Promises Make Finite (Constraint Satisfaction) Problems Infinitary

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***Abstract*—The fixed template Promise Constraint Satisfaction Problem (PCSP) is a recently proposed significant generalization of the fixed template CSP, which includes approximation variants of satisfiability and graph coloring problems. All the currently known tractable (i.e., solvable in polynomial time) PCSPs over finite templates can be reduced, in a certain natural way, to tractable CSPs. However, such CSPs are often over infinite domains. We show that the infinity is in fact necessary by proving that a specific finite-domain PCSP, namely (1-in-3-SAT, Not-All-Equal-3-SAT), cannot be naturally reduced to a tractable finite-domain CSP, unless P=NP.**

I. Introduction

Finding a 3-coloring of a graph or finding a satisfying assignment of a propositional 3-CNF formula (or rather the decision variants of these problems) are well-known and fundamental NP-complete computational problems. The latter problem, 3-SAT, has many restrictions still known to be NP-complete [1], two of which play a central role in this paper. The *positive 1-in-3-SAT*, denoted $\mathrm{1\text{-}in\text{-}3\text{-}SAT}$, can be defined as follows. The instance is a list of triples of variables and the problem is to find a mapping from the set of variables to $\{0,1\}$ such that exactly one variable in each triple is assigned 1. In the *positive Not-All-Equal-3-SAT*, denoted $\mathrm{NAE\text{-}3\text{-}SAT}$, instances are triples of variables as well, but the mapping is only required to assign not-all-equal elements to each triple.

There are two ways how to relax the requirement on the assignment in order to get a potentially simpler problem. The first one is to require a specified fraction of the constraints to be satisfied. For example, given a satisfiable 3-SAT instance, is it easier to find an assignment satisfying at least 90% of clauses? A celebrated result of Håstad [2] proves that the answer is “No.” – it is still an NP-complete problem. (Actually, any fraction greater than $7/8$ gives rise to an NP-complete problem while the fraction $7/8$ is achievable in polynomial time.)

The second type of relaxation is to require that a specified weaker version of every constraint is satisfied. For example, we want to find a 100-coloring of a 3-colorable graph, or we want to find a valid $\mathrm{NAE\text{-}3\text{-}SAT}$ assignment to a 1-in-3-satisfiable instance. The complexity of the former problem is a notorious open question (for a recent development see [3], [4], but even 6-coloring a 3-colorable graph is not known to be NP-complete). On the other hand, the latter problem admits an elegant polynomial time algorithm [5], [6], which we now describe.

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We take a satisfiable instance of $\mathrm{1\text{-}in\text{-}3\text{-}SAT}$ and replace each triple of variables (x,y,z) in the instance by the linear equation $x+y+z=1$ over $\mathbb{Z}$ (the integers). The obtained system is solvable (by the original 0,1 assignment) and it is known that finding a solution to a system of linear equations over $\mathbb{Z}$ is in P (see [7]). Now, if ϕ is any solution to the system, then

$$\psi(x) = \begin{cases} 0 & \text{if } \phi(x) \leq 0 \\ 1 & \text{if } \phi(x) > 0 \end{cases}$$

is a valid $\mathrm{NAE\text{-}3\text{-}SAT}$ assignment.

Alternatively, one can solve the system over $\mathbb{Q} \setminus \{1/3\}$ by a simple adjustment of Gaussian elimination and define $\psi(x) = 0$ iff $\phi(x) < 1/3$. (A more general class of problems can be solved, e.g., by restricting the domain $\mathbb{Q} \setminus \{c\}$ to the interval $[0,1]$ and using an adjustment of linear programming rather than Gaussian elimination; see [5], [6].)

It is remarkable that both polynomial algorithms transfer the original problem over a finite domain to a problem over an infinite domain. The main result of this paper shows that this finite-to-infinite transition is unavoidable. This result is stated more precisely below, as Theorem I.1, but let us first describe its background.

A. Constraint Satisfaction Problems

It will be convenient in this paper to use a formalization of CSP and PCSP via homomorphisms of relational structures. We refer to [8], [6] for translations to the other standard definitions.

Let $\mathbb{A}$ be a relational structure of a finite signature, often called *template* in this context. The *Constraint Satisfaction Problem (CSP) over* $\mathbb{A}$, denoted $\mathrm{CSP}(\mathbb{A})$, is the problem of deciding whether a given finite relational structure $\mathbb{X}$ (similar to $\mathbb{A}$) has a homomorphism to $\mathbb{A}$. The *search problem* for $\mathrm{CSP}(\mathbb{A})$ is to find such a homomorphism. Examples of CSPs include the 3-coloring problem (where $\mathbb{A}$ is a structure with a three-element domain and the binary disequality relation), 3-SAT (where $\mathbb{A}$ consists of 8 ternary relations of the form

$(\neg)x \vee (\neg)y \vee (\neg)z$ on the domain $\{0,1\}$), the problems 1-in-3-SAT, NAE-3-SAT for which the templates are structures with domain $\{0,1\}$ and a single ternary relation

$$\text{1-in-3} = (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\})$$
$$\text{NAE-3} = (\{0,1\}; \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}) \ ,$$

and the infinite-domain CSPs over $(\mathbb{Z}; x+y+z=1)$ and $(\mathbb{Q}\setminus\{1/3\}; x+y+z=1)$ that were used to solve the relaxed 1-in-3-SAT.

The complexity of the CSP over finite templates (modulo polynomial time reductions) is fully classified by a recent deep result of Bulatov [9] and, independently, Zhuk [10]. The classification is a culmination of an active research program, so called algebraic approach to CSPs, inspired by the landmark paper of Feder and Vardi [11], where the authors conjectured that each finite template CSP is either tractable or NP-complete, and observed that the tractability is often tied to closure properties of relations in the template. General theory of CSPs, whose basics were developed in [12], [13], [14], [15], confirmed this observation by closely linking CSPs to Universal Algebra (the theory of general algebraic systems) and provided guidance and tools for the eventual resolution of the dichotomy conjecture in [9], [10].

