_{a'}(y,x)\}$ (R1)
A.3	$i \neq k, j \neq l$	$\{x_i, x_l, x_o\}$	$\{d_a(x,y), d_{a'}(y,x)\}$ (R1)
B.1	$(\exists k)\ 0 \neq e_k \notin \mathcal{P}$	$\{x_k\}$	$\{c_a(e_k x, \ldots y), \ldots\}$ (R2)
B.2	$(\exists k,l)\ e_k, e_l \in \mathcal{P}$	$\{x_k, x_l, x_p\}$	$\{c_a((e_k + e_l)x, \ldots y), \ldots\}$ 7.1 (R2)
C.1	$(\exists k)\ e_k \neq 1$	Similar to B	
C.2	otherwise	$\{x_j\}$	$\{d_a(x,y), d_{a'}(y,x)\}$ (R1)
D.1	$(\exists i)\ e_i \neq 1$	Similar to B	
D.2	otherwise	$\{x_0\}$	$\{d_a(x,y), d_{a'}(x,y)\}$ (R1)
F.1	$i = k$	$\{x_i\}$	$\{d_a(x,y), x, y\}$ (R3)
F.2	$i = l$ or $j \in \{k,l\}$	Similar to F.1.	
F.3	otherwise	$\{x_i, x_k, x_o\}$	$\{d_a(x,y), x, y\}$ (R3)
G.1	$(\exists k)\ e_k \neq 1$	Similar to B	
G.2	otherwise	$\{x_i\}$	$\{d_a(x,y), x, y\}$ (R1)
H.1	$i \neq k, j \neq l$	See(A)	
H.2	$i \neq k, j = l, m = i$	$\{x_i\}$	$\{d_a(x,y), y, x\}$ (R3)
H.3	$i \neq k, j = l, m = k$	$\{x_k\}$	$\{y, d_{a'}(x,y), x\}$ (R3)
H.4	$i \neq k, j = l, m \notin \{i,k\}$	$\{x_i, x_j, x_o\}$	$\{x, d_{a'}(y,x), y\}$ (R3)
H.5	$j \neq k$	Similar to H.1-4	

It remains to prove (E). Let us continue writing the table.

E.1	$(\exists j)\ e_j \neq f_j, e_j \notin \mathcal{P}$	$\{x_j\}$	$\{c_{a'}(e_j x, ...y), ...\}$ (R2)
E.2	$(\exists j,k,l)\ e_j = e_k = e_l = 1$ $f_j = 0, f_k, f_l \in \mathcal{P}$	$\{x_j, x_k, x_l\}$	$\{d_{\ldots}(x,y),$ $c_{a''}((f_k + f_l)x, \ldots)\}$ 7.1 (R2)
E.3	$(\exists j,k,l)\ e_j = e_k = f_l = 0$ $f_j, f_k, e_l \in \mathcal{P}$	$\{x_j, x_k, x_l\}$	$\{c_{a'}(e_l x, ...y),$ $c_{a''}((f_j + f_k)x, \ldots)\}$ 7.1 (R2)

First suppose that for all j either $e_j \neq 0$ or $f_j \neq 0$. If $e_j = 1$ for all j (or $f_j = 1$ for all j), then we can use either E.1 (eventually with e and f interchanged) or E.2. Otherwise we can use either E.3 (in case there is more than one zero among the numbers $e_0, \ldots$ or $f_0, \ldots$) or E.1 (in case that $(\exists j)\ 2 = e_j \neq f_j$ or $2 = f_j \neq e_j$) or E.2.

Now, assume $e_j = f_j = 0$ for some j and $e_k = f_k \neq 0$ for some k and take i such that $e_i \neq f_i$.

E.4	$a' \neq a''$, $e_k \notin \mathcal{P}$	$\{x_k\}$	$\{c_{a'}(e_k x, ...y), c_{a''}(e_k x, ...)\}$
E.5	$a' = a''$, $e_i, f_i \in \mathcal{P}$	$\{x_i\}$	$e_i < 10, f_i > 10 \ldots \{d(x,y), d(y,x)\}$ $e_i > 10, f_i < 10 \ldots \{d(y,x), d(x,y)\}$ $e_i, f_i < 10 \ldots \{d_r(x,y), d_s(x,y)\}$ $r \neq s$ from properties of $\mathbf{Alg}_*(1,1)$ $e_i, f_i > 10 \ldots \{d_r(y,x), d_s(y,x)\}$ $r \neq s$ from properties of $\mathbf{Alg}_*(1,1)$
E.6	$a' = a''$, $e_i, f_i, e_k \in \mathcal{P}$	$\{x_i, x_j, x_k\}$	$\{c_{a'}((e_i + e_k)x, ...),$ 7.1 $c_{a''}((f_i + f_k), ...)\}$

We can further assume $e_i, f_i \in \mathcal{P}$ (otherwise use E.1), $a' \neq a''$ (otherwise E.5), $e_k \in \mathcal{P}$ (otherwise E.4). Now, we can use E.6.

The last case is that $e_j = f_j = 0$ and $e_k \neq f_k$ for all k for which $e_k \neq 0$ or $f_k \neq 0$. We can assume $e_k, f_k \in \mathcal{P} \cup \{0\}$ for all k (otherwise E.1). We can find l, m, n such that $e_l, e_m, e_n \in \mathcal{P}$ (otherwise the term $c_{a'}(e_0 x_0, \dots)$ is not reduced). It is easy to see that either $\{e_l, e_m\} \neq \{f_l, f_m\}$ or $\{e_l, e_n\} \neq \{f_l, f_n\}$ or $\{e_m, e_n\} \neq \{f_m, f_n\}$. In the first case put $P = \{x_j, x_l, x_m\}$. Then $g_P(H)$ will satisfy (R2) according to Lemma 7.2. The other two cases are analogical. $\square$

Now, we are ready to finish the proof of Fact 3.

The first possibility is $t = d_a(t_0, t_1)$. The only remaining cases, where we can't use Lemma 7.4 are in the following table (o is again an element of 19 distinct from i, j, k).

Case	$P =$	$\mathrm{Red}(g_P(t)) =$
$t_0 = c_{a'}(x_0, \dots, x_{18})$ $t_1 = x_i$	$\{x_i\}$	$d_a(d_{a'}(x,y), x)$
$t_0 = x_i$ $t_1 = c_{a'}(x_0, \dots, x_{18})$	$\{x_o\}$	$d_a(y, d_{a'}(x,y))$
$t_0 = d_{a'}(x_i, x_j)$ $t_1 = d_{a'}(x_i, x_k), j \neq k$	$\{x_i, x_k, x_o\}$	$d_a(d_{a'}(x,y), x)$
$t_0 = d_{a'}(x_i, x_j)$ $t_1 = d_{a'}(x_k, x_j), i \neq k$	$\{x_i, x_j, x_o\}$	$d_a(x, d_{a'}(y,x))$
$t_0 = d_{a'}(x_i, x_j)$ $t_1 = x_k, k \neq j$	$\{x_j\}$	$d_a(d_{a'}(y,x), y)$
$t_0 = d_{a'}(x_i, x_j)$ $t_1 = x_k, k = j, a \neq a'$	$\{x_i\}$	$d_a(d_{a'}(x,y), x)$

The second possibility is $t = c_a(t_0, \dots, t_{18})$.

Suppose that there exists $i \in 19$ such that $t_i = c_{a'}(...)$. The only case, where we can't apply Lemma 7.4 is $t = c_a(j c_{a'}(x_0, x_1, \dots, x_{18}), ... x_k)$ for some $j \notin \mathcal{P}$. Let $P = \{x_0\}$. We have $g_P(t) = c_a(j d_{a'}(x,y), ...)$.

The remaining possibilities are (up to a permutation of variables)

$$t = c(e_0 x_0, e_1 d(x_0, x_1), \dots, e_{18} d(x_0, x_{18}))$$

and

$$t = c(e_0 x_0, e_1 d(x_1, x_0), \dots, e_{18} d(x_{18}, x_0)),$$

where the indexes of c and d are arbitrary, $e_i \in 19$. Consider the first case, the second one is similar.

$(\exists i)\ 0 \neq e_i \notin \mathcal{P}$	$\{x_i\}$	
$i = 0$		$c(e_0 x, \ldots d(x, y))$
$i \neq 0$		$c(\ldots y, e_i d(y, x))$
$(\exists i, j, k)\ e_i = 0, e_j, e_k \in \mathcal{P}$	$\{x_i, x_j, x_k\}$	
$0 \in \{i, j, k\}$		$c((e_0 + e_i + e_j)x, \ldots d(x, y))$ (7.1)
$0 \notin \{i, j, k\}$		$c(\ldots y, (e_i + e_j + e_k)d(y, x))$
$(\forall i)\ e_i = 1$	$\{x_1, x_2, x_3\}$	$d(d(y, x), y)$

The proof of Fact 3 is concluded. □

Chapter IV

Functor slices

For every ordinal α we introduce a basket $\mathbb{E}_\alpha$, prove that every essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$, characterize small slices of $\mathbb{E}_\alpha$ and give a common generalization of known results about slices of the algebraic basket $\mathbb{A}$.

In [39], J. Sichler and V. Trnková introduced a concept of functor slices. Their theory yields a quasiorder (i.e. a reflexive and transitive relation) $\leq_s$ on the collection of all faithful functors and thus determines an equivalence $\sim_s$ by $U \sim_s V$ iff $U \leq_s V$ and $V \leq_s U$. If $U \leq_s V$, they say that U is a *slice* of V. See Section 2 for the corresponding definitions.

The results in [39] and more recent investigations [44], [30], [10] have shown an interesting and surprising phenomenon: Forgetful functors of many familiar concrete categories belong to one of five $\sim_s$ equivalence "classes", which were named *baskets*. These baskets together with $\leq_s$ inequalities between them are indicated in Figure 1 (an arrow stands for $\leq_s$; none of the arrows reverses and no arrow can be added, except the arrows implied by transitivity and reflexivity, of course).

$$\begin{tikzcd}
 & \mathbb{R} & \\
 & \mathbb{A} \arrow[u] & \\
\mathbb{P} \arrow[ur] & & \mathbb{P}^{op} \arrow[ul] \\
 & \mathbb{T} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 1: The five basic baskets

Loosely speaking, the basket $\mathbb{R}$ contains the concrete categories (we mean their forgetful functors) which choose their morphisms "in a relational way"; those categories which choose their morphisms "algebraically" are in the basket $\mathbb{A}$; the baskets $\mathbb{P}, \mathbb{P}^{op}$ consist of "degenerate" cases of categories from $\mathbb{A}$; the trivial basket $\mathbb{T}$ contains precisely full embeddings.

However, as was observed later by J. Sichler and V. Trnková, there are many "natural" baskets which lie strictly between $\mathbb{A}$ and $\mathbb{R}$. For example, the category whose objects are sets with two unary operation, the first one total and the second one partial, defined precisely where the first operation has a fix-point. This category determines the basket $\mathbb{E}_2$. We can add a third unary operation defined on fix-points of the second one and we obtain the basket $\mathbb{E}_3$. Continuing in a similar fashion, we get a basket $\mathbb{E}_\alpha$ for every ordinal α. The slice ordering between $\mathbb{E}_\alpha$ and their duals is shown in Figure 2.

Figure 2: Baskets of essentially algebraic categories

$$\begin{tikzcd}
 & \mathbb{R} & \\
\vdots \arrow[ur] & & \vdots \arrow[ul] \\
\mathbb{E}_\omega & & \mathbb{E}_\omega^{op} \\
\vdots & & \vdots \\
\mathbb{E}_2 \arrow[u] & & \mathbb{E}_2^{op} \arrow[u] \\
 & \mathbb{A} = \mathbb{E}_1 = \mathbb{E}_1^{op} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

These categories are special cases of so called essentially algebraic categories (see [2], Section 3). Our first major theorem says that every essentially algebraic category is a slice of some $\mathbb{E}_\alpha$. An important example of an essentially algebraic category is the category of small categories. We show that it belongs to the basket $\mathbb{E}_2$.

The reason why no arrow in Figure 1 can be reversed or added is that certain properties of faithful functors are inherited to slices: Every slice of any member of $\mathbb{R}$ is SSF (strongly small fibered, [44], see Section 2), every slice of (any member of) $\mathbb{A}$ obeys Isbell's [23, 24] zig-zag condition (zz) [39], every slice of $\mathbb{P}$ obeys (p), every slice of $\mathbb{P}^{op}$ obeys $(p)^{op}$ [39]. Conditions $(zz), (p), (p)^{op}$ are recalled in Section 4, we call them "closure rules". We introduce "multiple zig-zag closure rules" (zz^α) which are obeyed by all slices of $\mathbb{E}_\alpha$ and show that no arrow in Figure 2 can be reversed or added (except the obvious arrows, again).

On the other hand, these properties are known to be sufficient in the following cases: every SSF faithful functor is a slice of $\mathbb{R}$, every SSF faithful functor which obeys (p) (resp. (p^{op})) is a slice of $\mathbb{P}$ (resp. $\mathbb{P}^{op}$) (see [44]). Only partial results are known about the basket $\mathbb{A}$: If $U : \mathbf{K} \to \mathbf{H}$ is a faithful functor which obeys (zz) and either $\mathbf{K}$ and $\mathbf{H}$ are small [39], or U is SSF and $\mathbf{H} = \mathbf{Set}$ [36], then U is a slice of $\mathbb{A}$. We prove in Section 5 that every faithful functor between small categories which obeys (zz^α) is a slice of $\mathbb{E}_\alpha$. We also give a slight generalization of both above mentioned results about the basket $\mathbb{A}$.

The chapter is organized as follows:

Section 1	Preliminaries and notation.
Section 2	The concept of a functor slice, equivalent formulations; the baskets $\mathbb{R}, \mathbb{A}, \mathbb{P}, \mathbb{P}^{op}, \mathbb{T}$; SSF condition.
Section 3	The definition of essentially algebraic category of height α; the baskets $\mathbb{E}_\alpha$; every essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$.
Section 4	Closure rule, obeying a closure rule, semantic consequence; the closure rules (zz^α); every essentially algebraic category of height α obeys (zz^α); no arrow in Figure 2 can be added or reversed; syntactic and semantic consequences of closure rules.
Section 5	Known results about universality with respect to closure rules; every faithful functor between small categories which obeys (zz^α) is a slice of $\mathbb{E}_\alpha$; slices of $\mathbb{A}$.

1 Preliminaries and notation

Category theory

See also I.1.

