The algebraic dichotomy conjecture for infinite domain Constraint Satisfaction Problems

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Main result and outline

Main result as a dichotomy theorem:
For any countable ω-categorical $\mathbb{A}$, its core

- **either** has a huge expressive power,
- **or** has nontrivial symmetries

Other formulation: Topology is irrelevant

Outline:

- 1st formulation
- 2nd formulation
- motivation – infinite domain CSP
- proof via a pseudo-loop lemma

Note: All structures at most countable

1st formulation: the dichotomy

Huge expressive power

"huge expressive power"
$=$ pp-interprets all finite structures (with parameters)
$=$ pp-interprets $\mathbb{K}_3$ (with parameters)

Def: primitive positive (pp-) formula over $\mathbb{A} = (A; R_1, R_2, \dots)$
uses $\exists, \wedge, =$
eg. $(\exists x_1)(\exists x_2) R_3(x_1, x_3) \wedge R_2(x_1, x_1, x_4)$

Def: pp-formula with parameters can use elements of A

Def: $\mathbb{A}$ pp-interprets $\mathbb{B}$ if $\exists$ map from $S \subseteq A^k$ to B such that S, preimage of $=$, and preimages of relations in $\mathbb{B}$ are pp-definable from $\mathbb{A}$

ie. $\mathbb{B}$ is obtained:

- take A^n and some relations pp-definable from $\mathbb{A}$
- take an induced substructure on a pp-def set
- take a quotient modulo a pp-def equivalence

Unary symmetries = endomorphisms

$\mathrm{End}(\mathbb{A}) = \{f \mid f : \mathbb{A} \to \mathbb{A} \text{ homo}\}$ the set of endomorphisms
$\mathrm{Aut}(\mathbb{A}) = \{f', \mid f, f^{-1} \in \mathrm{End}(\mathbb{A})\}$ the set of automorphisms

Equipped with the topology of pointwise convergence:

(f_i) converges to f iff $f(a) = f_i(a)$ eventually, for all $a \in A$
or A discrete top., A^A product top., $\mathrm{End}(\mathbb{A})$ subspace top.

Thm: [Engeler, Ryll-Nardzewski, Svenonius'59] $\mathbb{A}$ is ω-categorical iff $\forall n$ $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ has finitely many orbits
(Aut($\mathbb{A}$) is oligomorphic)

Def: $\mathbb{A}$ is a core if $\mathrm{Aut}(\mathbb{A})$ dense in $\mathrm{End}(\mathbb{A})$

Thm: [Bodirsky'07] Each ω-categorical $\mathbb{A}$ is homomorphically equivalent to a unique ω-categorical core.

Symmetries = polymorphisms

$\mathrm{Pol}(\mathbb{A}) = \{f \mid f : \mathbb{A}^n \to \mathbb{A} \text{ homo}\}$ the set of polymorphisms

equipped with the topology of pointwise convergence
(will appear later)

it is a clone: contains projections, closed under composition
= closed under forming term operations

Def: Homomorphism between clones $\mathcal{A} \to \mathcal{B}$ is a mapping from operations in $\mathcal{A}$ to operations in $\mathcal{B}$ that
preserves projections and composition

equivalently, preserves (universally quantified) equations
eg. an associative/commutative operation is mapped to
an associative/commutative operation

Nontrivial symmetries

$\mathbb{A}$ has "nontrivial symmetries"
$=$ there is no homomorphism from Pol($\mathbb{A}$) to the trivial clone $\mathcal{P}$

trivial clone $\mathcal{P}$: clone of projections on (say) 2-element set

Note: There exists no homo from Pol($\mathbb{A}$) to $\mathcal{P}$ (the trivial clone)

iff the set of equations satisfied by polymorphisms of $\mathbb{A}$ is not satisfiable by projections

The dichotomy result

Theorem (Barto, Pinsker'06)

Let $\mathbb{A}$ be the core of a countable ω-categorical structure. Then

(1) **either** $\mathbb{A}$ *pp-interprets, with parameters, every finite structure*

(2) **or** *there is no homo* $\mathrm{Pol}(\mathbb{A}, a_1, \ldots, a_n) \to \mathcal{P}$ *for any* $a_1, \ldots, a_n \in A$

Moreover, (2) is equivalent to

(2') $\mathrm{Pol}(\mathbb{A})$ *contains unary* α, β *and* 6*-ary* s *(a pseudo–Siggers) such that*

$$\alpha(s(x, y, x, z, y, z)) = \beta(s(y, x, z, x, z, y))$$

The constants in (2) because of parameters in (1).