The theory of CSPs is based on a connection between constructions on relational structures (that lead to polynomial time reductions) and properties of their *polymorphisms* – multivariate functions preserving the structures. So far, the most general relational construction introduced in [15] is the so called pp-construction. Roughly, we say that $\mathbb{A}$ *pp-constructs* $\mathbb{B}$ if $\mathbb{B}$ can be obtained from $\mathbb{A}$ by a sequence of first order interpretations restricted to primitive positive formulae and replacements by homomorphically equivalent structures (see Section II for a more detailed definition). In this situation, there is a natural gadget reduction of $\mathrm{CSP}(\mathbb{B})$ to $\mathrm{CSP}(\mathbb{A})$. It follows, for instance, that $\mathrm{CSP}(\mathbb{A})$ is NP-complete whenever $\mathbb{A}$ pp-constructs a template of 3-SAT. It turned out [9], [10], confirming the *algebraic dichotomy conjecture* from [14], that this is exactly the borderline between tractable and NP-complete CSPs: all templates that do not pp-construct the template of 3-SAT have tractable CSPs. The algebraic part of the theory will not be discussed here, let us just mention that the strongest available algebraic characterization of the borderline by means of cyclic operations [16] is essential for the proof of the main result, Theorem I.1.

B. Promise CSPs

A *template* for the *Promise CSP* (PCSP) is a pair $(\mathbb{A},\mathbb{B})$ of similar relational structures of finite signature such that $\mathbb{A}$ has a homomorphism to $\mathbb{B}$. The PCSP over such a pair, denoted $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$, is the following promise problem: given a relational structure $\mathbb{X}$ (similar to $\mathbb{A}$ and $\mathbb{B}$) output "Yes." if $\mathbb{X}$ has a homomorphisms to $\mathbb{A}$, and "No." if $\mathbb{X}$ does not have a homomorphism to $\mathbb{B}$. The *search problem* for $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ is to find a homomorphism $\mathbb{X} \to \mathbb{B}$ given an input structure $\mathbb{X}$ that has a homomorphism to $\mathbb{A}$. Notice that $\mathrm{CSP}(\mathbb{A})$ is the same as $\mathrm{PCSP}(\mathbb{A},\mathbb{A})$. Examples of PCSPs include the 100-coloring of a 3-colorable graph, where the two structures are $(\{1,2,3\},\neq)$ and $(\{1,2,\dots,100\},\neq)$, and $\mathrm{PCSP}(\text{1-in-3},\text{NAE-3})$ – the central computational problem in this paper. Similar examples were the motivation for introducing the PCSP framework in [17], [3], [5], [6].

The current knowledge of the complexity of finite-domain PCSPs beyond CSPs is very much limited. For example, the Feder-Vardi dichotomy conjecture for CSPs was inspired by two earlier classification results: for CSPs over a Boolean (i.e., two-element) domain [1] and for CSPs over graphs [18]. In PCSPs, even the analogues of these early results are challenging. PCSPs over graphs include, as a very special case, the PCSP over a pair of complete graphs, the problem of l-coloring a k-colorable graph. A systematic study of Boolean PCSPs (where both structures have a two-element domain) was initiated in [5], e.g., $\mathrm{PCSP}(\text{1-in-3},\text{NAE-3})$ is included in the classification, but the general Boolean case is still wide open.

Fortunately, building on the initial insights and results in [17], [3], [5], [6], it was observed in [4] (among many other important results such as the NP-hardness of 5-coloring a 3-colorable graph) that the basics of the CSP theory from [15] generalize to PCSPs. In particular, the notions of pp-constructions and polymorphisms have their PCSP counterparts and the connection between relational and algebraic structures works just as well as in the CSP. This is especially interesting because some hardness and algorithmic results in PCSP require techniques used in approximation. PCSP thus might help building a bridge between the discrete, universal algebraic world of (exact) CSPs and analytical world of approximation.

C. Finite PCSPs are infinitary

The main result of this paper says that it is impossible to reduce $\mathrm{PCSP}(\text{1-in-3},\text{NAE-3})$ to a tractable finite-domain CSP by means of a pp-construction, unless P=NP.

Theorem I.1. *Let $\mathbb{C}$ be a finite relational structure that pp-constructs* $(\text{1-in-3},\text{NAE-3})$. *Then* $\mathrm{CSP}(\mathbb{C})$ *is NP-complete.*

A fundamental question is whether each finite tractable PCSP template can be pp-constructed from an infinite tractable CSP template. In [6], the authors conjectured that the answer is positive and even suggested a family of tractable CSPs that might solve all Boolean PCSPs.

The class of all infinite-domain CSPs is very broad. In fact, each computational problem is equivalent to an infinite-domain CSP [19]. However, some parts of the CSP theory can be extended to a quite rich class of structures, namely, reducts of finitely bounded homogeneous structures in finite signature, and some general results even to the broader class of ω-categorical structures [20], [21]. In particular, an algebraic criterion for NP-hardness is available [22], so it might be possible to generalize Theorem I.1 to this setting (possibly with different template). As ω-categorical structures are, in a sense, close to finite and CSPs over them are solved by "finitary" algorithms, such a generalization would show that a polynomial time algorithm for some PCSP must be "truly" infinitary.

Let us make a final remark before starting with the technicalities. Both algorithms [9], [10] for the finite-domain CSP are extremely complex and simplifications are much desired. Theorem I.1 supports the intuition that a simpler algorithm may require infinitary methods, such as CSPs over numerical domains [23] ($\mathbb{Z}$, $\mathbb{Q}$, ...).

II. Preliminaries

In this section we give formal definitions of the concepts essential for the proof. For an in depth introduction to CSP and PCSP, see [8], [4] and references therein.

A. PCSP

A *relational structure (of finite signature)* is a tuple $\mathbb{A} = (A; R_1, \ldots, R_n)$ where each $R_i \subseteq A^{\text{arity}(R_i)}$ is a relation on A of arity $\text{arity}(R_i) \geq 1$. The structure $\mathbb{A}$ is *finite* if A is finite.