Let $\mathbf{H}$ be a category and $F, G : \mathbf{H} \to \mathbf{Set}$ be functors. The category $\mathbf{A}[F, G]$ is defined as follows: Objects are pairs (H, α), where $H \in \mathrm{Obj}(\mathbf{H})$ and $\alpha \in \mathbf{Set}(FH, GH)$. An $\mathbf{H}$-morphism $h : H \to H'$ is an $\mathbf{A}[F, G]$-morphism from (H, α) to (H', α'), if $Gh \circ \alpha = \alpha' \circ Fh$. We have a natural forgetful functor $\mathbf{A}[F, G] \to \mathbf{H}$ sending (H, α) to H.

Set theory

A partially ordered set $(P, <)$ (= poset) is said to be *well-founded* provided that every nonempty subset has a $<$-minimal element. The *rank* function from P to the class of ordinals is the unique function which satisfy (see [25])

$$\mathrm{rank}_P(p) = \begin{cases} 0 & \text{there is no } q < p, \\ \sup\{\mathrm{rank}_P(q) + 1 \mid q < p\} & \text{otherwise.} \end{cases}$$

By the *height* of P is meant the ordinal number $\sup\{\mathrm{rank}_P(p) + 1 \mid p \in P\}$, or 0 if P is empty. The subscripts will be omitted, if they are clear from the context. A *tree* is a well-founded poset such that the set $\{q \mid q < p\}$ is well-ordered for all $p \in P$.

The symbols $\sqcup, \coprod$ are used for the *coproduct* of sets, i.e. the disjoint union. Since, as I hope, there is no danger of confusion, we identify components of a coproduct with the sets from which the coproduct is formed, so that $A, B \subseteq A \sqcup B$, for instance.

Algebra

The notation here follows the monograph [2].

Let S be a set (of *sorts*). By an *S-sorted signature* is understood a set Σ of *operational symbols* together with an *arity* function assigning to every $\sigma \in \Sigma$ a κ-tuple $(s_i)_{i<\kappa}$ of sorts for some cardinal number κ and a sort s. Notation:

$$\sigma : \prod_{i<\kappa} s_i \to s.$$

A signature is called *nullary*, if it contains nullary operational symbols only. Otherwise, the signature is *nonnullary*.

By an S-sorted set is meant a family $(A_s)_{s\in S}$ of sets. A partial algebra $\mathcal{A}$ of the signature Σ is a pair $((A_s)_{s\in S}, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma})$, where A_i are sets and $\sigma^{\mathcal{A}}$ are partial operations

$$\sigma^{\mathcal{A}} : \mathrm{Def}(\sigma^{\mathcal{A}}) \subseteq \prod_{i<\kappa} A_{s_i} \to A_s.$$

Operations with the *definition domain* $\mathrm{Def}(\sigma^{\mathcal{A}})$ equal to $\prod_{i<\kappa} A_{s_i}$ are called *total.*

A *homomorphism* from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a family of mappings $f = (f_s)_{s\in S}, f_s : A_s \to B_s$ preserving the operations in the following sense: If $\sigma : \prod_{i<\kappa} s_i \to s$ and $(a_i)_{i<\kappa} \in \mathrm{Def}(\sigma^{\mathcal{A}})$, then $(f(a_i))_{i<\kappa} \in \mathrm{Def}(\sigma^{\mathcal{B}})$ and

$$f_s(\sigma^{\mathcal{A}}(a_i)) = \sigma^{\mathcal{B}}(f(a_i)).$$

This yields the category $\mathbf{Palg}(\Sigma)$ of all partial algebras of the signature Σ and their homomorphisms, $\mathbf{Alg}(\Sigma)$ is its full subcategory formed by algebras with all operations total.

The set of *terms* (or Σ-terms) over an S-sorted set X of variables is the smallest S-sorted set such that

- each variable of sort s is a term of sort s,
- for each operational symbol $\sigma : \prod_{i<\kappa} s_i \to s$ and κ-tuple of terms τ_i of sort s_i, we conclude that $\sigma(\tau_i)$ is a term of sort s.

Given an algebra $\mathcal{A}$, term t and a family $(a_x)_{x\in X}$ of elements of the underlying S-sorted set of $\mathcal{A}$ we can naturally define the *value* $t^{\mathcal{A}}(a_x)$ of $t^{\mathcal{A}}$ in (a_x) for those (a_x) which are in the *definition domain* $\mathrm{Def}(t^{\mathcal{A}})$ of the term $t^{\mathcal{A}}$.

In this paragraph we assume that the signature Σ contains no nullary operational symbol. By an *address* we mean a finite (possible empty) sequence of ordinal numbers. The concatenation of addresses R, S is denoted by $R\hat{\ }S$. By a subterm of a term t at the address R, we mean the term $t[R]$ defined inductively by

1. $\tau[\emptyset] = \tau$.
2. If $R = S\hat{\ }i$, $\tau[S] = \sigma(\tau_i)_{i<\kappa}$ and $i < \kappa$, then $\tau[R] = \tau_i$; otherwise $\tau[R]$ is undefined.

If $\tau[R]$ is defined, we say that R is a *valid* address of τ. The valid addresses which have maximal length are addresses of *leaves*, i.e. variables in τ. The operational symbol at a valid address R of τ is denoted by $\tau\langle R\rangle$.

An *(S-)equation* is a pair (τ_1, τ_2) of terms over X of the same sort. Notation: $\tau_1 = \tau_2$. An equation $\tau_1 = \tau_2$ is *satisfied* by an algebra $\mathcal{A}$ in the elements $(a_x)_{x\in X}$ provided that $\tau_1^{\mathcal{A}}(a_x)$, $\tau_2^{\mathcal{A}}(a_x)$ are defined and equal. An algebra $\mathcal{A}$ satisfies $\tau_1 = \tau_2$ provided that $\tau_1^{\mathcal{A}}(a_x) = \tau_2^{\mathcal{A}}(a_x)$ whenever $(a_x)_{x\in X} \in \mathrm{Def}(t_1^{\mathcal{A}}), \mathrm{Def}(t_2^{\mathcal{A}})$.

2 Slices

The notion of a functor slice was introduced in [39]:

Definition 2.1. *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be faithful functors. A pair (Φ, F) of functors $\Phi : \mathbf{K} \to \mathbf{K}'$, $F : \mathbf{H} \to \mathbf{H}'$ is said to be an* s-embedding *of U to U', if $FU = U'\Phi$ and for every $A, B \in \mathrm{Obj}(\mathbf{K})$, $f \in \mathbf{H}(A, B)$*

$$\text{if } Ff \text{ carries a } \mathbf{K}'\text{-morphism } \Phi A \to \Phi B, \text{ then } f \text{ carries a } \mathbf{K}\text{-morphism } A \to B. \tag{1}$$

$$\begin{tikzcd} \mathbf{K} \arrow[r, "U"] \arrow[d, "\Phi"'] & \mathbf{H} \arrow[d, "F"] \\ \mathbf{K}' \arrow[r, "U'"'] & \mathbf{H}' \end{tikzcd} \tag{2}$$

If there exists an s-embedding of U to U', we say that U is a slice *of U' and write $U \leq_s U'$. If $U \leq_s U'$ and $U' \leq_s U$, we say that U and U' are* s-equivalent *and write $U \sim_s U'$. The equivalence "classes" of $\sim_s$ are called* baskets.

Remark 2.2.

1. *In the original definition from [39], the functor F (and thus the functor Φ) was assumed to be faithful. I think that the present definition is more workable and almost equally strong.*
2. *It is easy to see that $\leq_s$ is a quasiorder (reflexive and transitive) and thus $\sim_s$ is an equivalence relation. The notation $X \leq_s Y$ can (and will) be used, if X, Y are baskets, or if X is a faithful functor and Y is a basket, etc.*
3. *(Φ, Id) is an s-embedding iff Φ is concrete (that means $U'\Phi = U$), full and faithful.*
4. *If U, U' are concrete categories (see the introduction), what we can (and often will) assume, the condition (1) can be formulated as follows:*
$$\text{If } Ff \in \mathbf{K}'(\Phi A, \Phi B) \text{ then } f \in \mathbf{K}(A, B). \tag{3}$$
5. *An s-embedding is a weaker notion than a strong embedding: If (Φ, F) is an s-embedding and F is faithful, then (Φ, F) is a strong embedding iff every $\mathbf{K}'$-morphism $g : \Phi A \to \Phi B$ is of the form $g = Ff$ for some $\mathbf{H}$-morphism $f : UA \to UB$.*

To avoid verbose statements, we will often say that "a category $\mathbf{K}$ is a slice of a category $\mathbf{H}$", in place of "the natural forgetful functor of $\mathbf{K}$ is a slice of the natural forgetful functor of $\mathbf{H}$", if the meaning of "natural" is clear.

The diagram (2) is called a *subpullback* for the following reason (see [39]).

Proposition 2.3. *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be faithful functors and $(\Phi : \mathbf{K} \to \mathbf{K}', F : \mathbf{H} \to \mathbf{H}')$ be a pair of functors such that $FU = U'\Phi$. Then the following statements are equivalent.*

(i) (Φ, F) *is an s-embedding.*

(ii) For every $A, B \in \mathrm{Obj}(\mathbf{K})$, the following diagram is a pullback in **Set**.

$$\begin{tikzcd}
\mathbf{K}(A,B) \arrow[r, "U"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H}(UA,UB) \arrow[d, "F"] \\
\mathbf{K}'(\Phi A, \Phi B) \arrow[r, "U'"'] & \mathbf{H}'(U'\Phi A, U'\Phi B)
\end{tikzcd}$$

(iii) The functor I in the following commutative diagram is a full embedding.

$$\begin{tikzcd}
\mathbf{K} \arrow[dr, dashed, "I"] \arrow[drr, bend left, "U"] \arrow[ddr, bend right, "\Phi"'] & & \\
& \mathbf{K}' * \mathbf{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{H} \arrow[d, "F"] \\
& \mathbf{K}' \arrow[r, "U'"'] & \mathbf{H}'
\end{tikzcd}$$

Corollary 2.4. *Let $U : \mathbf{K} \to \mathbf{H}$, $V : \mathbf{H} \to \mathbf{L}$ be faithful functors. Then $U \leq_s VU$.*

Proof. It is easy to see that (Id, V) is an s-embedding. $\square$

Corollary 2.5. *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be faithful functors. Then $U \leq_s V$ iff $U^{op} : \mathbf{K}^{op} \to \mathbf{H}^{op} \leq_s V^{op} : \mathbf{K}'^{op} \to \mathbf{H}'^{op}$.*

Now, we mention some members of the baskets in Figure 1.

Basket $\mathbb{R}$ contains (see [39]) the category $\mathbf{Rel}(\Sigma)$ of relational structures and their homomorphisms for every nonnulary mono-sorted signature; the category $\mathbf{Palg}(\Sigma)$ for every nonnullary mono-sorted signature; the category **Pos** of all partially ordered sets (posets) and order preserving mappings; the category **Top** of all topological spaces and continuous mappings and all its full subcategories down to the category of all metrizable spaces; the category **Unif** of all uniform spaces and uniformly continuous mappings and all its full subcategories down to the category of all complete metrizable spaces; the category **Metr** of all metric spaces and maps which do not increase the distance and all its full subcategories down to the category of all complete metric spaces of diameter at most one; all their duals.

Basket $\mathbb{A}$ contains the category $\mathbf{Alg}(\Sigma)$ for every nonnullary mono-sorted signature (see [39]); more generally the category $\mathbf{Set}^T$ of all monadic algebras for any non-degenerate monad T over **Set** (see [30]; a monad is non-degenerate iff its functor part T is neither the identity nor a constant nor their coproduct); the category $\mathbf{Set}_T$ of all comonadic coalgebras for any non-degenerate comonad T over **Set** (see [10]); all their duals [39].

Basket $\mathbb{P}$ contains the category $\mathbf{Alg}(\Sigma)$ for a nullary nonempty mono-sorted signature [39].

Basket $\mathbb{P}^{op}$ contains precisely the duals of categories in $\mathbb{P}$ [39].

Basket $\mathbb{T}$ consists of all full and faithful functors [39].

An important property which is inherited to slices is the SSF condition (see [1]):

Definition 2.6. *A concrete category $U : \mathbf{K} \to \mathbf{H}$ is said to be* SSF (strongly small fibered)*, if for every $H \in \mathrm{Obj}(\mathbf{H})$, the following equivalence $\sim_{SSF}$ on the class of all pairs (K, f), where $K \in \mathrm{Obj}(\mathbf{K})$, $f \in \mathbf{H}(K, H)$, has only set-many equivalence classes:*

$$(K, f) \sim_{SSF} (K', f')$$

iff

$$(\forall L \in \mathrm{Obj}(\mathbf{K}))\ (\forall g \in \mathbf{H}(H, L))\quad gf \in \mathbf{K}(K, L) \Leftrightarrow gf' \in \mathbf{K}'(K', L)$$

Most of "everyday life" categories are SSF. All categories mentioned in this thesis, for instance.

Proposition 2.7 (See [44])**.** *A slice of SSF concrete category is SSF.*

On the other hand, every SSF concrete category is a slice of $\mathbb{R}$. See Section 5 for this and similar results.

3 Essentially algebraic categories

As mentioned, the category $\mathbf{Palg}(\Sigma)$ of all partial algebras with given (nonnullary) signature and their homomorphisms belongs to the relational basket (we mentioned the mono-sorted case only, but this can be easily generalized). However, these categories have important full subcategories called essentially algebraic. These categories substantially enrich our five-member collection of baskets.