The version without constants is open.

An equivalent formulation

Theorem

Let $\mathbb{A}$ be a core ω-categorical structure. TFAE

(1) $\mathbb{A}$ *does not pp-interpret* $\mathbb{K}_3$ *(with parameters).*

(2) $\mathbb{A}$ *has polymorphisms* α, β, s *satisfying* $\alpha(s(x,y,x,z,y,z)) = \beta(s(y,x,z,x,z,y))$.

2nd formulation: topology is irrelevant

pp-interpretations and clone homomorphisms; finite case

Thm: Finite $\mathbb{A}$ pp-interprets finite $\mathbb{B}$
iff $\exists$ homo $\mathsf{Pol}(\mathbb{A}) \to \mathsf{Pol}(\mathbb{B})$

Why:

- pp-definitions $\leftrightarrow$ polymorphisms [Geiger'68, Bodnarčuk, Kalužnin, Kotov, Romov'69]
- pp-interpretations $\leftrightarrow$ standard algebraic constructions (powers, subalgebras, quotients) [Bodirsky ???]
- standard constructions $\leftrightarrow$ equations (clone homomorphisms) [Birkhoff'35]

Cor: Finite $\mathbb{A}$ pp-interprets all finite
iff $\exists$ homo $\mathsf{Pol}(\mathbb{A}) \to \mathcal{P}$

pp-interpretations and clone homomorphisms; infinite

Thm: ω-categorical $\mathbb{A}$ pp-interprets finite $\mathbb{B}$
iff $\exists$ **continuous** homo $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$

Why:

- pp-definitions $\leftrightarrow$ polymorphisms [Bodirsky, Nešetřil'06]
- pp-interpretations $\leftrightarrow$ **finite** powers, subalgebras, quotients
- these constructions $\leftrightarrow$ continuous clone homomorphisms [Bodirsky, Pinsker'15]

Cor: ω-categorical $\mathbb{A}$ pp-interprets all finite
iff $\exists$ homo $\mathrm{Pol}(\mathbb{A}) \to \mathcal{P}$

Cor: ω-categorical $\mathbb{A}$ pp-interprets, with parameters, all finite
iff $\exists$ homo $\mathrm{Pol}(\mathbb{A}, a_1, \ldots, a_n) \to \mathcal{P}$ for some $a_1, \ldots, a_n \in A$

The dichotomy theorem rephrased

Theorem (B,P'06)

Let $\mathbb{A}$ be a core countable ω-categorical structure.

If $\exists$ *homo* $\mathrm{Pol}(\mathbb{A}, a_1, \ldots, a_n) \to \mathcal{P}$ *for some* $a_i \in A$

Then $\exists$*continuous homo* $\mathrm{Pol}(\mathbb{A}, a_1, \ldots, a_n) \to \mathcal{P}$ *for some* $a_i \in A$

Moreover, in the opposite case, $\mathrm{Pol}(\mathbb{A})$ *contains a pseudo-Siggers operation.*

- Is the implication true without the constants?
- There are discontinuous homomorphisms to $\mathcal{P}$ [Bodirsky, Pinsker, Pongrácz'?]

Motivation: Infinite domain CSP

CSP over a structure

Def: Constraint Satisfaction Problem over $\mathbb{A}$, $\mathrm{CSP}(\mathbb{A})$,
INPUT: pp-sentence over $\mathbb{A}$
QUESTION: is it true in $\mathbb{A}$?

- $\mathbb{A}$ pp-interprets $\mathbb{B} \Rightarrow \mathrm{CSP}(\mathbb{B})$ easier than $\mathrm{CSP}(\mathbb{A})$
- $\mathbb{A}$ finite: $\mathrm{CSP}(\mathbb{A})$ in NP
  - core of $\mathbb{A}$ pp-interprets (w/params) $\mathbb{K}_3 \Rightarrow \mathrm{CSP}(\mathbb{A})$ NP-complete
  - **conjecture:** otherwise $\mathrm{CSP}(\mathbb{A})$ in P [Bulatov,Jeavons,Krokhin'00]
  - in any case, the complexity of $\mathrm{CSP}(\mathbb{A})$
    depends only on equations true in Pol($\mathbb{A}$)
- For infinite $\mathbb{A}$ (important problems!)
  - $\mathrm{CSP}(\mathbb{A})$ can be anything [Bodirsky, Grohe'08]
  - For ω-categorical $\mathbb{A}$: still bad [Bodirsky, Grohe'08]
  - ...but the complexity of $\mathrm{CSP}(\mathbb{A})$
    depends only on equations and topology of Pol($\mathbb{A}$)
  - For nicer infinite $\mathbb{A}$: $\mathrm{CSP}(\mathbb{A})$ in NP, the dichotomy NP-complete/P plausible