Two relational structures $\mathbb{A} = (A; R_1, \ldots, R_n)$ and $\mathbb{B} = (B; S_1, \ldots, S_n)$ are *similar* if they have the same number of relations and $\text{arity}(R_i) = \text{arity}(S_i)$ for each $i \in \{1, \ldots, n\}$.

For two such similar relational structures $\mathbb{A}$ and $\mathbb{B}$, a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a mapping $f : A \to B$ such that $(f(a_1), f(a_2), \ldots, f(a_k)) \in S_i$ whenever $i \in \{1, \ldots, n\}$ and $(a_1, a_2, \ldots, a_k) \in R_i$ where $k = \text{arity}(R_i)$.

We write $\mathbb{A} \to \mathbb{B}$ if there exists a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, and $\mathbb{A} \not\to \mathbb{B}$ if there is none.

Definition II.1. A *PCSP template* is a pair $(\mathbb{A}, \mathbb{B})$ of similar relational structures such that $\mathbb{A} \to \mathbb{B}$.

The decision version of PCSP over $(\mathbb{A}, \mathbb{B})$, written $\text{PCSP}(\mathbb{A}, \mathbb{B})$, is the following promise problem. Given a finite structure $\mathbb{X}$ similar to $\mathbb{A}$ (and $\mathbb{B}$), output "Yes." if $\mathbb{X} \to \mathbb{A}$ and output "No." if $\mathbb{X} \not\to \mathbb{B}$.

The search version of $\text{PCSP}(\mathbb{A}, \mathbb{B})$ is, given a structure $\mathbb{X}$ similar to $\mathbb{A}$ such that $\mathbb{X} \to \mathbb{A}$, find a homomorphisms $\mathbb{X} \to \mathbb{B}$.

In the case $\mathbb{A} = \mathbb{B}$ we talk about a CSP template (and simply write $\mathbb{A}$ instead of $(\mathbb{A}, \mathbb{A})$) and define $\text{CSP}(\mathbb{A}) = \text{PCSP}(\mathbb{A}, \mathbb{A})$.

The decision version of the PCSP over $(\mathbb{A}, \mathbb{B})$ can be reduced to the search version. For CSPs, it is known [14] that these two versions are in fact equivalent, but it is an open problem whether they are equivalent for PCSPs as well.

B. Constructions

The two ingredients of a pp-construction are pp-powers and homomorphic relaxations.

Homomorphic relaxation, called *homomorphic sandwiching* in [6], is a generalization of the concept of homomorphic equivalence between CSP templates.

Definition II.2. Let $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{A}', \mathbb{B}')$ be PCSP templates. We that that $(\mathbb{A}', \mathbb{B}')$ is a *homomorphic relaxation* of $(\mathbb{A}, \mathbb{B})$ if there exist homomorphisms $f : \mathbb{A}' \to \mathbb{A}$ and $g : \mathbb{B} \to \mathbb{B}'$.

If $(\mathbb{A}', \mathbb{B}')$ is a homomorphic relaxation of $(\mathbb{A}, \mathbb{B})$, then the trivial reduction, which does not change the input structure $\mathbb{X}$, reduces (the decision or search version of) $\text{PCSP}(\mathbb{A}', \mathbb{B}')$ to $\text{PCSP}(\mathbb{A}, \mathbb{B})$. Both polynomial algorithms for $\text{PCSP}(\text{1-in-3}, \text{NAE-3})$ shown in the introduction come from this reduction with

$$\mathbb{A} = \mathbb{B} = (A; R), \quad (x, y, z) \in R \text{ iff } x + y + z = 1 \ ,$$

where $A = \mathbb{Z}$ in the first version of the algorithm and $A = \mathbb{Q} \setminus \{1/3\}$ in the second. In both cases, the mapping f was the inclusion and the "rounding" mapping g is defined by $g(x) = 0$ iff $x < 1/3$.

In order to define the other ingredient of a pp-construction, recall that a *primitive positive formula* over a relational structure $\mathbb{A}$ is an existential quantified conjunction of atomic formulas of the form $x_1 = x_2$ or $(x_{i_1}, \ldots, x_{i_k}) \in R$ where x_j's are variables and R is a relation in $\mathbb{A}$ of arity k.

Definition II.3. Let $(\mathbb{A}, \mathbb{B})$ and $(\mathbb{A}' = (A'; R_1, \ldots, R_n), \mathbb{B}' = (B', S_1, \ldots, S_n))$ be PCSP templates.

We say that $(\mathbb{A}', \mathbb{B}')$ is *pp-definable* from $(\mathbb{A}, \mathbb{B})$ if, for each $i \in \{1, \ldots, n\}$, there exists a primitive positive formula ϕ over $\mathbb{A}$ such that ϕ defines R_i and the formula, obtained by replacing each occurrence of a relation of $\mathbb{A}$ by the corresponding relation in $\mathbb{B}$, defines S_i.

We say that $(\mathbb{A}', \mathbb{B}')$ is an n-th *pp-power* of $(\mathbb{A}, \mathbb{B})$ if $A' = A^n$, $B' = B^n$, and, if we view k-ary relations on $\mathbb{A}'$ and $\mathbb{B}'$ as kn-ary relations on A and B, respectively, then $(\mathbb{A}', \mathbb{B}')$ is pp-definable from $(\mathbb{A}, \mathbb{B})$.

By combining these two constructions we get the notion of pp-construction.

Definition II.4. We say that a PCSP template $(\mathbb{A}, \mathbb{B})$ pp-constructs a PCSP template $(\mathbb{A}', \mathbb{B}')$ if there exists a sequence

$$(\mathbb{A}, \mathbb{B}) = (\mathbb{A}_1, \mathbb{B}_1), \ldots, (\mathbb{A}_k, \mathbb{B}_k) = (\mathbb{A}', \mathbb{B}')$$

of PCSP templates such that each $(\mathbb{A}_{i+1}, \mathbb{B}_{i+1})$ is a pp-power or a homomorphic relaxation of $(\mathbb{A}_i, \mathbb{B}_i)$.