Definition 3.1. *Let α be an ordinal, S be a set. An S-sorted* essentially algebraic theory *of height α is given by a quadruple $\Gamma = (\Sigma, \mathrm{level}, E, \mathrm{Def})$ where:*

- *Σ is an S-sorted signature (finitary or infinitary).*
- *$\mathrm{level} : \Sigma \to \alpha$ is a mapping assigning a* level *to each operational symbol $\sigma \in \Sigma$. The set of all operational symbols of level β is denoted by Σ_β. Analogically we define $\Sigma_{<\beta}$, $\Sigma_{\leq\beta}$.*

- E *is a set of* Σ*-equations.*

- Def *assigns to each* κ*-ary operational symbol* $\sigma \in \Sigma$ *a set of* $\Sigma_{<\mathrm{level}(\sigma)}$*-equations over a* κ*-indexed set* $X = (x_i)_{i<\kappa}$ *(where the variables have the right sorts). For all* σ *such that* $\mathrm{level}(\sigma) = 0$*, we assume* $\mathrm{Def}(\sigma) = \emptyset$*.*

By a model *of* Γ *(or a* Γ-algebra*) we mean a partial* S*-sorted algebra* $\mathcal{A} = ((A_s)_{s\in S}, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma})$ *such that* $\mathcal{A}$ *satisfies all equations of* E *and* $\sigma^{\mathcal{A}}(a_i)_{i<\kappa}$ *is defined iff* $\mathcal{A}$ *satisfies all equations from* $\mathrm{Def}(\sigma)$ *in the elements* $(a_i)_{i<\kappa}$*.*

The category of all Γ*-algebras and homomorphisms is called an* S*-sorted* essentially algebraic category *of height* α*.*

Remark 3.2. 1. *Locally presentable categories are, up to equivalence, precisely essentially algebraic categories (see [2]). In fact, essentially algebraic categories of height 2 suffice to describe all locally presentable categories at the abstract level (i.e. up to equivalence), but the height is significant at the concrete level (i.e. when considering forgetful functors).*

2. *Operations of level 0 are total. Operations of level 1 are defined where certain equations in total operational symbols are satisfied, and so on. This guarantees the following pleasant property of homomorphisms: Let* ρ *be a* κ*-ary operational symbol of level* β*. If a mapping* $f : \mathcal{A} \to \mathcal{B}$ *preserves all operations* $\sigma \in \Sigma_{<\beta}$*, then* $(a_i)_{i<\kappa} \in \mathrm{Def}(\rho^{\mathcal{A}})$ *implies* $(f(a_i))_{i<\kappa} \in \mathrm{Def}(\rho^{\mathcal{B}})$*.*

3. *An* S*-sorted essentially algebraic category of height 0 is (isomorphic to) the category* $\mathbf{Set}^S$ *of* S*-sorted sets.*

4. S*-sorted essentially algebraic categories of height 1 are precisely varieties of* S*-sorted algebras.*

5. *Let* $\mathbf{K}$ *be an* S*-sorted essentially algebraic category. We have two "natural" forgetful functors* U, V*:* $U : \mathbf{K} \to \mathbf{Set}^S$ *sends an algebra* $\mathcal{A} = ((A_s)_{s\in S}, ...)$ *to* $(A_s)_{s\in S}$*.* $V : \mathbf{K} \to \mathbf{Set}$ *sends* $\mathcal{A}$ *to* $\coprod_{s\in S} A_s$*.*

For every well-founded poset P we now define a mono-sorted essentially algebraic category $\mathbf{Fix}(P)$ of height equal to the height of P. The important cases are $P = \alpha$ for an ordinal α with its natural ordering.

Definition 3.3. *Let* $(P, <)$ *be a well-founded poset.* $\mathbf{Fix}(P)$ *is the category of models of the essentially algebraic theory* $\Gamma = (\Sigma, \mathrm{level}, E, \mathrm{Def})$*, where* Σ *is mono-sorted and consists of unary operational symbols* ϕ_p*,* $p \in P$*;* $\mathrm{level}(p)$ *is the rank of* p *in the poset* P*;* $E = \emptyset$*;* $\mathrm{Def}(\phi_p) = \{\phi_q(x_0) = x_0 \,|\, q < p\}$*.*

So, an algebra $\mathcal{A} \in \mathrm{Obj}(\mathbf{Fix}(P))$ *is a set* A *together with partial unary operations* $\phi_p^{\mathcal{A}}$*,* $p \in P$ *such that* $\mathrm{Def}(\phi_p^{\mathcal{A}}) = \mathrm{Fix}\{\phi_q^{\mathcal{A}} \,|\, q < p\}$*, where*

$$\mathrm{Fix}\{\phi_q^{\mathcal{A}} \,|\, q < p\} = \{a \in A \,|\, \phi_q^{\mathcal{A}}(a) = a \text{ for all } q < p\}$$

is the set of common fix-points of all operations $\phi_q^{\mathcal{A}}$*,* $q < p$*.*

Let $\mathbb{E}_\alpha$ *denote the basket determined by* $\mathbf{Fix}(\alpha)$*.*

We will see that (any of the two forgetful functors of) each essentially algebraic category of height α is a slice of $\mathbb{E}_\alpha$ (Theorem 3.5) and we will characterize those functors between small categories which are slices of $\mathbb{E}_\alpha$ (Theorem 5.2).

We will show that the inequalities marked in Figure 2 hold and no arrow can be added or reversed: $\mathbb{E}_\alpha \leq_s \mathbb{E}_\beta$ for $\alpha \leq \beta$ (3.4) and the inequality is strict if $\alpha < \beta$ (4.9); $\mathbb{E}_2 \not\leq_s \mathbb{E}_\alpha^{op}$ for every α (4.10); of course, $\mathbb{E}_\alpha \leq_s \mathbb{R}$, since every essentially algebraic category is a concrete full subcategory of $\mathbf{Palg}(\Sigma)$; $\mathbb{E}_\alpha \not\sim_S \mathbb{R}$ follows from 4.4, 4.8, 5.3.1., for instance.

Proposition 3.4. *Let P be a subposet of a poset Q. Then $\mathbf{Fix}(P) \leq_s \mathbf{Fix}(Q)$. In particular $\mathbb{E}_\alpha \leq_s \mathbb{E}_\beta$ for arbitrary ordinals $\alpha \leq \beta$.*

Proof. Let $F = \mathrm{Id}$. For an algebra $\mathcal{A} = (A, (\phi_p)^{\mathcal{A}}_{p\in P}) \in \mathbf{Fix}(P)$ let $\Phi\mathcal{A} = (A, (\phi_q)^{\Phi\mathcal{A}}_{q\in Q})$, where

$$\begin{aligned}
\mathrm{Def}(\phi_q^{\Phi\mathcal{A}}) &= \begin{cases} A & \text{if } \{p \in P \,|\, p < q\} \text{ is empty,} \\ \bigcap_{p\in P,\ p<q} \mathrm{Fix}(\phi_p^{\mathcal{A}}) & \text{otherwise,} \end{cases} \\
\phi_q^{\Phi\mathcal{A}}(a) &= \begin{cases} \phi_q^{\mathcal{A}}(a) & \text{if } q \in P, \\ a & \text{otherwise} \end{cases}
\end{aligned}$$

for all $a \in \mathrm{Def}(\phi_q^{\Phi\mathcal{A}})$.

Clearly, $\Phi\mathcal{A}$ is a $\mathbf{Fix}(Q)$-object, Φ is a functor and (Φ, F) is an s-embedding. $\square$

Theorem 3.5. *Let $\mathbf{K}$ be an S-sorted essentially algebraic category of height α with its theory $\Gamma = (\Sigma, \mathrm{level}, E, \mathrm{Def})$. Then $U \leq_s V \leq_s \mathbb{E}_\alpha$ where $U : \mathbf{K} \to \mathbf{Set}^S$, $V : \mathbf{K} \to \mathbf{Set}$ are the natural forgetful functors.*

Proof. $U \leq_s V$ follows from 2.4 since V is the composition of U and the coproduct functor $\mathbf{Set}^S \to \mathbf{Set}$.

We can assume that $E = \emptyset$ (because concrete full subcategory is a slice) and that Σ contains no nullary operational symbol (we can replace them by unary operational symbols).

We can further assume that Σ is mono-sorted:

Claim 1. *V is a slice of a mono-sorted essentially algebraic category of height α.*

Proof. Let
$$\overline{\Gamma} = (\overline{\Sigma} = \Sigma \sqcup \{\rho\}, \overline{\mathrm{level}}, \emptyset, \overline{\mathrm{Def}}),$$
where operational symbols from $\Sigma \subseteq \overline{\Sigma}$ have the same arities, levels and defining identities, but are considered as mono-sorted (we forget sorts). The operational symbol ρ is unary and total (of level 0). The category of $\overline{\Gamma}$-algebras will be denoted by $\mathbf{L}$.

Now, we are going to define an s-embedding of V to (the natural forgetful functor of) $\mathbf{L}$. The functor F from the subpullback square (2) is defined by

$$\begin{aligned}
FA &= A \sqcup S \sqcup \{c\}, \\
Ff &= f \sqcup \mathrm{id}_s \sqcup \mathrm{id}_c,
\end{aligned}$$

where A is a set and $f : A \to B$ is a mapping.

The functor Φ is defined for an algebra $\mathcal{A} \in \mathbf{K}$ by

$$\Phi\mathcal{A} = \Phi((A_s)_{s\in S}, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma}) = (\coprod_{s\in S} A_s \sqcup S \sqcup \{c\}, (\sigma^{\Phi\mathcal{A}})_{\sigma\in\Sigma}, \rho^{\Phi\mathcal{A}}),$$

where $\rho^{\Phi\mathcal{A}}(a_s) = s$ for $a_s \in A_s$, $\rho^{\Phi\mathcal{A}}(s) = \rho^{\Phi\mathcal{A}}(c) = c$ for $s \in S$. For an operational symbol $\sigma : \prod_{i<\kappa} s_i \to s$, the operation $\sigma^{\Phi\mathcal{A}} : \prod_{i<\kappa} FV\mathcal{A} \to FV\mathcal{A}$ is given by

$$\sigma^{\Phi\mathcal{A}}(a_i)_{i<\kappa} = \begin{cases} \sigma^{\mathcal{A}}(a_i)_{i<\kappa} & \text{if } a_i \in A_{s_i}, i < \kappa \text{ and } (a_i)_{i<\kappa} \in \mathrm{Def}(\sigma^{\mathcal{A}}), \\ c & \text{otherwise (on the def. dom.).} \end{cases}$$

It is easy to see that $\Phi\mathcal{A} \in \mathbf{L}$ for any $\mathcal{A} \in \mathbf{K}$. Let $\mathcal{A} = ((A_s)_{s\in S}, \dots), \mathcal{B} = ((B_s)_{s\in S}, \dots) \in \mathbf{K}$. A mapping $f : \coprod_{s\in S} A_s \to \coprod_{s\in S} B_s$ carries a $\mathbf{K}$-homomorphism $\mathcal{A} \to \mathcal{B}$, iff $f(A_s) \subseteq B_s$ (for all $s \in S$) and f preserves all operations $\sigma \in \Sigma$. This arises precisely when $Ff : \Phi\mathcal{A} \to \Phi\mathcal{B}$ preserves ρ and all $\sigma \in \Sigma$. Hence (Φ, F) is an s-embedding. $\square$

To formulate and prove the next two claims, we need to introduce further notation. For a set X, let $Q_X : \mathbf{Set} \to \mathbf{Set}$ be the covariant hom-functor:

$$\begin{aligned} Q_X A &= \{(a_x)_{x\in X} \mid a_x \in A\}, \quad \text{where } A \text{ is a set,} \\ Q_X f(a_x)_{x\in X} &= (f(a_x))_{x\in X}, \quad \text{where } f : A \to B \text{ is a mapping.} \end{aligned}$$

Given a set Y a subset $D \subseteq Q_Y A$ and a set $X \subseteq Y$ we define a set $\mathrm{Proj}(D; Y \to X) \subseteq Q_X A$ by

$$\mathrm{Proj}(D; Y \to X) = \{(a_x)_{x\in X} \mid (\exists\ (b_y)_{y\in Y} \in D)\ (\forall x \in X)\ a_x = b_x\}.$$

Given a partial unary operation $\rho : \mathrm{Def}(\rho) \subseteq Q_X A \to Q_X A$ we define a partial unary operation $\mathrm{Ext}(\rho; X \to Y) : D \subseteq Q_Y A \to Q_Y A$ by

$$\begin{aligned} (a_y)_{y\in Y} \in D \quad &\text{iff} \quad (a_x)_{x\in X} \in \mathrm{Def}(\rho), \\ (\mathrm{Ext}(\rho; X \to Y)(a_y)_{y\in Y})_k \quad &= \quad \begin{cases} (\rho(a_x)_{x\in X})_k & \text{if } k \in X, \\ a_k & \text{otherwise.} \end{cases} \end{aligned}$$

Given a subset $D \subseteq Q_X A$, an element $r \in X$ and a partial mapping $e : D \to A$ we define a partial unary operation $\mathrm{Ope}(D; e(a_x)_{x\in X} \to a_r)$ by

$$\begin{aligned} \mathrm{Def}(\mathrm{Ope}(D; e(a_x)_{x\in X} \to a_r)) &= D, \\ (\mathrm{Ope}(D; e(a_x)_{x\in X} \to a_r)(a_x)_{x\in X})_k &= \begin{cases} e(a_x)_{x\in X} & \text{if } k = r, \\ a_k & \text{otherwise.} \end{cases} \end{aligned}$$

Let P be a poset. We say that P satisfy (P1), if

(P1) P is well-founded and the dual poset is a tree.

The following two claims will be proved simultaneously by induction on β.

Claim 2. *Let $\beta \leq \alpha$ be an ordinal. Let τ be a term over X in operational symbols from $\Sigma_{<\beta}$. Then there exists a poset P_τ of height $\leq \beta$ satisfying (P1), a set Y_τ and a functor $\Phi_\tau : \mathbf{K} \to \mathbf{Fix}(P_\tau)$ such that*

(A1) $W\Phi_\tau = Q_{X \sqcup Y_\tau} U$*, where* $W : \mathbf{Fix}(P_\tau) \to \mathbf{Set}$ *is the forgetful functor.*

(A2) There is an element $z_\tau \in Y_\tau$ *such that for each algebra* $\mathcal{A} \in \mathbf{K}$

$$\mathrm{Proj}(\mathrm{Fix}\{\phi_p^{\Phi_\tau \mathcal{A}} \,|\, p \in P_\tau\}; X \sqcup Y_\tau \to X \sqcup \{z_\tau\}) =$$

$$= \{(a_j)_{j \in X \sqcup \{z_\tau\}} \,|\, (a_x)_{x \in X} \in \mathrm{Def}(\tau^{\mathcal{A}}),\ a_{z_\tau} = \tau^{\mathcal{A}}(a_x)_{x \in X}\}.$$

(A3) Let $\mathcal{A} = (A, \dots), \mathcal{B} = (B, \dots) \in \mathbf{K}$*. Let* $f : A \to B$ *be a mapping such that* $Q_{X \sqcup Y_\tau} f : \Phi_\tau \mathcal{A} \to \Phi_\tau \mathcal{B}$ *is a* $\mathbf{Fix}(P_\tau)$*-morphism.*
Then $f(\tau^{\mathcal{A}}(a_x)_{x \in X}) = \tau^{\mathcal{B}}(f(a_x))_{x \in X}$ *for any* $(a_x)_{x \in X} \in \mathrm{Def}(\tau^{\mathcal{A}})$.