The dichotomy conjectyre for infinite template CSPs

Conjecture (Bodirsky, Pinsker'11)

Let $\mathbb{A}$ be the core of $\mathbb{B}$– a reduct of a finitely bounded homogeneous structure in finite language (such structures are ω-categorical)

If there is no continuous homo $\mathrm{Pol}(\mathbb{A}, a_1, \ldots, a_n) \to \mathcal{P}$ for any $a_i \in A$, then $\mathrm{CSP}(\mathbb{B})$ is in P. (Otherwise it is NP-complete.)

- Our theorem removes the topology
- It gives a positive alternative (perhaps useful for attacking the conjecture)
- ... there are many such alternatives in the finite case
- It gives a tool for proving NP-completeness
- There is a better conjecture [Barto, Opršal, Pinsker: Wonderland of reflections'??]
- The two conjectures are equivalent [Olšák]

Proof via a pseudo-loop lemma

Loop lemmata

Loop lemma: Let $R \subseteq B^2$, B finite.
Then R contains a loop (a, a) provided

- certain structural assumption is satisfied, like
  (1) R is symmetric and contains a triangle
  (2) R is symmetric and contains an odd cycle
  (3) R is strongly connected, GCD of cycles lengths $= 1$
  (4) R has no sources or sinks and has algebraic length 1
- and (B, R) does not pp-interpret, with parameters, $\mathbb{K}_3$

Each loop lemma gives equations [Siggers'10] using a standard universal algebraic argument [Kearnes,Marković,McKenzie'14]

Loop lemma proved by

- ∼[Hell, Nešetřil'90] assuming (2): purely relational proof
- [Bulatov'05] assuming (2): relations + operations
- [Barto, Kozik, Niven'09] assuming (4): purely algebraic proof

Equations from loop lemma

Theorem (Siggers'10)

TFAE for finite core $\mathbb{A}$

(1) *$\mathbb{A}$ does not pp-interpret $\mathbb{K}_3$ (with parameters).*

(2) *$\mathbb{A}$ has a polymorphism s satisfying*
$s(x,y,x,z,y,z) = s(y,x,z,x,z,y)$.

Only (1) $\Rightarrow$ (2) interesting.

- Define $B = A^{A^3}$ (element is a 3-ary operation on A)
- Define
$$R = \left\{ \begin{pmatrix} (x,y,z) \mapsto s(x,y,x,z,y,z) \\ (x,y,z) \mapsto s(y,x,z,x,z,y) \end{pmatrix} \mid s \text{ a 6-ary polymorphism} \right\}$$
- Observe
  - R is symmetric
  - R contains a triangle (x,y,z form a triangle)
  - $\mathbb{A}$ pp-interprets $(B;R)$, thus $(B;R)$ does not interpret $\mathbb{K}_3$
- Loop in R gives the Siggers operation

To the infinity

Two issues with generalization to ω-categorical:

- A^{A^3} is not a finite power
  - Use A^X for X finite subsets of A^3
  - This gives "locally nice operations"
  - Compactness argument using ω-categoricity $\to$ globally nice operations
- loop lemma does not hold (eg. $(\mathbb{N}; \neq)$)
  - loop lemma $\to$ pseudo–loop lemma
  - pseudo–loop lemma: the technical core

We get:

Theorem

Let $\mathbb{A}$ be a core ω-categorical structure. TFAE

(1) *$\mathbb{A}$ does not pp-interpret $\mathbb{K}_3$ (with parameters).*

(2) *$\mathbb{A}$ has polymorphisms α, β, s satisfying* $\alpha s(x, y, x, z, y, z) = \beta s(y, x, z, x, z, y)$.

Pseudo–loop lemma

Theorem

Let $R \subseteq B^2$, B countable.
Let a group G acts on B oligomorphically, let R be G-invariant.
Then R contains a pseudo–loop (a, b), a, b in the same G-orbit provided

- *R is symmetric and contains a triangle*
- *and $(B; R, G\text{–orbits of pairs})$ does not pp-interpret, with parameters, $\mathbb{K}_3$*

- First attempt: use B,K,N algebraic approach. Not successful yet, but **very** interesting "side product" – Olšák's equations
- Second attempt: use Bulatov's relational/algebraic approach. Success, proof requires generalizations and extra work

Thank you!