It is not hard to see that if $(\mathbb{A}, \mathbb{B})$ pp-constructs $(\mathbb{A}', \mathbb{B}')$, then $\text{PCSP}(\mathbb{A}', \mathbb{B}')$ reduces (even in log-space) to $\text{PCSP}(\mathbb{A}, \mathbb{B})$. The proof is similar to the analogous proof for CSP (see [8]). An interesting alternative way for PCSP was given (but explicitly proved only for finite templates) in [4].

However, in this paper, pp-constructions make only a cosmetic difference in the statement of Theorem I.1 – it is enough to prove the theorem for homomorphic relaxations. Indeed, it is well known (see [15]) that if $(\mathbb{A}, \mathbb{B})$ pp-constructs $(\mathbb{A}', \mathbb{B}')$, then $(\mathbb{A}', \mathbb{B}')$ is a homomorphic relaxation of a pp-power of $(\mathbb{A}, \mathbb{B})$. Therefore, if a finite $\mathbb{C}$ pp-constructs $(\text{1-in-3}, \text{NAE-3})$, then $(\text{1-in-3}, \text{NAE-3})$ is a homomorphic relaxation of a template $(\mathbb{D}, \mathbb{D}')$, which is a pp-power of $\mathbb{C}$. Then, clearly, $\mathbb{D} = \mathbb{D}'$ are finite and $\text{CSP}(\mathbb{D})$ reduces to $\text{CSP}(\mathbb{C})$.

C. Cyclic polymorphisms

For a PCSP template $(\mathbb{A}, \mathbb{B})$, a function $f : A^n \to B$ is called a *polymorphism* of the template if it is a homomorphism from the n-th categorical power of $\mathbb{A}$ to $\mathbb{B}$. The basic fact of the algebraic theory of (P)CSP is that the set of polymorphisms determine the complexity of $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ ([13], cf. [8]).

We will only work with polymorphisms of CSP templates and we spell out the definition of a polymorphism in a more elementary way for this case.

Definition II.5. Let $\mathbb{C}$ be a CSP template and $s : C^n \to C$ a function (also called an *operation* in this context). We say that s is a *polymorphism* of $\mathbb{C}$ if, for each relation R in $\mathbb{C}$ with $k = \mathrm{arity}(R)$ and all tuples $(a_1^1, \dots, a_k^1), \dots, (a_1^n, \dots, a_k^n) \in R$, we have

$$(s(a_1^1, \dots, a_1^n), \dots, s(a_k^1, \dots, a_k^n)) \in R \ .$$

The proof of the main theorem is based on the following result from [16].

Definition II.6. An operation $s : C^n \to C$ is called *cyclic* if, for all $(a_1, \dots, a_n) \in C^n$, we have

$$s(a_1, a_2, \dots, a_n) = s(a_2, \dots, a_n, a_1) \ .$$

Theorem II.7. *Let $\mathbb{C}$ be a finite CSP template. If $\mathrm{CSP}(\mathbb{C})$ is not NP-complete, then $\mathbb{C}$ has a cyclic polymorphism of arity p for every prime number $p > |C|$.*

We remark that cyclic operations characterize the borderline between NP-complete and tractable CSPs – whenever $\mathbb{C}$ has a cyclic polymorphism of arity at least 2, then $\mathrm{CSP}(\mathbb{C})$ is tractable [9], [10]. In fact, cyclic polymorphisms provide currently the strongest characterization of the borderline in the sense that the other important types of operations (such as the Sigger's operations [24], [25] or the weak near-unanimity operations [26]) can be obtained from a cyclic operation by an identification of variables. The proof of Theorem I.1 could still be simplified having a yet stronger (or alternative) characterization at hand. See Section IV for a concrete open problem in this direction.

III. Infinity is necessary

In this section we prove the main theorem. As explained in Subsection II-B, it is enough to prove the following result.

Theorem III.1. *Let $\mathbb{C} = (C; R)$ be a finite relational structure with ternary $R \subseteq C^3$ such that $(\text{1-in-3}, \text{NAE-3})$ is a homomorphic relaxation of $\mathbb{C}$. Then $\mathrm{CSP}(\mathbb{C})$ is NP-complete.*

Assume that $\mathrm{CSP}(\mathbb{C})$ is not NP-complete and let f : 1-in-3 $\to \mathbb{C}$ and $g : \text{NAE-3} \to \mathbb{C}$ be homomorphisms from the definition of homomorphic relaxation, Definition II.2.

Since gf is a homomorphism, this mapping applied component-wise to the 1-in-3 tuple $(0, 0, 1)$ is a not-all-equal tuple. In particular $f(0) \neq f(1)$. We rename the elements of C so that $\{0, 1\} \subseteq C$ and f is the inclusion. As f and g are homomorphisms, we get

$$\{0, 1\} \subseteq C, \quad \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq R$$

and

$$\neg(g(a) = g(b) = g(c)) \text{ whenever } (a, b, c) \in R \ .$$

By Theorem II.7, $\mathbb{C}$ has a cyclic polymorphism of any prime arity $p > |C|$. We fix a cyclic polymorphism

$$s \text{ of prime arity } p > 60|C| \ .$$

Next we define an operation t on C of arity p^2 by

$$\begin{aligned} t(x_{11}, x_{12}, \dots, x_{1p}, x_{21}, x_{22}, \dots x_{2p}, x_{31}, \dots, \dots, x_{pp}) \\ = s(s(x_{11}, x_{21}, \dots, x_{p1}), \\ s(x_{12}, x_{22}, \dots, x_{p2}), \\ \dots \\ s(x_{1p}, x_{2p}, \dots, x_{pp})) \ . \end{aligned}$$

It will be convenient to organize the arguments of t into a $p \times p$ matrix X whose entry in the i-th row and j-the column is x_{ij}, so the value

$$t\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix}$$

is obtained by applying s to the columns and then s to the results.