Claim 3. *Let $\beta < \alpha$ be an ordinal, $\sigma \in \Sigma_{\leq \beta}$ be an operational symbol of arity κ, $X = (x_i)_{i < \kappa}$ be a κ-indexed set. Then there exists a poset P_σ of height $\leq \beta$ satisfying (P1), a set Y_σ and a functor $\Phi_\sigma : \mathbf{K} \to \mathbf{Fix}(P_\sigma)$ such that*

(B1) $W\Phi_\sigma = Q_{X \sqcup Y_\sigma} U$*, where* $W : \mathbf{Fix}(P_\sigma) \to \mathbf{Set}$ *is the forgetful functor.*

(B2) For each algebra $\mathcal{A} \in \mathbf{K}$ *we have*

$$\mathrm{Proj}(\mathrm{Fix}\{\phi_p^{\Phi_\sigma \mathcal{A}} \,|\, p \in P_\sigma\}; X \sqcup Y_\sigma \to X) = \{(a_x)_{x \in X} \,|\, (a_{x_i})_{i < \kappa} \in \mathrm{Def}(\sigma^{\mathcal{A}})\}.$$

Proof of Claim 2. Since the statement is empty for $\beta = 0$, we assume $\beta \geq 1$. Assume that Claim 3 holds for all $\gamma < \beta$. We denote

$$\begin{aligned} \mathrm{Leaves} &= \{R \,|\, R \text{ is an address of a leaf of } \tau\}, \\ \mathrm{Addr} &= \{R \,|\, R \text{ is a valid address of } \tau,\ R \notin \mathrm{Leaves}\}, \\ \mathrm{Succ}(R) &= \{i \,|\, R\hat{\ }i \text{ is a valid address of } \tau\}, \qquad R \in \mathrm{Addr}, \\ Z_R &= \{z_{R\hat{\ }i} \,|\, i \in \mathrm{Succ(R)}\}, \qquad \mathrm{R} \in \mathrm{Addr}. \end{aligned}$$

For all $R \in \mathrm{Addr}$ let Y_R be a set, P_R be a poset satisfying (P1) and $\Phi_R : \mathbf{K} \to \mathbf{Fix}(P_R)$ be a functor such that

- $W_R \Phi_R = Q_{Z_R \sqcup Y_R} U$, where $W_R : \mathbf{Fix}(P_R) \to \mathbf{Set}$ is the forgetful functor.
- For each algebra $\mathcal{A} \in \mathbf{K}$

$$\mathrm{Proj}(\mathrm{Fix}\{\phi_p^{\Phi_R \mathcal{A}} \,|\, p \in P_R\}; Z_R \sqcup Y_R \to Z_R) =$$

$$= \{(a_z)_{z \in Z_R} \,|\, (a_{z_{R\hat{\ }i}})_{i < \kappa} \in \mathrm{Def}(t\langle R \rangle^{\mathcal{A}})\}.$$

Let

$$P_\tau = \coprod_{R \in \mathrm{Addr}} P_R \sqcup \{q_R \,|\, R \in \mathrm{Addr} \cup \mathrm{Leaves}\},$$

where the ordering of P_τ on the set P_R coincides with the ordering of P_R, q_R is a new greatest element of P_R for $R \in \mathrm{Addr}$ and q_R is of rank 0 for $R \in \mathrm{Leaves}$. The poset P_τ clearly satisfy (P1) and its height is not greater than β.

Let

$$\begin{aligned} Z &= \{z_R \mid R \in \mathrm{Addr} \cup \mathrm{Leaves}\}, \\ Y_\tau &= \coprod_{R \in \mathrm{Addr}} Y_R \sqcup Z = \\ &= \coprod_{R \in \mathrm{Addr}} Y_R \sqcup \coprod_{R \in \mathrm{Addr}} Z_R \sqcup \{z_\emptyset\}. \end{aligned}$$

Finally we have to define the functor Φ_τ. For an algebra $\mathcal{A} = (A, (\sigma^{\mathcal{A}})_{\sigma \in \Sigma}) \in \mathbf{K}$ we put

$$\Phi_\tau \mathcal{A} = (Q_{X \sqcup Y_\tau} A, (\phi_p^{\Phi_\tau \mathcal{A}})_{p \in P_\tau}),$$

where

$$\begin{aligned} \phi_p^{\Phi_\tau \mathcal{A}} &= \mathrm{Ext}(\phi_p^{\Phi_R \mathcal{A}}; Z_R \sqcup Y_R \to X \sqcup Y_\tau), \quad p \in P_R, \\ \phi_{q_R}^{\Phi_\tau \mathcal{A}} &= \mathrm{Ope}(\mathrm{Fix}\{\phi_p^{\Phi_\tau \mathcal{A}} \mid p \in P_R\}; \tau\langle R\rangle^{\mathcal{A}}(a_{z_{R\hat{}i}})_{i \in \mathrm{Succ}(R)} \to a_{z_R}), \quad R \in \mathrm{Addr}, \\ \phi_{q_R}^{\Phi_\tau \mathcal{A}} &= \mathrm{Ope}(Q_{X \sqcup Y_\tau} A; a_{\tau\langle R\rangle} \to a_{z_R}), \quad R \in \mathrm{Leaves}. \end{aligned}$$

From the properties of Φ_R we know that the definition of $\phi_{q_R}^{\Phi_\tau \mathcal{A}}$ makes sense. Clearly, if $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $Q_{X \sqcup Y_\tau} f : \Phi_\tau \mathcal{A} \to \Phi_\tau \mathcal{B}$ preserves the operation ϕ_p for all $p \in P$. Thus Φ_τ is a functor.

For $R \in \mathrm{Leaves}$ we have

$$\mathrm{Proj}(\mathrm{Fix}\{\phi_{q_R}^{\Phi_\tau \mathcal{A}}\}; X \sqcup Y_\tau \to X \sqcup Z) = \{(a_j)_{j \in X \sqcup Z} \mid a_{z_R} = a_{\tau\langle R\rangle}\}$$

and for $R \in \mathrm{Addr}$ we have

$$\mathrm{Proj}(\mathrm{Fix}\{\phi_{q_R}^{\Phi_\tau \mathcal{A}}\}; X \sqcup Y_\tau \to X \sqcup Z) =$$

$$= \{(a_j)_{j \in X \sqcup Z} \mid (a_{z_{R\hat{}i}})_{i \in \mathrm{Succ}(R)} \in \mathrm{Def}(\tau\langle R\rangle^{\mathcal{A}}) \text{ and } a_{z_R} = \tau\langle R\rangle^{\mathcal{A}}(a_{z_{R\hat{}i}})_{i \in \mathrm{Succ}(R)}\}.$$

Therefore

$$\mathrm{Proj}(\mathrm{Fix}\{\phi_p^{\Phi_\tau \mathcal{A}} \mid p \in P_\tau\}; X \sqcup Y_\tau \to X \sqcup Z) =$$

$$= \{(a_j)_{j \in X \sqcup Z} \mid (\forall R \in \mathrm{Leaves} \cup \mathrm{Addr})\ (a_x)_{x \in X} \in \mathrm{Def}(\tau[R]^{\mathcal{A}}),\ a_{z_R} = \tau[R]^{\mathcal{A}}(a_x)_{x \in X}\}$$

and thus the property (A2) is satisfied for $z_\tau = z_\emptyset$ and (A3) is clear. $\square$

Proof of Claim 3. The statement is clear for $\beta = 0$, thus we can assume $\beta \geq 1$. Assume that Claim 2 holds for all $\gamma \leq \beta$. Let $\mathrm{Def}(\sigma)$ consist of equations $\tau_i = \xi_i$, $i \in \lambda$, where τ and ξ are $\Sigma_{<\beta}$-terms over X. Let $Y_{\tau_i}, Y_{\xi_i}, z_{\tau_i}, z_{\xi_i}, P_{\tau_i}, P_{\xi_i}, \Phi_{\tau_i}, \Phi_{\xi_i}$ be from the induction hypothesis.

Let

$$\begin{aligned} Y_\sigma &= (\coprod_{i<\lambda} Y_{\tau_i} \sqcup \coprod_{i<\lambda} Y_{\xi_i}) / \approx \\ P_\sigma &= \coprod_{i<\lambda} P_{\tau_i} \sqcup \coprod_{i<\lambda} P_{\xi_i} \end{aligned}$$

where the ordering of P_σ on the sets P_{τ_i} and P_{ξ_i} coincides with the original one and no other inequalities are added; the equivalence $\approx$ glues z_{τ_i} with z_{ξ_i} and nothing else. The element $[z_{\tau_i}] = [z_{\xi_i}]$ of Y_σ will be denoted by z_i.

Now we define the functor Φ_σ. For an algebra $\mathcal{A} = (A, (\sigma^{\mathcal{A}})_{\sigma\in\Sigma}) \in \mathbf{K}$ we put

$$\Phi_\sigma \mathcal{A} = \{Q_{X\sqcup Y_\sigma} A, (\phi_p^{\Phi_\sigma \mathcal{A}})_{p\in P_\sigma}\},$$

where

$$\begin{aligned} \phi_p^{\Phi_\sigma \mathcal{A}} &= \operatorname{Ext}(\phi_p^{\Phi_{\tau_i} \mathcal{A}}; X \sqcup Y_{\tau_i} \to X \sqcup Y_\sigma), \quad p \in P_{\tau_i}, \\ \phi_p^{\Phi_\sigma \mathcal{A}} &= \operatorname{Ext}(\phi_p^{\Phi_{\xi_i} \mathcal{A}}; X \sqcup Y_{\xi_i} \to X \sqcup Y_\sigma), \quad p \in P_{\xi_i}. \end{aligned}$$

Evidently, Φ_σ is a functor.

We have

$$(a_j)_{j\in X\sqcup\{z_i \,|\, i<\lambda\}} \in \operatorname{Proj}(\operatorname{Fix}\{\phi_p^{\Phi_\sigma \mathcal{A}} \,|\, p \in P_\sigma\}; X \sqcup Y_\sigma \to X \sqcup \{z_i \,|\, i < \lambda\})$$

iff

$$(\forall i < \lambda)\ (a_x)_{x\in X} \in \operatorname{Def}(\tau_i^{\mathcal{A}}) \cap \operatorname{Def}(\xi_i^{\mathcal{A}}) \text{ and } a_{z_i} = \tau_i^{\mathcal{A}}(a_x)_{x\in X} = \xi_i^{\mathcal{A}}(a_x)_{x\in X}$$

and (B2) follows. □

From Claim 2 we can now easily deduce:

Claim 4. $\mathbf{K} \leq_s \mathbf{Fix}(P)$ *for a poset* P *of height* $\leq \alpha$ *satisfying (P1).*

Proof. For every operational symbol $\sigma \in \Sigma$ we can use Claim 2 for the term $\sigma(x_i^\sigma)_{i\in\mathrm{arity}(\sigma)}$ over $X_\sigma = \{x_i^\sigma\}_{i\in\mathrm{arity}(\sigma)}$. We obtain a set Y_σ a poset P_σ of height at most α satisfying (P1) and a functor $\Phi_\sigma : \mathbf{K} \to \mathbf{Fix}(P_\sigma)$ such that

- $W_\sigma \Phi_\sigma = Q_{X_\sigma\sqcup Y_\sigma} V$,
- A mapping $f : \mathcal{A} \to \mathcal{B}$ preserves the operation σ whenever $Q_{X_\sigma\sqcup Y_\sigma} : \Phi_\sigma\mathcal{A} \to \Phi_\sigma\mathcal{B}$ is a $\mathbf{Fix}(P_\sigma)$-morphism.

Let

$$P = \coprod_{\sigma\in\Sigma} P_\sigma, \quad F = \coprod_{\sigma\in\Sigma} Q_{X_\sigma\sqcup Y_\sigma},$$

where the ordering of P on each component P_σ coincides with the original one and no other inequalities are added. Recall that the coproduct of functors is computed componentwise.

For an algebra $\mathcal{A} = (A, \dots) \in \mathbf{K}$, let $\Phi\mathcal{A} = (FA, (\phi_p^{\Phi\mathcal{A}})_{p\in P})$, where the operation $\phi_p^{\Phi\mathcal{A}}$ agrees with $\phi_p^{\Phi_\sigma\mathcal{A}}$ on the component $Q_{X_\sigma\sqcup Y_\sigma} A$ and $\phi_p^{\Phi\mathcal{A}}(x) = x$ for every $p \in P_\sigma$, $x \in FA - Q_{X_\sigma\sqcup Y_\sigma} A$. It is clear that Φ is a correctly defined functor and (Φ, F) is an s-embedding. □

To finish the proof we first adjust properties of the poset P and then find an s-embedding to $\mathbf{Fix}(\alpha)$. The wanted properties are:

(P2) *P is well-founded and $\{q \mid q > p\}$ is linearly (and hence well) ordered for every $p \in P$.*

(P3) *For every $p \in P$ and every ordinal β such that $\operatorname{rank}(p) < \beta < \alpha$, there exists a (unique) $q \in P$ for which $p < q$, $\operatorname{rank}(q) = \beta$.*

(P4) *For every $p, p', q \in P$ such that $p, p' < q$ and $\operatorname{rank}(q)$ is a limit ordinal, there exists $r \in P$ such that $p, p' < r < q$;*

Claim 5. *Every poset P of height $\leq \alpha$ satisfying (P1) is a subposet of some poset Q of height α which satisfy (P2), (P3) and (P4).*

Proof. Let $\overline{P}$ be the poset P with a new greatest element ∞:

$$\overline{P} = P \sqcup \{\infty\}, \quad p < \infty,\ p \in P.$$

Since the dual of $\overline{P}$ is a tree, we know that the interval $\langle p, p') = \{p'' \mid p \leq p'' < p'\}$ has a unique maximal element (for arbitrary $p, p' \in \overline{P}$, $p < p'$). Let

$$Q = P \sqcup \coprod_{p \in \overline{P}} Q_p,$$

where

$$\begin{aligned} Q_p &= \{q_{p,\beta} \mid 0 \leq \beta < \operatorname{rank}(p) \text{ is an ordinal } \}, \quad p \in P, \\ Q_\infty &= \{q_{\infty,\beta} \mid 0 \leq \beta < \alpha \text{ is an ordinal } \}. \end{aligned}$$

The ordering $<_Q$ is given by

$$\begin{array}{llll} p <_Q p' & \text{iff} & p <_P p' & \text{where} \quad p, p' \in P \\ q_{p,\beta} <_Q p' & \text{iff} & p \leq_P p' & \text{where} \quad p \in \overline{P}, p' \in P \\ p <_Q q_{p',\beta} & \text{iff} & p <_P p', & \text{where} \quad p \in P,\ p' \in \overline{P}, \\ & & \beta > \operatorname{rank}_{\overline{P}}(\max_{\overline{P}}\langle p, p')) & \qquad\qquad q_{p',\beta} \in Q_{p'} \\ q_{p,\beta} <_Q q_{p',\beta'} & \text{iff} & p = p',\ \beta < \beta' \text{ or} & \text{where} \quad p, p' \in \overline{P} \\ & & p <_{\overline{P}} p',\ p <_Q q_{p',\beta'} & \qquad\qquad q_{p,\beta} \in Q_p,\ q_{p',\beta'} \in Q_{p'} \end{array}$$

It is straightforward to verify that

- $<_Q$ is a partial ordering on Q.
- The function rank_Q given by $\operatorname{rank}_Q(p) = \operatorname{rank}_P(p)$ for $p \in P$ and $\operatorname{rank}_Q(q_{p,\beta}) = \beta$ for $p \in \overline{P}$, $q_{p,\beta} \in Q_p$ is the rank function of the poset Q. Hence Q is well-founded.
- If $q \in Q$ and β is an ordinal such that $\alpha > \beta > \operatorname{rank}(q)$, then there exists a unique $q' \in Q$ of Q-rank β such that $q <_Q q'$. Thus the properties (P2), (P3) are satisfied.

