Topology is relevant in infinite-domain constraint satisfaction

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- Each problem associated with a relational structure: its template.
- A class of decision problems.
- For a large class of templates, the complexity of each problem depends solely on the symmetries of the template.

non-trivial symmetries $\rightsquigarrow$ easier problems

Examples

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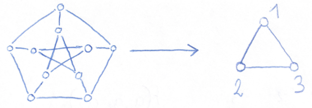

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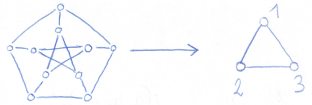

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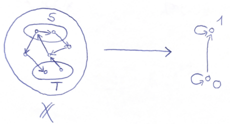

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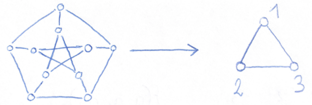

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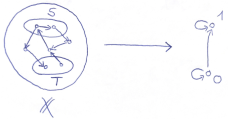
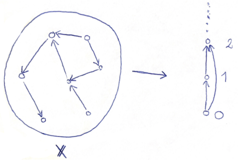


- CSP($\mathbb{N}, <$): acyclicity



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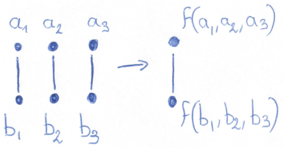

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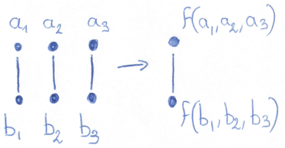

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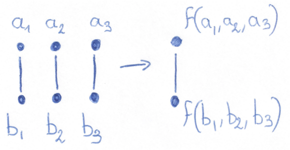

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Finite-Domain Dichotomy

Theorem (Bulatov '17, Zhuk '17)

$\mathbb{A}$ *finite structure. Exactly one of the following holds:*

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Infinite-domain constraint satisfaction problems:

- Step 0: what relational structures to look at?
- Step 1: identify the borderline,
- Step 2: find a useful characterisation of the borderline.

Step 0: the scope

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- This class contains all finite structures,
- all such CSPs are in NP,
- all such templates are ω-categorical, in particular polymorphisms still capture complexity.

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- Complexity still captured by polymorphisms: $\mathrm{Pol}(\mathbb{A}) = \mathrm{Pol}(\mathbb{B}) \Rightarrow$ ptime equivalent CSPs
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Ways to make the conjecture easier to work with:

- Is there a weakest system of nontrivial height 1 identities for such structures?
- Can topology be dropped in the statement?

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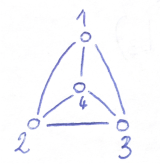

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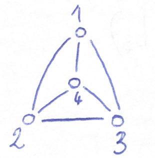

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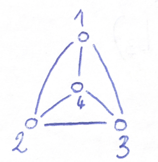

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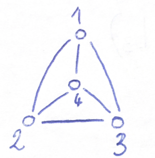

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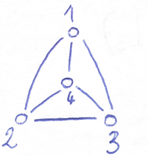

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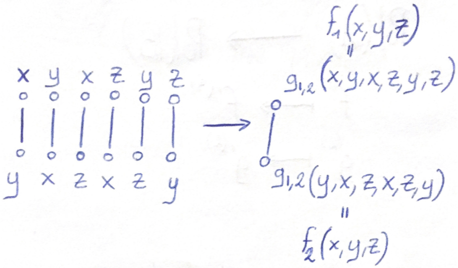

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Moreover:

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Theorem (Cherlin-Shelah-Shi, Hubička-Nešetřil)

Let $\mathbb{G}$ be a finite connected graph. There exists an ω-categorical $CSS(\mathbb{G})$ such that for all $\mathbb{X}$, $\mathbb{X} \to CSS(\mathbb{G})$ iff $\mathbb{G} \not\to \mathbb{X}$.

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Theorem (BMOOPW)

There is no weakest non-trivial system of global height 1 identities. In particular, one cannot replace $\mathrm{Pol}(\mathbb{A}) \overset{u.c.}{\not\longrightarrow} \mathscr{P}$ by some global identities in the statement of the dichotomy conjecture.

Conjecture (Barto, Opršal, Pinsker)

$\mathbb{A}$ *definable over a finitely bounded homogeneous structure. One of the following holds:*

- $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(SAT)$ *uniformly continuously and* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete,*
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Local $\neq$ Global

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More recently:

Theorem

There exists an ω-categorical structure with finite signature with the same properties.

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Combine several ω-categorical structures (oligomorphic groups) and get another ω-categorical structure (oligomorphic group) in a generic way.

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 # orbits of pairs in $\mathbb{A}(\mathbb{G}_1) \oplus \mathbb{A}(\mathbb{G}_2) =$ *(# ... in* $\mathbb{A}(\mathbb{G}_1)$*)* $\times$ *(#... in* $\mathbb{A}(\mathbb{G}_2)$*)*

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Final trick: first encode $\mathbb{A}(\mathbb{G}_n)$ as a graph on n-tuples.

Conclusion

Theorem (BMOOPW)

There is no weakest non-trivial system of global height 1 identities. In particular, one cannot replace $\mathrm{Pol}(\mathbb{A}) \overset{u.c.}{\not\longrightarrow} \mathscr{P}$ *by some global identities in the statement of the dichotomy conjecture.*

Theorem (BMOOPW)

There exists a closed oligomorphic clone $\mathscr{C}$ *such that* $\mathscr{C} \overset{u.c.}{\not\longrightarrow} \mathscr{P}$ *and* $\mathscr{C} \longrightarrow \mathscr{P}$*.*