We introduce several concepts for zero-one matrices, the only important arguments of t for the proof.

Definition III.2. Let $X = (x_{ij}), Y$ be $p \times p$ zero-one matrices. The *area of* X is the fraction of ones and is denoted

$$\lambda(X) = \left(\sum_{i,j} x_{ij} \right) / p^2 \ .$$

The matrices X, Y are called *g-equivalent*, denoted $X \sim Y$, if $g(t(X)) = g(t(Y))$.

The matrix X is called *tame* if

$$\begin{aligned} X \sim 0_{p\times p} \quad &\text{if} \quad \lambda(X) < 1/3 \\ \text{and } X \sim 1_{p\times p} \quad &\text{if} \quad \lambda(X) > 1/3 \end{aligned}$$

where $0_{p\times p}$ stands for the zero matrix and $1_{p\times p}$ for the all-ones matrix.

Observe that the equivalence $\sim$ has two blocks, so, e.g., $X \not\sim Y \not\sim Z$ implies $X \sim Z$. Also recall that $p > 3$ is a prime number, so the area of X is never equal to $1/3$.

The proof now proceeds as follows. We show that certain matrices, called "almost rectangles", are tame. The proof is by induction (although the proof logic, as presented, is a bit different). Subsection III-B provides the base case and Subsection III-C handles the induction step. In Subsection III-D, we construct two tame matrices X_1, X_2 such that $\lambda(X_1) < 1/3$ and $\lambda(X_2) > 1/3$, but $t(X_1) = t(X_2)$ (because the corresponding columns of X_1 and X_2 will be evaluated by s to the same elements). This gives us a contradiction since $0_{p\times p} \not\sim 1_{p\times p}$ as we shall see.

A. Covers

Before launching into the proof, we introduce an additional concept and state a consequence of the fact that s is a polymorphism.

Definition III.3. A triple X, Y, Z of $p \times p$ zero-one matrices is called a *cover* if, for every $1 \leq i, j \leq p$, exactly one of x_{ij}, y_{ij}, z_{ij} is equal to one.

Lemma III.4. *If X, Y, Z is a cover, then X, Y, Z are not all g-equivalent.*

Proof. By the definition of a cover, the ij-th coordinates of X, Y, Z are in $\{(0,0,1),(0,1,0),(1,0,0)\} \subseteq R$ for each i, j. Since t preserves R (because s does), the triple $(t(X), t(Y), t(Z))$ is in R as well. Finally, g is a homomorphism from $\mathbb{C}$ to NAE-3, therefore $g(t(X)), g(t(Y)), g(t(Z))$ are not all equal. In other words, X, Y, Z are not all g-equivalent, as claimed. □

B. Line segments are tame

In this subsection it will be more convenient to regard the arguments of t as a tuple $\mathbf{x} = (x_{11}, x_{12}, \ldots)$ of length p^2 rather than a matrix. The concepts of the area, g-equivalence, tameness, and cover is extended to tuples in the obvious way. Since $p > 3$ is a prime number, p^2 is 1 modulo 3. Let q be such that

$$p^2 = 3q + 1 \ .$$

Moreover, let $\langle i \rangle$ denote the following tuple of length p^2.

$$\langle i \rangle = (\underbrace{1, 1, \cdots, 1}_{i\times}, 0, 0, \cdots 0)$$

We prove in this subsection that all such tuples are tame. We first recall a well-known fact.

Lemma III.5. *The operation t is cyclic.*

Proof. By cyclically shifting the arguments we get the same result:

$$t(x_{12}, \cdots, x_{pp}, x_{11}) = t\begin{pmatrix} x_{12} & x_{13} & \cdots & x_{1p} & x_{21} \\ x_{22} & x_{23} & \cdots & x_{2p} & x_{31} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{p2} & x_{p3} & \cdots & x_{pp} & x_{11} \end{pmatrix}$$
$$= t\begin{pmatrix} x_{21} & x_{12} & x_{13} & \cdots & x_{1p} \\ x_{31} & x_{22} & x_{23} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{11} & x_{p2} & x_{p3} & \cdots & x_{pp} \end{pmatrix}$$
$$= t\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} = t(x_{11}, x_{12}, \cdots, x_{pp}) \ ,$$

where the second equality uses the cyclicity of the outer "s" in the definition of t, while the third one the cyclicity of the first inner "s". □

The following lemma is proved by induction on $i = 0, 1, \ldots, q$.

Lemma III.6. *For each $i \in \{0, 1, \ldots, q\}$, we have*

$$\begin{aligned} &\langle q-i \rangle \sim \langle q-i+1 \rangle \sim \cdots \sim \langle q \rangle \\ &\not\sim \langle q+1 \rangle \sim \cdots \sim \langle q+i \rangle \sim \langle q+i+1 \rangle \ . \end{aligned}$$

Proof. For the first induction step, $i = 0$, let $\mathbf{x} = \langle q \rangle$, let $\mathbf{y}$ be $\langle q \rangle$ (cyclically) shifted q times to the right (so the first 1 is at the $(q+1)$-st position), and let $\mathbf{z}$ be $\langle q+1 \rangle$ shifted $2q$ times to the right. The tuples $\mathbf{x}, \mathbf{y}, \mathbf{z}$ form a cover, therefore they are not all g-equivalent by Lemma III.4. But t is cyclic, thus $t(\mathbf{x}) = t(\mathbf{y}) = t(\langle q \rangle)$ and $t(\mathbf{z}) = t(\langle q+1 \rangle)$. It follows that $\langle q \rangle, \langle q \rangle, \langle q+1 \rangle$ are not all g-equivalent and we get $\langle q \rangle \not\sim \langle q+1 \rangle$.