- Q satisfy (P4). This follows easily from the following fact: If $p, p', r \in P$, $p, p' <_P r$ and β is an ordinal such that $\mathrm{rank}_P(r) > \beta > \mathrm{rank}_P(\max\langle p, r\rangle)$, $\mathrm{rank}_P(\max\langle p', r\rangle)$, then $p, p' <_Q q_{r,\beta}$.

$\square$

From the last claim and Proposition 3.4 we get $\mathbf{Fix}(P) \leq_s \mathbf{Fix}(Q)$.

Now it suffices to prove:

Claim 6. *Let P be a poset of height α satisfying (P2), (P3) and (P4). Then $\mathbf{Fix}(P) \leq_s \mathbf{Fix}(\alpha)$.*

Proof. For $\beta < \alpha$ let $P_\beta = \{p \in P \mid \mathrm{rank}(p) = \beta\}$ and for every $\beta < \gamma < \alpha$ let $s_{\beta,\gamma} : P_\beta \to P_\gamma$ be the mapping satisfying $p < s_{\beta,\gamma}(p)$, $p \in P_\beta$. Since P satisfy (P2) and (P3), $s_{\beta,\gamma}$ is a correctly defined surjective mapping and $p < q$ iff $s_{\mathrm{rank}(p),\mathrm{rank}(q)}(p) = q$.

Let $g : P_0 \to A$ be a mapping, $0 \leq \beta < \alpha$. If g factorizes through $s_{0,\beta}$, i.e. $g = g_\beta s_{0,\beta}$ for a mapping $g_\beta : P_\beta \to A$, we say that g_β exists. Since $s_{0,\beta}$ is surjective, if g_β exists then it is necessarily unique. From (P4) it follows that, for a limit β, g_β exists iff g_γ exists for all $\gamma < \beta$.

For sets A, B and a mapping $f : A \to B$ let

$$\begin{aligned} FA &= \{(g, \beta) \mid g : P_0 \to A,\ \beta \leq \alpha\}/\approx, \\ Ff[g, \beta]_\approx &= [fg, \beta]_\approx. \end{aligned}$$

The equivalence $\approx$ is given by

$$(g, \beta) \approx (h, \gamma) \text{ iff both } g_{\max(\beta,\gamma)}, h_{\max(\beta,\gamma)} \text{ exist and } g = h.$$

F is clearly correctly defined and $\approx$ is an equivalence. We write $[\dots]$ instead of $[\dots]_\approx$.

Given $\mathcal{A} = (A, (\phi_p^{\mathcal{A}})_{p\in P}) \in \mathbf{Fix}(P)$, let

$$\Phi\mathcal{A} = (FA, (\phi_\beta^{\Phi\mathcal{A}})_{\beta<\alpha}),$$

where

$$\phi_0^{\Phi\mathcal{A}}[g, \beta] = [\overline{g}, 1], \quad \overline{g}(p) = \phi_p^{\mathcal{A}}(g(p)), \quad p \in P_0$$

and for $0 < \beta < \alpha$

$$\begin{aligned} \mathrm{Def}(\phi_\beta^{\Phi\mathcal{A}}) &= \{[g, \beta] \mid g_\beta \text{ exists, } (\forall p \in P_\beta)\ g_\beta(p) \in \mathrm{Def}(\phi_p^{\mathcal{A}})\}, \\ \phi_\beta^{\Phi\mathcal{A}}[g, \beta] &= [\overline{g}, \beta^+], \quad \overline{g}(p) = \phi_{s_{0,\beta}(p)}^{\mathcal{A}}(g(p)), \quad p \in P_0. \end{aligned}$$

To verify that $\Phi\mathcal{A}$ is a $\mathbf{Fix}(\alpha)$-object, we must check the following: For every $0 < \beta < \alpha$ we have $\mathrm{Fix}\{\phi_\gamma^{\Phi\mathcal{A}} \mid \gamma < \beta\} = \mathrm{Def}(\phi_\beta^{\Phi\mathcal{A}})$. By induction on β:

First step, $\beta = 1$: The element $[g, \beta] \in FA$ is a fix-point of $\phi_0^{\Phi\mathcal{A}}$ iff $[g, \beta] = [\overline{g}, 1]$, i.e. iff g_1 exists (which means that $g(p) = g(q)$ whenever $s_{0,1}(p) = s_{0,1}(q)$, where $p, q \in P_0$) and $\overline{g}(p) = \phi_p^{\mathcal{A}}(g(p)) = g(p)$ for all $p \in P_0$. This happens precisely when g_1 exists and $g_1(p) \in \mathrm{Fix}\{\phi_q^{\mathcal{A}} \mid q \in s_{0,1}^{-1}(p)\} = \mathrm{Def}(\phi_p^{\mathcal{A}})$ for all $p \in P_1$.

Isolated step is similar to the first step, limit step follows from the observation above: For a limit β, g_β exists iff g_γ exists for all $\gamma < \beta$.

Let f be a mapping $\mathcal{A} = (A, \dots) \to \mathcal{B} = (B, \dots)$. The mapping Ff preserves ϕ_0, iff for all $[g, \beta] \in FA$

$$\phi_0^{\Phi\mathcal{B}}(Ff[g, \beta]) = \phi_0^{\Phi\mathcal{B}}[fg, \beta] = [\overline{fg}, 1] =$$

$$= Ff(\phi_0^{\Phi\mathcal{A}}[g, \beta]) = Ff[\overline{g}, 1] = [f\overline{g}, 1].$$

For all $p \in P_0$

$$\overline{fg}(p) = \phi_p^{\mathcal{B}}(f(g(p)))$$

and

$$f(\overline{g}(p)) = f(\phi_p^{\mathcal{A}}(g(p)).$$

This means that Ff preserves ϕ_0, iff f preserves ϕ_p for all $p \in P_0$. Similarly, Ff preserves ϕ_β, iff f preserves $\phi_{s_{0,\beta}(p)}$ for all $p \in P_0$, i.e. iff f preserves ϕ_q for all $q \in P_\beta$. We can now see that Φ is a functor and (Φ, F) is an s-embedding. $\square$

The proof of Theorem 3.5 is concluded. $\square$

Open problem. *Find all baskets of essentially algebraic categories.*

Remark 3.6. *As mentioned, every mono-sorted essentially algebraic category of height 1 (i.e. a variety) belongs to one of the baskets $\mathbb{T}, \mathbb{P}, \mathbb{A}$. So that the first step could be to generalize this result to many-sorted signatures and then to look at (mono-sorted) essentially algebraic categories of height 2.*

A natural example of an essentially algebraic category of height 2 is the category **Cat** of all small categories and functors (the forgetful functor **Cat** $\to$ **Set** assigns the set of all morphisms to a category). Indeed, **Cat** can be described as (i.e., is concretely equivalent to) the category of models of $\Gamma = (\{\circ, d, c\}, \mathrm{level}, E, \mathrm{Def})$, where

$$\mathrm{level}(d) = \mathrm{level}(c) = 0, \quad \mathrm{level}(\circ) = 1$$

are the operations of domain, codomain and comoposition, respectively.

$$\begin{aligned}
E &= \{dd(x) = cd(x) = d(x),\ cc(x) = dc(x) = c(x), \\
&\quad d(x \circ y) = d(y),\ c(x \circ y) = c(x), \\
&\quad c(x) \circ x = x = x \circ d(x), \\
&\quad x \circ (y \circ z) = (x \circ y) \circ z\}, \\
\mathrm{Def}(\circ) &= \{d(x_0) = c(x_1)\}.
\end{aligned}$$

This is just an object free definition of a category.

Proposition 3.7. *The category* **Cat** *is a member of* $\mathbb{E}_2$.

Proof. Since **Cat** $\leq_s$ **Fix**(2) follows from 3.5, it suffices to find an s-embedding (Φ, F) of **Fix**(2) to **Cat**.

The functor $F : \mathbf{Set} \to \mathbf{Set}$ is defined by

$$\begin{aligned} FA &= \{m_{a,b},\ \mathrm{id}_{a,b,i} \,|\, a,b \in A,\ i \in 2\}/\approx, \\ Ff[m_{a,b}] &= [m_{f(a),f(b)}], \\ Ff[\mathrm{id}_{a,b,i}] &= [\mathrm{id}_{f(a),f(b),i}], \end{aligned}$$

where A is a set, $f : A \to B$ is a mapping, the equivalence $\approx$ is generated by $\mathrm{id}_{a,a,0} \approx \mathrm{id}_{a,a,1}$ for all $a \in A$, and $[\dots]$ means $[\dots]_{\approx}$.

For an algebra $\mathcal{A} = (A, (\phi_i^{\mathcal{A}})_{i\in 2}) \in \mathbf{Fix}(2)$ we put

$$\Phi\mathcal{A} = (FA, d^{\Phi\mathcal{A}}, c^{\Phi\mathcal{A}}, \circ^{\Phi\mathcal{A}}),$$

where

$$d[m_{a,b}] = [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),0}], \quad d[\mathrm{id}_{a,b,i}] = [\mathrm{id}_{a,b,i}],$$
$$c[m_{a,b}] = [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),1}], \quad c[\mathrm{id}_{a,b,i}] = [\mathrm{id}_{a,b,i}]$$

for every $a, b \in A$, $i \in 2$.

The operation $x \circ y$ is to be defined iff $d(x) = c(y)$. The interesting case is $x = m_{a,b}$, $y = m_{c,d}$. In this case $d(x) = c(y)$ iff $a = c$ and $\phi_0^{\mathcal{A}}(a) = a$. Let

$$\begin{aligned} [\mathrm{id}_{a,b,i}] \circ [\mathrm{id}_{a,b,i}] &= [\mathrm{id}_{a,b,i}], \\ [m_{a,b}] \circ [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),0}] &= [m_{a,b}], \\ [\mathrm{id}_{a,\phi_0^{\mathcal{A}}(a),1}] \circ [m_{a,b}] &= [m_{a,b}], \\ [m_{a,b}] \circ [m_{a,c}] &= [m_{a,\phi_1^{\mathcal{A}}(a)}], \quad \text{if } \phi_0^{\mathcal{A}}(a) = a, \end{aligned}$$

where $a, b \in A$, $i \in 2$.

It is straightforward to verify that the equations from E are satisfied and that $Ff : \Phi\mathcal{A} \to \Phi\mathcal{B}$ is a **Fix**(2)-morphism whenever $f : \mathcal{A} \to \mathcal{B}$ is a **Cat**-morphism. Hence Φ is a functor.

To prove that (Φ, F) is an s-embedding, let $\mathcal{A} = (A, (\phi_i^{\mathcal{A}})_{i\in 2}), \mathcal{B} = (B, (\phi_i^{\mathcal{B}})_{i\in 2}) \in$ **Fix**(2) and $f : FA \to FB$ be a **Cat**-homomorphism $\Phi\mathcal{A} \to \Phi\mathcal{B}$. For every $a \in A$ we have

$$[\mathrm{id}_{f(a),f(\phi_0^{\mathcal{A}}(a)),0}] = Ff(d[m_{a,a}]) = d(Ff[m_{a,a}]) = [\mathrm{id}_{f(a),\phi_0^{\mathcal{B}}(f(a)),0}],$$

hence $f(\phi_0^{\mathcal{A}}(a)) = \phi_0^{\mathcal{B}}(f(a))$.

For every $a \in A$ such that $\phi_0^{\mathcal{A}}(a) = a$ we have

$$[m_{f(a),f(\phi_1^{\mathcal{A}}(a))}] = Ff([m_{a,a}] \circ [m_{a,a}]) = Ff[m_{a,a}] \circ Ff[m_{a,a}] = [m_{f(a),\phi_1^{\mathcal{B}}(f(a))}],$$

hence $f(\phi_1^{\mathcal{A}}(a)) = \phi_1^{\mathcal{B}}(f(a))$. Therefore $f : \mathcal{A} \to \mathcal{B}$ is a **Cat**-morphism and the proof is concluded. $\square$

4 Closure rules

The following formalization of the "properties which are inherited to slices" was suggested by J. Sichler in an unpublished note.

Definition 4.1. *A triple* $\mathfrak{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ *is called a* closure rule*, if* $\mathbf{a}_i$ *(*$i = 0, 1, 2$*) are small categories with the same set of objects,* $\mathbf{a}_0$ *is a subcategory of* $\mathbf{a}_1$ *and* $\mathbf{a}_1$ *is a subcategory of* $\mathbf{a}_2$*.*

Definition 4.2. *Let* $\mathfrak{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ *be a closure rule and* $i_0 : \mathbf{a}_0 \to \mathbf{a}_1$ *and* $i_1 : \mathbf{a}_1 \to \mathbf{a}_2$ *denote the inclusion functors. We say, that a faithful functor* $U : \mathbf{K} \to \mathbf{H}$ obeys $\mathfrak{a}$*, if for every pair of functors* $G_0 : \mathbf{a}_0 \to \mathbf{K}$*,* $G_2 : \mathbf{a}_2 \to \mathbf{H}$ *such that* $G_2 i_1 i_0 = U G_0$*, there exists a functor* $G_1 : \mathbf{a}_1 \to \mathbf{K}$ *such that* $G_1 i_0 = G_0$ *and* $U G_1 = G_2 i_1$*. Notation:* $U \vDash \mathfrak{a}$*.*

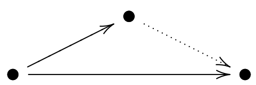
$$\begin{tikzcd} \mathbf{a}_0 \arrow[r, hook, "i_0"] \arrow[d, "G_0"'] & \mathbf{a}_1 \arrow[r, hook, "i_1"] \arrow[dl, dashed, "G_1"] & \mathbf{a}_2 \arrow[d, "G_2"] \\ \mathbf{K} \arrow[rr, "U"'] & & \mathbf{H} \end{tikzcd} \tag{4}$$

A closure rule $\mathfrak{a}$ *is said to be* trivial *provided that* $U \vDash \mathfrak{a}$ *for every faithful functor* U*.*

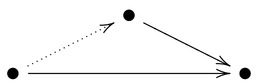
Let $\mathfrak{a}, \mathfrak{b}$ *be closure rules. We say that* $\mathfrak{b}$ *is a (semantic) consequence of* $\mathfrak{a}$*, if* $U \vDash \mathfrak{a}$ *implies* $U \vDash \mathfrak{b}$ *for every faithful functor* U*. Notation:* $\mathfrak{a} \vDash \mathfrak{b}$*.*

All closure rules used in this chapter have the property that the category $\mathbf{a}_2$ is a quasiordered set, i.e. there is at most one arrow between any two objects of $\mathbf{a}_2$.