Now we prove the claim for $i > 0$ assuming it holds for $i-1$. To verify $\langle q-i \rangle \sim \langle q-i+1 \rangle$ consider $\langle q-i \rangle$, $\langle q+1 \rangle$, $\langle q+i \rangle$. Since $(q-i)+(q+1)+(q+i) = 3q+1 = p^2$, these tuples can be shifted to form a cover and then the same argument as above gives us that $\langle q-i \rangle$, $\langle q+1 \rangle$, $\langle q+i \rangle$ are not all g-equivalent. But $\langle q+1 \rangle \sim \langle q+i \rangle$ by the induction hypothesis, therefore $\langle q-i \rangle \not\sim \langle q+1 \rangle$. Since $\langle q+1 \rangle \not\sim \langle q-i+1 \rangle$ (again by the induction hypothesis), we get $\langle q-i \rangle \sim \langle q-i+1 \rangle$, as required.

It remains to check $\langle q+i \rangle \sim \langle q+i+1 \rangle$. This is done in a similar way, using the tuples $\langle q-i \rangle$, $\langle q \rangle$, $\langle q+i+1 \rangle$. □

We have proved that $\langle 0 \rangle \sim \cdots \sim \langle q \rangle \not\sim \langle q+1 \rangle \sim \cdots \sim \langle 2q+1 \rangle$. Using the same argument as in the previous lemma once more for $\langle 0 \rangle, \langle p^2 - i \rangle, \langle i \rangle$ with $p^2 \geq i > 2q+1$ we get $\langle i \rangle \not\sim \langle 0 \rangle$. In summary, $\langle i \rangle \sim \langle 0 \rangle$ whenever $i \leq q$ and $\langle i \rangle \sim \langle p^2 \rangle \not\sim \langle 0 \rangle$ when $i \geq q+1$. Observing that $\lambda(\langle i \rangle) < 1/3$ iff $i \leq q$ we obtain the following lemma.

Lemma III.7. *Each $\langle i \rangle$, $i \in \{0, 1, \cdots, p^2\}$, is tame and $\langle 0 \rangle \not\sim \langle p^2 \rangle$.*

C. Almost rectangles are tame

We start by introducing a special type of zero-one matrices.

Definition III.8. Let $1 \leq k_1, \ldots, k_p \leq p$. By

$$[k_1, k_2, \ldots, k_p]$$

we denote the matrix whose i-th column begins with k_i ones followed by $(p - k_i)$ zeros, for each $i \in \{1, \ldots, p\}$.

An *almost rectangle* is a matrix of the form $[k, k, \ldots, k, l, l, \ldots, l]$ (the number of k's can be arbitrary, including 0 or p) where $0 \leq k - l \leq 5|C|$. The quantity $k - l$ is referred to as the *size of the step.*

In the remainder of this subsection we prove the following proposition.

Proposition III.9. *Each almost rectangle is tame.*

Let

$$X = [\underbrace{k, k, \cdots, k}_{m\times}, l, l, \ldots, l]$$

be a minimal counterexample in the following sense.

- X has the minimum size of the step and,
- among such counterexamples, $|\lambda(X) - 1/3|$ is maximal.

Lemma III.10. *The size of the step of X is at least 2.*

Proof. This lemma is just a different formulation of Lemma III.7 since an almost rectangle with step of size 0 or 1 represents the same choice of arguments as $\langle i \rangle$ for some i. □

We handle two cases $\lambda(X) \geq 5/12$ and $\lambda(X) \leq 5/12$ separately, but the basic idea for both of them is the same as in the proof of Lemma III.7. To avoid puzzles, let us remark that any number strictly between $1/3$ and $1/2$ (instead of $5/12$) would work with a sufficiently large p.

Lemma III.11. *The area of X is less than $5/12$.*

Proof. Assume that $\lambda(X) \geq 5/12$. Let k_1, k_2, l_1, and l_2 be the non-negative integers such that

$$l_1 + l_2 + k = p = k_1 + k_2 + l, \tag{1}$$

$$1 \geq k_1 - k_2 \geq 0, \text{ and } 1 \geq l_1 - l_2 \geq 0 \ . \tag{2}$$

We have $k_1 \geq l_1$ and $k_2 \geq l_2$. Moreover, since $k - l \geq 2$ by the previous lemma, it follows that both $k_1 - l_1$ and $k_2 - l_2$ are strictly smaller than $k - l$.

Consider the matrices

$$Y_i = [\underbrace{l_i, l_i, \ldots, l_i}_{m\times}, k_i, k_i, \ldots, k_i], \quad i = 1, 2 \ .$$

By shifting all the rows of Y_i, $i \in \{1, 2\}$, m times to the left we obtain an almost rectangle with a smaller step size than X, which is thus tame by the minimality assumption on X. Since such a shift changes neither the value of t (as the outer "s" in the definition of t is cyclic) nor the area, both Y_1 and Y_2 are tame matrices.

Let Y_1' (Y_2', resp.) be the matrices obtained from Y_1 (Y_2, resp.) by shifting the first m columns k times ($k + l_1$ times, resp.) down and the remaining columns l times ($l + k_1$ times, resp.) down. Since X, Y_1', Y_2' is a cover (by (1)) and cyclically shifting columns does not change the value of t (as the inner occurrences of "s" in the definition of t are cyclic), Lemma III.4 implies that X, Y_1, Y_2 are not all g-equivalent.

From X, Y_1', Y_2' being a cover, it also follows that

$$\lambda(X) + \lambda(Y_1') + \lambda(Y_2') = \lambda(X) + \lambda(Y_1) + \lambda(Y_2) = 1 \ .$$

Moreover, by (2), we have $\lambda(Y_2) \leq \lambda(Y_1)$ and these areas differ by at most $p/p^2 = 1/p$. Therefore

$$\lambda(Y_1) = 1 - \lambda(X) - \lambda(Y_2) \leq 1 - 5/12 - \lambda(Y_1) + 1/p$$

and, since $p > 12$ by the choice of p, we obtain

$$\lambda(Y_2) \leq \lambda(Y_1) < 1/3 \ .$$

The tameness of Y_i now gives us $Y_1 \sim Y_2 \sim 0_{p\times p}$ and then, since Y_1, Y_2, X are not all g-equivalent and $0_{p\times p} \not\sim 1_{p\times p}$ (by the second part of Lemma III.7), we get $X \sim 1_{p\times p}$. But $\lambda(X) \geq 5/12 > 1/3$, hence X is tame, a contradiction with the choice of X. □

It remains to handle the case $\lambda(X) < 5/12$.

We first claim that $2k$ (and thus $k + l$ and $2l$) is less than p. Indeed, since the step size of X is at most $5|C|$ (by the definition of an almost rectangle) and $p > 60|C|$, we get

$$5/12 > \lambda(X) \geq \frac{p(k - 5|C|)}{p^2}$$
$$k \leq 5p/12 + 5|C| < 5p/12 + p/12 = p/2 \ .$$

We now again need to distinguish two cases. Assume first that $m < p/2$.

Let

$$Y = [\underbrace{l, \cdots, l}_{m\times}, \underbrace{k, \cdots, k}_{m\times}, l, \cdots, l],$$
$$Z = [\underbrace{p-k-l, \cdots, p-k-l}_{2m\times}, p-2l, \cdots, p-2l] \ .$$

The definition of Z makes sense since $p - k - l, p - 2l \geq 0$ by the inequality $2k < p$ derived above.

The triple X, Y, Z (similarly to X, Y_1, Y_2 in the proof of Lemma III.11) is such that we can obtain a cover by shifting the columns down. Therefore X, Y, Z are not all g-equivalent and $\lambda(X) + \lambda(Y) + \lambda(Z) = 1$.

On the other hand, by shifting all the rows of Y m times to the left we obtain X. We get $\lambda(X) = \lambda(Y)$ and $t(X) = t(Y)$, therefore $Z \not\sim X$ by the previous paragraph.

Moreover, by shifting all the rows of Z $2m$ times to the left we obtain an almost rectangle Z' with $t(Z) = t(Z')$ and $\lambda(Z) = \lambda(Z')$. The step size of Z' is $(p-2l) - (p-k-l) = k - l$, which is the same as the step size of X. However, the distance of its area from $1/3$ is strictly greater as shown by the following calculation.

$$\frac{|\lambda(Z) - 1/3|}{|\lambda(X) - 1/3|} = \frac{|(1 - 2\lambda(X)) - 1/3|}{|\lambda(X) - 1/3|}$$
$$= \frac{|2(1/3 - \lambda(X)|}{|\lambda(X) - 1/3|} = 2 > 1 \ .$$

By the minimality of X, the almost rectangle Z' is tame and so is Z. It is also apparent from the calculation that the signs of $\lambda(X) - 1/3$ and $\lambda(Z) - 1/3$ are opposite. Combing these two facts with $Z \not\sim X$ derived above, we obtain that X is tame, a contradiction.

In the other case, when $m > p/2$, the proof is similar using the tuples

$$Y = (l, \cdots, l, \underbrace{k, \cdots, k}_{m\times}),$$
$$Z = (\underbrace{p-k-l, \cdots, p-k-l}_{(p-m)\times}, p-2k, \cdots, p-2k,$$
$$\underbrace{p-k-l, \cdots, p-k-l}_{(p-m)\times}) \ .$$

The proof of Proposition III.9 is concluded.

D. Contradiction

Let

$$m = (p-1)/2$$

and choose natural numbers l_1 and l_2 so that

$$p/3 - 2|C| < l_1 < l_2 < p/3$$

and

$$s(\underbrace{1,\cdots,1}_{l_1\times},0,\cdots,0) = s(\underbrace{1,\cdots,1}_{l_2\times},0,\cdots,0) \ .$$

This is possible by the pigeonhole principle since there are $2|C| > C$ integers in the interval and $p/3 - 2|C| > 0$ by the choice of p.

The sought after contradiction will be obtained by considering the two matrices

$$X_i = [\underbrace{k,\dots,k}_{m\times}, l_i,\dots,l_i],\ i = 1,2 \ ,$$

where k will be specified soon.

Before choosing k, we observe that $t(X_1) = t(X_2)$. Indeed, the first m columns of these matrices are the same (and thus so are their images under s) and the remaining columns have the same image under s by the choice of l_1 and l_2. The claim thus follows from the definition of t.

Next, note that for $k \le p/3$ the area of both matrices is less than $1/3$ since $l_i < p/3$. On the other hand, for $k \ge p/3+3|C|$ the area is greater:

$$\begin{aligned}\lambda(X_i) &= \frac{mk + (p-m)l_i}{p^2} \\ &\ge \frac{\frac{p-1}{2}(p/3+3|C|) + \frac{p+1}{2}(p/3-2|C|)}{p^2} \\ &= \frac{p^2/3 + |C|(p-5)/2}{p^2} > 1/3 \ .\end{aligned}$$

Choose the maximum k so that $\lambda(X_1) < 1/3$. The derived inequalities and the choice of l_i implies

$$l_1 < l_2 \le k < p/3 + 3|C| \le l_1 + 5|C| < l_2 + 5|C| \ ,$$

therefore both X_1 and X_2 are almost rectangles. By Proposition III.9, X_1 and X_2 are tame.

Since the area of X_1 is less than $1/3$, we get $X_1 \sim 0_{p\times p}$. We chose k so that increasing k by 1 makes the area of X_1 greater than $1/3$. From $m < p/2$ it follows that increasing l_1 by 1 makes the area even greater, hence $\lambda(X_2) > 1/3$ (recall that $l_2 > l_1$) and we obtain $X_2 \sim 1_{p\times p}$.

Recall that $0_{p\times p} \not\sim 1_{p\times p}$ by the second part of Lemma III.7. Therefore $X_1 \not\sim X_2$, contradicting $t(X_1) = t(X_2)$.