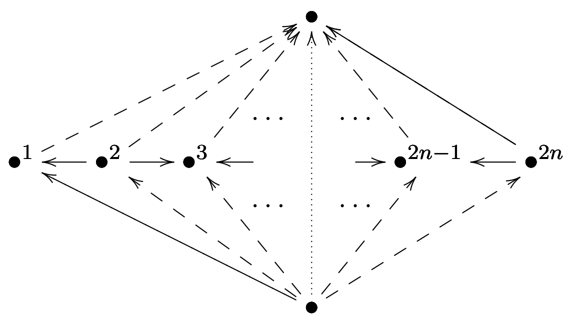
Examples of closure rules:



(p)



(p^{op})



(zz_n^1), $n \geq 1$

The nodes in the picture denote elements of the common set of objects of the closure rule. Arrows are $\mathbf{a}_2$-morphisms (identities are not drawn), solid arrows are $\mathbf{a}_0$-morphisms and dotted arrows are $\mathbf{a}_1$-morphisms.

Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category. The definition of $U \vDash \mathfrak{a}$ says the following: Whenever we have objects of $\mathbf{K}$ and $\mathbf{H}$-morphisms between the respective underlying $\mathbf{H}$-objects, as in the picture, such that the diagram is commutative and solid arrows are $\mathbf{K}$-morphisms, then the dotted arrows are $\mathbf{K}$-morphisms as well.

Remark 4.3. *1. It can be readily seen that a faithful functor $U : \mathbf{K} \to \mathbf{H}$ obeys each of closure rules $\mathfrak{a}^i = (\mathbf{a}_0^i, \mathbf{a}_1^i, \mathbf{a}_2^i), i \in I$ iff U obeys its coproduct*

$$\coprod_{i\in I} \mathfrak{a}^i = (\coprod_{i\in I} \mathbf{a}_0^i, \coprod_{i\in I} \mathbf{a}_1^i, \coprod_{i\in I} \mathbf{a}_2^i).$$

By (zz^1) is meant the coproduct of the closure rules (zz_n^1).

2. *A faithful functor $U : \mathbf{K} \to \mathbf{H}$ obeys a closure rule $\mathfrak{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ iff U^{op} obeys the dual closure rule $\mathfrak{a}^{op} = (\mathbf{a}_0^{op}, \mathbf{a}_1^{op}, \mathbf{a}_2^{op})$.*

3. *It can be easily checked that the (forgetful functor of the) category of algebras with one nullary operation obeys (p), and the category of algebras with one unary operation obeys (zz^1) (this fact is a special case of Proposition 4.8). Obviously $(p) \models (zz^1)$, $(p^{op}) \models (zz^1)$ and $(zz_{n+1}^1) \models (zz_n^1)$.*

If a faithful functor U obeys a closure rule a, then so does every slice of U:

Proposition 4.4. *Let $U : \mathbf{K} \to \mathbf{H}$, $U' : \mathbf{K}' \to \mathbf{H}'$ be concrete categories, $\mathfrak{a}$ be a closure rule. If $U \leq_s U'$ and $U' \models \mathfrak{a}$, then $U \models \mathfrak{a}$.*

Proof. Let be G_0, G_2 be functors such that diagram (4) is commutative, (Φ, F) be an s-embedding of U to U'. Let $A, B \in \mathrm{Obj}(\mathbf{a}_0)$ and $f \in \mathbf{a}_1(A, B)$ (dotted arrow). Since U' obeys a, $FG_2 f$ is a $\mathbf{K}$-morphism from $\Phi G_0 A$ to $\Phi G_0 B$, hence $G_2 f$ is a $\mathbf{K}$-morphism from $G_0 A$ to $G_0 B$, because $U \leq_s U'$. □

Remark 4.5. *1. An easy consequence of Proposition 4.4 is that s-equivalent faithful functors obey the same closure rules. Therefore the formulation "the basket ... obeys ..." makes sense. From Remark 4.3 it follows that $\mathbb{P} \models (p)$, $\mathbb{P}^{op} \models (p)^{op}$, $\mathbb{A} \models (zz^1)$.*

2. *Proposition 4.4 enables us to show that certain s-inequality $U \leq_s U'$ doesn't hold: It suffices to find a closure rule which is obeyed by U' but it is not obeyed by U.*

3. *The notion of a closure rule could be generalized and Proposition 4.4 would remain true. For instance, consider a concrete category $U : \mathbf{K} \to \mathbf{H}$. The condition "the composition of two $\mathbf{H}$-morphism which are not $\mathbf{K}$-morphisms is not a $\mathbf{K}$-morphism" inherits also to slices of U. However we have no application of such generalizations.*

Now we are going to define inductively closure rules (zz^α) (for every ordinal α) which are obeyed by essentially algebraic categories of height α.

Definition 4.6. *Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category, A, B be $\mathbf{K}$-objects, $f \in \mathbf{H}(A, B)$.*

- *f is called (zz^0)-morphism.*

- *Let α be an ordinal; f is said to be a (zz^{α^+})-morphism, if there exists a commutative diagram*

$$\begin{tikzcd}
 & & & B & & \\
C_1 \arrow[urrr, dashed, Rightarrow, "h_1"] & C_2 \arrow[l, "l_1"'] \arrow[r, "l_2"] \arrow[urr, dashed, Rightarrow, "h_2"] & C_3 \arrow[ur, dashed, Rightarrow, "h_3"] & \cdots \arrow[l] \arrow[r] & C_{2n-1} \arrow[ul, dashed, Rightarrow] & C_{2n} \arrow[l, "l_{2n-1}"'] \arrow[ull, "h_{2n}"'] \\
 & & & A \arrow[uu, dotted, "f"'] \arrow[ulll, "g_1"] \arrow[ull, dashed, Rightarrow, "g_2"] \arrow[ul, dashed, Rightarrow, "g_3"] \arrow[ur, dashed, Rightarrow, "g_{2n-1}"'] \arrow[urr, dashed, Rightarrow, "g_{2n}"'] & &
\end{tikzcd}$$

(zz^α)

where points are **K**-*objects, arrows are* **H**-*morphisms, solid arrows are* **K**-*morphisms and dashed double arrows are (zz^α)-morphisms.*

- *Let α be a limit ordinal; f is said to be a (zz^α)-morphism, if it is a (zz^β)-morphism for every $\beta < \alpha$.*

We say that U obeys (zz^α), if every (zz^α)-morphism is a **K**-*morphism.*

Remark 4.7. *1. For any α, every* **K**-*morphism is a (zz^α)-morphism.*

2. *Note that (zz^α) can be written in the form of a closure rule. The rule (zz^1) coincides with the earlier defined version. If $\alpha \leq \beta$, then $(zz^\alpha) \models (zz^\beta)$.*

3. *It can be easily verified that the composition of a (zz^α)-morphism and a (zz^β)-morphism is a $(zz^{\min(\alpha,\beta)})$-morphism. In particular (zz^α)-morphisms are closed under composition.*

Proposition 4.8. *Let α be an ordinal. Let* **K** *be an essential algebraic category of height α with any of the two natural forgetful functors. Then $\mathbf{K} \models (zz^\alpha)$. In particular $\mathbb{E}_\alpha \models (zz^\alpha)$ and dually $\mathbb{E}_\alpha^{op} \models (zz^\alpha)^{op}$.*

Proof. Since both forgetful functors of **K** are slices of $\mathbf{Fix}(\alpha)$ (Theorem 3.5), it suffices to prove $\mathbf{Fix}(\alpha) \models (zz^\alpha)$. We proof by induction on $\beta \leq \alpha$ that every (zz^β)-morphism $f : \mathcal{A} = (A, (\phi_\gamma^{\mathcal{A}})_{\gamma<\alpha}) \to \mathcal{B} = (B, (\phi_\gamma^{\mathcal{B}})_{\gamma<\alpha})$ is a $\mathbf{Fix}(\beta)$-morphism $(A, (\phi_\gamma^{\mathcal{A}})_{\gamma<\beta}) \to (B, (\phi_\gamma^{\mathcal{B}})_{\gamma<\beta})$.

For $\beta = 0$ the statement is empty, for limit β it is clear. Now we assume that the statement holds for β and we will prove it for β^+. Since f is a (zz^{β^+})-morphism, we can find $\mathbf{Fix}(\beta^+)$-objects $\mathcal{C}_i$ and mappings g_i, h_i, l_i as in the diagram in Definition 4.6.

From the induction hypothesis we know that f preserves the operations ϕ_γ for all $\gamma < \beta$. Let $a \in A$ be in the definition domain of $\phi^{\mathcal{A}}_\beta$. We have

$$\begin{aligned}
f\phi^{\mathcal{A}}_\beta(a) &= h_1 g_1 \phi^{\mathcal{A}}_\beta(a) = \quad [g_1 \text{ is a } \mathbf{Fix}(\beta)\text{-morphism}]\\
&= h_1 \phi^{\mathcal{C}_1}_\beta g_1(a) = \\
&= h_1 \phi^{\mathcal{C}_1}_\beta l_1 g_2(a) = \ [g_2 \text{ is a } (zz^\beta)\text{-morphism and } l_1 \text{ a } \mathbf{Fix}(\beta)\text{-morphism}]\\
&= h_1 l_1 \phi^{\mathcal{C}_2}_\beta g_2(a) = \\
&= h_3 l_2 \phi^{\mathcal{C}_2}_\beta g_2(a) = \\
&= h_3 \phi^{\mathcal{C}_3}_\beta l_2 g_2(a) = h_3 \phi^{\mathcal{C}_3}_\beta l_3 g_4(a) = \\
&\dots\\
&= h_{2n} \phi^{\mathcal{C}_{2n}}_\beta g_{2n}(a) = \phi^{\mathcal{B}}_\beta h_{2n} g_{2n}(a) = \phi^{\mathcal{B}}_\beta f(a).
\end{aligned}$$

$\square$

Proposition 4.9. *Let α be an ordinal. Then $\mathbb{E}_\alpha \neq \mathbb{E}_{\alpha^+}$.*

Proof. According to Propositions 4.4, 4.8 it suffices to construct a mapping between algebras in $\mathbf{Fix}(\alpha^+)$ which is not a $\mathbf{Fix}(\alpha^+)$-homomorphisms, but it is a (zz^α)-morphism.

Let $\mathcal{A} = (1, (\phi^{\mathcal{A}}_\gamma)_{\gamma \le \alpha})$ be the unique $\mathbf{Fix}(\alpha^+)$-algebra on the set 1. For $0 \le \beta \le \alpha$, let $\mathcal{B}_\beta = (2, (\phi^{\mathcal{B}_\beta}_\gamma)_{\gamma \le \alpha})$, where

$$\phi^{\mathcal{B}_\beta}_\gamma(i) = \begin{cases} i & \text{if } \gamma < \beta,\ i \in 2\\ 1-i & \text{if } \gamma = \beta,\ i \in 2\\ \text{undefined} & \text{otherwise}\end{cases}.$$

In what follows, c_0 denotes the constant mapping with the value 0 (domains and codomains vary). The mapping $c_0 : \mathcal{A} \to \mathcal{B}_{\alpha^+}$ is not a homomorphism, because it doesn't preserve the operation ϕ_α. We will show by induction on $\beta \le \alpha$ that $c_0 : \mathcal{A} \to \mathcal{B}_\beta$ is a (zz^β)-morphism.

First step: Every mapping is a (zz^0)-morphism.

Isolated step: Suppose that $c_0 : \mathcal{A} \to \mathcal{B}_\beta$ is a (zz^β)-morphism. The following diagram shows that $c_0 : \mathcal{A} \to \mathcal{B}_{\beta^+}$ is a (zz^{β^+})-morphism

$$\begin{tikzcd}
 & \mathcal{B}_{\beta+} & \\
\mathcal{A} \arrow[ur, dashed, "c_0"] & & \mathcal{B}_\beta \arrow[ul, "c_0"'] \arrow[ll, "c_0"'] \\
 & \mathcal{A} \arrow[ul, "c_0"] \arrow[ur, Rightarrow, dashed, "c_0"'] &
\end{tikzcd}$$

Limit step: Suppose that $c_0 : \mathcal{A} \to \mathcal{B}_\delta$ is a (zz^δ)-morphism for every $\delta < \beta$, where β is a limit ordinal. Since $c_0 : \mathcal{B}_\delta \to \mathcal{B}_\beta$ is a $\mathbf{Fix}(\alpha^+)$-morphism, $c_0 : \mathcal{A} \to \mathcal{B}_\beta$ is a (zz^δ)-morphism following Remark 4.7.3. $\square$

Proposition 4.10. *Let α be an ordinal. Then $\mathbb{E}_2 \not\le_s \mathbb{E}^{op}_\alpha$ (and dually $\mathbb{E}^{op}_2 \not\le_s \mathbb{E}_\alpha$).*

Proof. According to 4.4, 4.8 it suffices to construct a mapping between algebras in $\mathbf{Fix}(2)$ which is not a $\mathbf{Fix}(2)$-homomorphism, but it is a $(zz^{\alpha})^{op}$-morphism.

Let $\mathcal{A} = (1, (\phi_i^{\mathcal{A}})_{i\in 2})$ be the $\mathbf{Fix}(2)$-algebra on the set 1. Let $\mathcal{B} = (2, (\phi_i^{\mathcal{B}})_{i\in 2})$ and $\mathcal{C} = (2, (\phi_i^{\mathcal{C}})_{i\in 2})$, where

$$\phi_0^{\mathcal{B}}(i) = 1 - i, \quad \phi_0^{\mathcal{C}}(i) = i, \quad \phi_1^{\mathcal{C}}(i) = 1 - i, \quad i \in 2.$$

Let c_0 be the constant mapping with the value 0 (domains and codomains vary again). The mapping $c_0 : \mathcal{A} \to \mathcal{C}$ is not a $\mathbf{Fix}(2)$-homomorphisms, since it doesn't preserve ϕ_1. By induction on β we prove that it is a $(zz^{\beta})^{op}$-morphism $\mathcal{A} \to \mathcal{C}$.

First step: Every mapping is a $(zz^0)^{op}$-morphism.

Isolated step: Suppose that $c_0 : \mathcal{A} \to \mathcal{C}$ is a $(zz^{\beta})^{op}$-morphism. The following diagram shows that it is a $(zz^{\beta+})^{op}$-morphism.

$$\begin{tikzcd}
 & \mathcal{C} & \\
\mathcal{A} \arrow[ur, Rightarrow, dashed, "c_0"] & & \mathcal{B} \arrow[ul, "c_0"'] \arrow[ll, "c_0"'] \\
 & \mathcal{A} \arrow[ul, "c_0"] \arrow[ur, dashed, "c_0"'] &
\end{tikzcd}$$

Limit step: Suppose that $c_0 : \mathcal{A} \to \mathcal{C}$ is a $(zz^{\gamma})^{op}$-morphism for all $\gamma < \beta$. Then it is a $(zz^{\beta})^{op}$-morphism. $\square$

The reasons for $(p) \vDash (zz_n^1), (zz_{n-1}^1) \vDash (zz_n^1)$ are syntactic – we can see it from the pictures of these closure rules. Theorem 4.13 bellow says that this is not by chance.

Definition 4.11. *Let $\mathfrak{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, $\mathfrak{b} = (\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$ be closure rules. Let $\mathbf{b}$ be the smallest subcategory of $\mathbf{b}_2$, such that $\mathbf{b}_0 \subset \mathbf{b}$ and the functor $\subseteq : \mathbf{b} \to \mathbf{b}_2$ obeys $\mathfrak{a}$. We say, that $\mathfrak{b}$ is a syntactic consequence of $\mathfrak{a}$, if $\mathbf{b}_1 \subset \mathbf{b}$. Notation: $\mathfrak{a} \vdash \mathfrak{b}$.*

Remark 4.12. *This smallest subcategory exists, it can be formed as the intersection of those satisfying the condition. This category can also be constructed by transfinite induction: We start with $\mathbf{b} = \mathbf{b}_0$. Then we repeat the following steps unless no new element can be added to $\mathbf{b}$ (at the limit step, we take the union, of course).*

1. *Take functors $H_0 : \mathbf{a}_0 \to \mathbf{b}$, $H_2 : \mathbf{a}_2 \to \mathbf{b}_2$ such that $jH_0 = H_2 i_1 i_0$, where j is the inclusion $j : \mathbf{b} \to \mathbf{b_2}$. Add all morphisms $H_2 f$ where f is a morphism of $\mathbf{a}_1$.*

2. *Make a closure of $\mathbf{b}$ with respect to composition.*

It's clear that this leads to the category $\mathbf{b}$ from the definition.

Theorem 4.13. *Let $\mathfrak{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, $\mathfrak{b} = (\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$ be closure rules. Then $\mathfrak{a} \vDash \mathfrak{b}$ if and only if $\mathfrak{a} \vdash \mathfrak{b}$.*

Proof. "$\Rightarrow$". Suppose $\mathtt{a} \not\vdash \mathtt{b}$. Let $\mathbf{b}$ be the smallest subcategory from Definition 4.11. The concrete category $\subseteq : \mathbf{b} \to \mathbf{b}_2$ obeys $\mathtt{a}$ (according to the definition) and doesn't obey $\mathtt{b}$: Put $G_0 : \mathbf{b}_0 \to \mathbf{b}$ to be the inclusion and $G_2 : \mathbf{b}_2 \to \mathbf{b}_2$ to be the identity. Now G_1 from Definition 4.2 doesn't exist, since $\mathbf{b}_1 \not\subseteq \mathbf{b}$. Hence $\mathtt{a} \not\models \mathtt{b}$.

"$\Leftarrow$". Assume $\mathtt{a} \vdash \mathtt{b}$ and let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category which obeys $\mathtt{a}$. Striving for a contradiction, assume that U doesn't obey $\mathtt{b}$, i.e. there exist functors $G_0 : \mathbf{b}_0 \to \mathbf{K}$, $G_2 : \mathbf{b}_2 \to \mathbf{H}$ such that $G_2 i_1 i_0 = U G_0$ and there is no functor $G_1 : \mathbf{b}_1 \to \mathbf{K}$ completing the commutative diagram (4). Let $\mathbf{b}'_0$ be the maximal subcategory of $\mathbf{b}_2$ such that there exists a functor $G'_0 : \mathbf{b}'_0 \to \mathbf{K}$ for which the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{b}_0 \arrow[r, hook] \arrow[d, "G_0"'] & \mathbf{b}'_0 \arrow[r, hook] \arrow[dl, "G'_0"'] & \mathbf{b}_2 \arrow[d, "G_2"] \\
\mathbf{K} \arrow[rr, "U"] & & \mathbf{H}
\end{tikzcd}$$

(The category $\mathbf{b}'_0$ thus consists precisely of those $\mathbf{b}_2$-morphisms $g : c \to d$ for which $G_2 g : G_0 c \to G_0 d$ is a $\mathbf{K}$-morphism.)

Since $\mathbf{b}_1 \not\subseteq \mathbf{b}'_0$ and $\mathtt{a} \vdash \mathtt{b}$, there exist functors $H_0 : \mathbf{a}_0 \to \mathbf{b}'_0$, $H_2 : \mathbf{a}_2 \to \mathbf{b}_2$ such that there is no $H_1 : \mathbf{a}_1 \to \mathbf{b}'_0$ for which the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{a}_0 \arrow[r, hook] \arrow[d, "H_0"'] & \mathbf{a}_1 \arrow[r, hook] \arrow[dl, "H_1"'] & \mathbf{a}_2 \arrow[d, "H_2"] \\
\mathbf{b}'_0 \arrow[rr, hook] & & \mathbf{b}_2
\end{tikzcd}$$

In other words, there are objects $c, d \in \mathrm{Obj}(\mathbf{a}_0)$ and $f \in \mathbf{a}_1(c, d)$ such that $H_2 f : H_0 c \to H_0 d$ isn't a $\mathbf{b}'_0$-morphism. But $G_2 H_2 f : G'_0 H_0 c \to G'_0 H_0 d$ is a $\mathbf{K}$-morphism, because $U \models \mathtt{a}$. This is a contradiction with the maximality of $\mathbf{b}'_0$. $\square$

An easy consequence of the proof is:

Corollary 4.14. *Let $\mathtt{a}, \mathtt{b}$ be closure rules. Then $\mathtt{a} \models \mathtt{b}$, iff $U \models \mathtt{a}$ implies $U \models \mathtt{b}$ for all faithful functors U between small categories.*

Corollary 4.15. *A closure rule $\mathtt{a}$ is trivial iff $\mathbf{a}_0 = \mathbf{a}_1$.*

Proof. If $\mathbf{a}_0 = \mathbf{a}_1$, then $\mathtt{a}$ is clearly trivial.

If $\mathbf{a}_0 \neq \mathbf{a}_1$, then the concrete category $\subseteq : \mathbf{a}_0 \to \mathbf{a}_2$ doesn't obey $\mathtt{a}$. $\square$

5 Universality with respect to closure rules

First recall relevant known results:

Theorem 5.1. *Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category. Then*

1. *$U \leq_s \mathbb{R}$ iff U is SSF (see [44]).*

2. $U \leq_s \mathbb{P}$ iff U is SSF and $U \vDash (p)$ (see [44]).

2^{op}. $U \leq_s \mathbb{P}^{op}$ iff U is SSF and $U \vDash (p^{op})$.

3. Let both $\mathbf{K}, \mathbf{H}$ be small; or $\mathbf{H} = \mathbf{Set}$. $U \leq_s \mathbb{A}$ iff U is SSF and $U \vDash (zz^1)$. (see [39] for the small case, [36] for the set case)

The following theorem characterizes small slices of $\mathbb{E}_\alpha$.

Theorem 5.2. *Let $U : \mathbf{k} \to \mathbf{h}$ be a concrete category, where $\mathbf{k}$, $\mathbf{h}$ are small. Then $U \leq_s \mathbb{E}_\alpha$ iff $U \vDash (zz^\alpha)$.*

Proof. If $U \leq_s \mathbb{E}_\alpha$ then $U \vDash (zz^\alpha)$ follows from Propositions 4.4, 4.8.

Suppose that $U \vDash (zz^\alpha)$. We will find an s-embedding (Φ, F) from U to $\mathbf{Fix}(P)$, where the poset P is the ordinal α plus a second minimal element $\overline{0}$:

$$P = \alpha \sqcup \{\overline{0}\}, \quad \overline{0} < \beta \text{ iff } 0 < \beta.$$

This is enough due to Theorem 3.5 (Claim 6 suffices).

First we define, for every $H \in \mathrm{Obj}(\mathbf{h})$ and $0 \leq \beta < \alpha$, a set $G^\beta H$ and an equivalence $\approx_\beta$ on $G^\beta H$:

$$\begin{aligned} G^\beta H \quad = \quad & \{(A, g, B, h, \beta) \mid A, B \in \mathrm{Obj}(\mathbf{k}),\ g \in \mathbf{h}(A, B) \text{ is a } (zz^\beta)\text{-morphism}, \\ & h \in \mathbf{h}(B, H)\}. \end{aligned}$$

The equivalence $\approx_\beta$ is given by

$$(A, g, B, h, \beta) \approx_\beta (A, g', B', h', \beta)$$

iff there exists a commutative diagram

$$\begin{tikzcd}
 & & & H & & \\
B \arrow[urrr, dashed, "h"] & B_2 \arrow[l] \arrow[r] \arrow[urr, dashed] & B_3 \arrow[ur, dashed] & \cdots \arrow[l] \arrow[r] & B_{2n-1} \arrow[ul, dashed] & B' \arrow[l] \arrow[ull, dashed, "h'"'] , \\
 & & & A \arrow[ulll, Rightarrow, dashed, "g"] \arrow[ull, Rightarrow, dashed] \arrow[ul, Rightarrow, dashed] \arrow[ur, Rightarrow, dashed] \arrow[urr, Rightarrow, dashed, "g'"'] & &
\end{tikzcd}$$

where B_i are $\mathbf{k}$ objects, arrows are $\mathbf{h}$-morphisms (between the respective objects), solid arrows are $\mathbf{k}$-morphisms and dashed double arrows are (zz^β)-morphisms.

The functor F is defined for $\mathbf{h}$-objects H, H' and $f \in \mathbf{h}(H, H')$ as follows:

$$\begin{aligned} FH &= \{[A, g, B, h, \beta]_{\approx_\beta} \mid \beta < \alpha,\ (A, g, B, h, \beta) \in G^\beta H, \}/\approx, \\ Ff[A, g, B, h, \beta]_\approx &= [A, g, B, fh, \beta]_\approx, \end{aligned}$$

where the equivalence $\approx$ is given by

$$[A, g, B, h, \beta]_{\approx_\beta} \approx [A, g', B', h', \beta']_{\approx_{\beta'}}$$

iff

$$(A, g, B, h, \beta) \approx_\beta (A, \mathrm{id}_{UA}, A, hg, \beta) \text{ and } (A, g', B', h', \beta') \approx_{\beta'} (A, \mathrm{id}_{UA}, A, hg, \beta').$$

In what follows we omit the subscript $\approx$.

We will use the following abbreviation:

$$(A, g) = [A, \mathrm{id}_{UA}, A, g, \beta],$$

where β is arbitrary (the right hand side does not depend on β). Note that $Ff(A, g) = (A, fg)$.

Observe that

- $Ff[A, g, B, h, \beta]$ doesn't depend on the choice of the representative and that F preserves the composition and identities. Thus F is a correctly defined functor $F : \mathbf{h} \to \mathbf{Set}$.
- Let A, K be $\mathbf{k}$-objects, $g \in \mathbf{h}(A, K)$ be a (zz^β)-morphism. Then $(A, g) = [A, g, K, \mathrm{id}_{UK}, \beta]$ iff g is a (zz^{β^+})-morphism.

Next we define the functor Φ. Let $K \in \mathrm{Obj}(\mathbf{k})$.

$$\Phi K = (FUK, (\phi_p^{\Phi K})_{p \in P}),$$

where the total operations $\phi_{\overline{0}}^{\Phi K}$, $\phi_0^{\Phi K}$ are given by

$$\begin{aligned} \phi_{\overline{0}}^{\Phi K}[A, f, B, g, \beta] &= (A, gf), \\ \phi_0^{\Phi K}[A, f, B, g, \beta] &= [A, gf, K, \mathrm{id}_{UK}, 0]. \end{aligned}$$

Let $0 < \beta < \alpha$. The operation $\phi_\beta^{\Phi K}$ is defined by

$$\begin{aligned} \mathrm{Def}(\phi_\beta^{\Phi K}) &= \{(A, g) \mid g : A \to K \text{ is a } (zz^\beta)\text{-morphism } \}, \\ \phi_\beta^{\Phi K}(A, g) &= [A, g, K, \mathrm{id}_{UK}, \beta]. \end{aligned}$$

To verify that ΦK is a $\mathbf{Fix}(P)$-object, we have to check the following:

Claim 1. *Let* $0 < \beta < \alpha$. *Then* $\mathrm{Fix}\{\phi_p^{\Phi K} \mid p < \beta\} = \mathrm{Def}(\phi_\beta^{\Phi K})$.

Proof. We proceed by induction on β.

First step: An element $x \in FUK$ is a fix-point of $\phi_{\overline{0}}^{\Phi K}$ iff $x = (A, g)$ for some $g \in \mathbf{H}(A, K)$. An element $(A, g) \in FUK$ is a fix-point of $\phi_0^{\Phi K}$ iff

$$(A, g) = [A, \mathrm{id}_{UA}, A, g, 0] = \phi_0^{\Phi K}[A, \mathrm{id}_{UA}, A, g, 0] = [A, g, K, \mathrm{id}_{UK}, 0].$$

This happens precisely when $g : A \to K$ is a (zz^1)-morphism.

Isolated step: Assume that $\mathrm{Fix}\{\phi_p^{\Phi K} \mid p < \beta\} = \mathrm{Def}(\phi_\beta^{\Phi K})$. The element (A, g), where $g \in \mathbf{H}(A, K)$ is a (zz^β)-morphism, is a fix-point of $\phi_\beta^{\Phi K}$ iff

$$(A, g) = \phi_\beta^{\Phi K}(A, g) = [A, g, K, \mathrm{id}_{UK}, \beta].$$

This happens precisely when g is a (zz^{β^+})-morphism $A \to K$ (see the observation above).

The limit step is obvious. $\square$

It is clear that Φ preserves the composition and identities. Therefore, to prove that Φ is a functor, we have to verify the following:

Claim 2. *Let $f : K \to L$ be a* **k***-morphism. Then $Ff : \Phi K \to \Phi L$ is a* **Fix**(P)*-morphism.*

Proof. Ff preserves $\phi_{\overline{0}}$:

$$Ff(\phi_{\overline{0}}^{\Phi K}[A, g, B, h, \beta]) = Ff(A, hg) = (A, fhg),$$

$$\phi_{\overline{0}}^{\Phi L}(Ff[A, g, B, h, \beta]) = \phi_{\overline{0}}^{\Phi L}[A, g, B, fh, \beta] = (A, fhg).$$

Ff preserves ϕ_0:

$$Ff(\phi_0^{\Phi K}[A, g, B, h, \beta]) = Ff[A, hg, K, \mathrm{id}_{UK}, 0] = [A, hg, K, f, 0],$$

$$\phi_0^{\Phi L}(Ff[A, g, B, h, \beta]) = \phi_0^{\Phi L}[A, g, B, fh, \beta] = [A, fhg, L, \mathrm{id}_{UL}, 0].$$

The right hand sides are equal, since $(A, hg, K, f, 0) \approx_0 (A, fgh, L, \mathrm{id}_{UL}, 0)$:

$$\begin{tikzcd}
 & UL & \\
K \arrow[ur, dashed, "f"] \arrow[rr, "f"] & & L \arrow[ul, dashed, "\mathrm{id}_{UL}"'] \\
 & A \arrow[ul, dashed, "hg"] \arrow[ur, dashed, "hgf"'] &
\end{tikzcd}$$

Ff preserves ϕ_β, $0 < \beta < \alpha$: Let $g : A \to K$ be a (zz^β)-morphism. Then

$$Ff(\phi_\beta^{\Phi K}(A, g)) = Ff[A, g, K, \mathrm{id}_{UK}, \beta] = [A, g, K, f, \beta],$$

$$\phi_\beta^{\Phi K}(Ff(A, g)) = \phi_\beta^{\Phi L}(A, fg) = [A, fg, L, \mathrm{id}_{UL}, \beta].$$

The right hand sides are equal, since $(A, g, K, f, \beta) \approx_\beta (A, fg, L, \mathrm{id}_{UL}, \beta)$:

$$\begin{tikzcd}
 & UL & \\
K \arrow[ur, dashed, "f"] \arrow[rr, "f"] & & L \arrow[ul, dashed, "\mathrm{id}_{UL}"'] \\
 & A \arrow[ul, Rightarrow, dashed, "g"] \arrow[ur, Rightarrow, dashed, "fg"'] &
\end{tikzcd}$$

(the dashed double arrows are (zz^β)-morphisms). $\square$

Finally, to show that (Φ, F) is an s-embedding, we prove

Claim 3. *Let $f : K \to L$ be a* **h***-morphism such that $Ff : \Phi K \to \Phi L$ is a* **Fix**(α)*-morphism. Then f is a* **k***-morphism.*

Proof. Let $\beta < \alpha$. The identity $\mathrm{id}_{UK} : K \to K$ is a $\mathbf{k}$-morphism, hence it is a (zz^β)-morphism. Thus $\phi_\beta^{\Phi K}(K, \mathrm{id}_{UK})$ is defined. We have

$$Ff(\phi_\beta^{\Phi K}(K, \mathrm{id}_{UK})) = Ff(K, \mathrm{id}_{UK}) = (K, f),$$

$$\phi_\beta^{\Phi L}(Ff(K, \mathrm{id}_{UK})) = \phi_\beta^{\Phi L}(K, f) = [K, f, L, \mathrm{id}_{UL}, \beta].$$

Thus $(K, f) = [K, f, L, \mathrm{id}_{UL}, \beta]$, hence $f : K \to L$ is a (zz^{β^+})-morphism.

Since $f : K \to L$ is a (zz^{β^+})-morphism for all $\beta < \alpha$, it is a (zz^α)-morphism. Because $U \vDash (zz^\alpha)$, we have $f \in \mathbf{K}(K, L)$. □

The proof of Theorem 5.2 is concluded. □

A consequence of Theorems 4.13, 5.1, 5.2 is that, loosely speaking, our baskets obey no other closure rule than we already know:

Corollary 5.3. *Let a be a closure rule. Then*

1. $\mathbb{R} \vDash \mathsf{a}$ *iff* a *is trivial.*
2. $\mathbb{P} \vDash \mathsf{a}$ *iff* $(p) \vdash \mathsf{a}$.
3. $\mathbb{E}_\alpha \vDash \mathsf{a}$ *iff* $(zz^\alpha) \vdash \mathsf{a}$.

Proof. We are going to prove 3. The remaining cases can be proved similarly.

If $(zz^\alpha) \vdash \mathsf{a}$, then $(zz^\alpha) \vDash \mathsf{a}$ (4.13), whence $\mathbb{E}_\alpha \vDash \mathsf{a}$.

If $\mathbb{E}_\alpha \vDash \mathsf{a}$ and $U \vDash (zz^\alpha)$, where U is a faithful functor between small categories, then $U \leq_s \mathbb{E}_\alpha$ (5.2), hence $U \vDash \mathsf{a}$. Thus $(zz^\alpha) \vdash \mathsf{a}$ due to 4.14, 4.13. □

Using a modification of the last proof, we are able to give a slight generalization of Theorem 5.1.3. To formulate this result, we need the following definition.

Definition 5.4. *We say that a category $\mathbf{H}$ satisfies (*), if for every $H \in \mathrm{Obj}(\mathbf{H})$ the following equivalence on the class of all morphism with codomain H has set-many equivalence classes only*

$$f : A \to H \sim_* g : B \to H \quad \textit{iff} \quad (\exists k : A \to B)\, (\exists l : B \to A)\ gk = f \ \textit{and}\ fl = g.$$

Theorem 5.5. *Let $U : \mathbf{K} \to \mathbf{H}$ be a concrete category, where both $\mathbf{H}$ and $\mathbf{H}^{op}$ satisfy (*). Then $U \leq_s \mathbb{A}$ iff U is SSF and obeys (zz^1).*

Proof. If $U \leq_s \mathbb{A}$, then U is SSF and $U \vDash (zz^1)$ (see 2.7, 4.8, 4.4).

Suppose that U is SSF and obeys (zz^1). We will find functors $F, G : \mathbf{H} \to \mathbf{Set}$ and a concrete full embedding $\Phi : \mathbf{K} \to \mathbf{A}[F, G]$. This suffices, since $\mathbf{A}[F, G] \leq_s \mathbb{A}$ (Let $C = F \sqcup G$, $\Psi(H, \alpha) = (FH \sqcup GH, \overline{\alpha})$, where $\overline{\alpha}$ coincides with α on FH and is identical on GH. (Ψ, C) is an s-embedding.)

For an $\mathbf{H}$-object, let

$$FH = \{(A, g) \mid A \in \mathrm{Obj}(\mathbf{K}),\, g \in \mathbf{H}(A, H)\}/\approx,$$

where

$$(A, g) \approx (A', g') \quad \text{iff} \quad (A, g) \sim_{SSF} (A', g') \text{ and } g \sim_* g'.$$

For a $\mathbf{H}$-morphism $f : H \to H'$ let

$$Ff[A,g]_{\approx} = [A,fg]_{\approx}.$$

Since U is SSF and $\mathbf{H}$ satisfy (*), there is set-many equivalence classes of $\approx$ only (for each H).

For $H \in \mathrm{Obj}(\mathbf{H})$ we now define a class $G'H$ by

$$G'H = \{(A,g,B,h) \mid A,B \in \mathrm{Obj}(\mathbf{K}),\ g \in \mathbf{H}(A,B),\ h \in \mathbf{H}(B,H)\}$$

and an equivalence $\approx_{zz}$ on $G'H$: $(A,g,B,h) \approx_{zz} (A',g',B',h')$, iff $(A,hg) \sim_{SSF} (A',h'g')$ and there exists a commutative diagram

$$\begin{tikzcd}
 & & & H & & \\
B \arrow[urrr, dashed, "h"] & B_2 \arrow[l] \arrow[r] \arrow[urr, dashed] & B_3 \arrow[ur, dashed] & \cdots \arrow[l] \arrow[r] & B_{2n-1} \arrow[ul, dashed] & B' \arrow[l] \arrow[ull, dashed, "h'"'] \\
A \arrow[u, dashed, "g"] \arrow[r, dashed, shift left] & A_2 \arrow[u, dashed] \arrow[l, dashed, shift left] \arrow[r, dashed, shift left] & A_3 \arrow[u, dashed] \arrow[l, dashed, shift left] & \cdots \arrow[l, dashed] \arrow[r, dashed, shift left] & A_{2n-1} \arrow[u, dashed] \arrow[l, dashed, shift left] \arrow[r, dashed, shift left] & A' \arrow[u, dashed, "g'"'] \arrow[l, dashed, shift left]
\end{tikzcd}$$

where, again, A_i, B_i are $\mathbf{K}$-objects, arrows are $\mathbf{H}$-morphism and solid arrows are $\mathbf{K}$-morphisms.

The functor $G : \mathbf{H} \to \mathbf{Set}$ is defined for $H, H' \in \mathrm{Obj}(\mathbf{H})$ and $f \in \mathbf{H}(H,H')$ by

$$\begin{aligned}
GH &= \{(A,g,B,h) \mid A,B \in \mathrm{Obj}(\mathbf{K}), g \in \mathbf{H}(A,B), h \in \mathbf{H}(B,H)\}/\equiv, \\
Gf[A,g,B,h]_{\equiv} &= [A,g,B,fh]_{\equiv}.
\end{aligned}$$

The equivalence $\equiv$ is given by

$$(A,g,B,h) \equiv (A',g',B',h') \quad \text{iff} \quad \mathrm{OUT}(A,g,B,h) = \mathrm{OUT}(A',g',B',h'),$$

where

$$\begin{aligned}
\mathrm{OUT}(A,g,B,h) = \ &\{l \in \mathbf{H}(H,H') \mid H' \in \mathrm{Obj}(H),\ (\exists A' \in \mathrm{Obj}(\mathbf{K})) \\
&(\exists m \in \mathbf{H}(A',H'))\ (A,g,B,lh) \approx_{zz} (A', \mathrm{id}_{UA'}, A', m)\}.
\end{aligned}$$

Observe that

- if $l \sim_* l'$ in $\mathbf{H}^{op}$, then $l \in \mathrm{OUT}(A,g,B,h)$ iff $l' \in \mathrm{OUT}(A,g,B,h)$. Thus $\equiv$ has set-many equivalence classes only.

- Let $C \in \mathrm{Obj}(\mathbf{K})$ be such that $UC = H$. If $\mathrm{id}_H \in \mathrm{OUT}(A,g,B,h)$ and $h \in \mathbf{K}(B,C)$, then $hg \in \mathbf{K}(A,C)$.

For each $K \in \mathrm{Obj}(K)$, let $\Phi K = (UK, \overline{K} : FUK \to GUK)$, where

$$\overline{K}[A,g]_{\approx} = [A,g,K,\mathrm{id}_{UK}]_{\equiv}.$$

The definition doesn't depend on the choice of the representative of $[A,g]$. Φ is a functor:

Claim 1. *Let $f : K \to L$ be a $\mathbf{K}$-morphism. Then $f : \Phi K \to \Phi L$ is an $\mathbf{A}[F, G]$-morphism.*

Proof. Let $[A, g]_\approx \in FUK$. Then

$$Gf(\overline{K}[A, g]_\approx) = Ff[A, g, K, \mathrm{id}_{UK}]_\equiv = [A, g, K, f]_\equiv$$

and

$$\overline{L}(Ff[A, g]_\approx) = \overline{L}[A, fg]_\approx = [A, fg, L, \mathrm{id}_{UL}]_\equiv.$$

Since $(A, g, K, f) \approx_{zz} (A, gf, L, \mathrm{id}_{UL})$ it follows that $(A, g, K, f) \equiv (A, gf, L, \mathrm{id}_{UL})$. □

Φ is full:

Claim 2. *Let $f : K \to L$ be a $\mathbf{H}$-morphism such that $f : \Phi K \to \Phi L$ is an $\mathbf{A}[F, G]$-morphism. Then f is a $\mathbf{K}$-morphism from K to L.*

Proof.

$$Gf(\overline{K}[K, \mathrm{id}_{UK}]_\approx) = Gf[K, \mathrm{id}_{UK}, K, \mathrm{id}_{UK}]_\equiv = [K, \mathrm{id}_{UK}, K, f]_\equiv$$

and

$$\overline{L}(Ff[K, \mathrm{id}_{UK}]_\approx) = \overline{L}[K, f]_\approx = [K, f, L, \mathrm{id}_{UL}]_\equiv.$$

Since $\mathrm{id}_{UL} \in \mathrm{OUT}(K, \mathrm{id}_{UK}, K, f)$, $\mathrm{id}_{UL} \in \mathrm{OUT}(K, f, L, \mathrm{id}_{UL})$. Thus $f : K \to L$ is a $\mathbf{K}$-morphism due to the observation above. □

The last claim finishes the proof of 5.5. □

The conditions on $\mathbf{H}$ are still very strong. However, they are satisfied by any small category, the category of sets, the category of pointed sets, the category of vector spaces.

Open problem. *Is it possible to generalize Theorem 5.2 to arbitrary concrete categories $U : \mathbf{K} \to \mathbf{H}$? Or, at least, answer the following particular questions:*

- *Is it possible to generalize Theorem 5.5 to arbitrary concrete categories $U : \mathbf{K} \to \mathbf{H}$? An attempt was made in author's master thesis: There exists a concrete category V such that $U \leq_s V$ iff U is SSF and $U \models (zz_1^1)$. This however doesn't seem to be the right direction.*

- *Is it possible to generalize Theorem 5.2 to concrete categories over* **Set***? If the answer is positive, Theorem 3.5 would easily follow, since the usage of 3.5 in Proposition 4.8 is inessential.*

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