IV. Conclusion

This paper shows that if 1-in-3 $\to \mathbb{C} \to$ NAE-3 and $\mathbb{C}$ is finite, then $\mathrm{CSP}(\mathbb{C})$ is NP-complete. The proof strategy is based on Theorem II.7 and a simple fact that, given $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$, each polymorphism of $\mathbb{C}$ induces a polymorphism of $(\mathbb{A},\mathbb{B})$ (by composition with the homomorphism $\mathbb{A} \to \mathbb{C}$ from the inside and with $\mathbb{C} \to \mathbb{B}$ from the outside).

There is an algebraic sufficient condition for NP-hardness for all ω-categorical structures [22] – $\mathrm{CSP}(\mathbb{C})$ is NP-hard whenever $\mathbb{C}$ does not have a *pseudo-Siggers* polymorphism, that is, a 6-ary polymorphism s such that

$$\alpha s(x,y,x,z,y,z) = \beta s(y,x,z,x,z,y) \text{ for all } x,y,z \in C \ ,$$

where α and β are unary polymorphisms of $\mathbb{C}$. Is it to possible to apply pseudo-Siggers operations to strengthen the main theorem?

Question IV.1. *Let $\mathbb{C}$ be an ω-categorical structure that pp-constructs* $(\text{1-in-3}, \text{NAE-3})$. *Is* $\mathrm{CSP}(\mathbb{C})$ *necessarily NP-hard?*

The proof of Theorem I.1 could be simplified if we had stronger or more suitable polymorphisms than cyclic operations. Alternative versions of Theorem II.7 could also help in simplifying the proof of the CSP dichotomy conjecture. In particular, the following question seems open.

Question IV.2. *Let $\mathbb{C}$ be a finite relational structure with a cyclic polymorphism of arity at least 2. Does $\mathbb{C}$ necessarily have a polymorphism s of arity $n > 1$ such that, for any $a,b \in C$ and $(x_1,\dots,x_n) \in \{a,b\}^n$, the value $s(x_1,\dots,x_n)$ depends only on the number of occurrences of a in $(x_1,\dots,x_n)$?*

Note that a more optimistic version involving evaluations with $|\{x_1,\dots,x_n\}| = 3$ is disproved by considering the polymorphisms of the disjoint union of a directed 2-cycle and a directed 3-cycle.

Let us finish with an optimistic outlook. While the main result of this paper is negative, its message is rather positive. It suggests that algebraic and analytical methods in the finite-domain CSP and PCSP should be combined with the model theoretic methods used for the infinite domains, and such a combination promises a significant synergy gain.

References

[1] T. J. Schaefer, "The complexity of satisfiability problems," in *Conference Record of the Tenth Annual ACM Symposium on Theory of Computing (San Diego, Calif., 1978)*. New York: ACM, 1978, pp. 216–226.

[2] J. Håstad, "Some optimal inapproximability results," *J. ACM*, vol. 48, pp. 798–859, 2001.

[3] J. Brakensiek and V. Guruswami, "New hardness results for graph and hypergraph colorings," in *Proceedings of the 31st Conference on Computational Complexity*, ser. CCC '16. Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016, pp. 14:1–14:27.

[4] J. Bulín, A. A. Krokhin, and J. Opršal, "Algebraic approach to promise constraint satisfaction," *CoRR*, vol. abs/1811.00970, 2018.

[5] J. Brakensiek and V. Guruswami, "Promise constraint satisfaction: Structure theory and a symmetric boolean dichotomy," in *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, ser. SODA '18. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2018, pp. 1782–1801.

[6] ——, "An algorithmic blend of LPs and ring equations for promise CSPs," *CoRR*, vol. abs/1807.05194, 2018, to appear in SODA'19.

[7] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric algorithms and combinatorial optimization*, 2nd ed., ser. Algorithms and Combinatorics. Springer-Verlag, Berlin, 1993, vol. 2.

[8] L. Barto, A. Krokhin, and R. Willard, "Polymorphisms, and How to Use Them," in *The Constraint Satisfaction Problem: Complexity and Approximability*, ser. Dagstuhl Follow-Ups, A. Krokhin and S. Zivny, Eds. Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017, vol. 7, pp. 1–44.

[9] A. A. Bulatov, "A dichotomy theorem for nonuniform CSPs," in *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, vol. 00, 2017, pp. 319–330.

[10] D. Zhuk, "A proof of CSP dichotomy conjecture," in *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, vol. 00, 2017, pp. 331–342.

[11] T. Feder and M. Y. Vardi, "The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory," *SIAM Journal on Computing*, vol. 28, no. 1, pp. 57–104, 1998.

[12] P. Jeavons, D. Cohen, and M. Gyssens, "Closure properties of constraints," *J. ACM*, vol. 44, no. 4, pp. 527–548, 1997.

[13] P. Jeavons, "On the algebraic structure of combinatorial problems," *Theoretical Computer Science*, vol. 200, no. 1–2, pp. 185 – 204, 1998.

[14] A. Bulatov, P. Jeavons, and A. Krokhin, "Classifying the complexity of constraints using finite algebras," *SIAM J. Comput.*, vol. 34, pp. 720–742, 2005.

[15] L. Barto, J. Opršal, and M. Pinsker, "The wonderland of reflections," *Israel Journal of Mathematics*, vol. 223, no. 1, pp. 363–398, 2018.

[16] L. Barto and M. Kozik, "Absorbing subalgebras, cyclic terms, and the constraint satisfaction problem," *Logical Methods in Computer Science*, vol. 8, no. 1, 2012.

[17] P. Austrin, V. Guruswami, and J. Håstad, "$(2+\varepsilon)$-Sat is NP-hard," *SIAM J. Comput.*, vol. 46, no. 5, pp. 1554–1573, 2017.

[18] P. Hell and J. Nešetřil, "On the complexity of H-coloring," *J. Combin. Theory Ser. B*, vol. 48, no. 1, pp. 92–110, 1990.

[19] M. Bodirsky and M. Grohe, "Non-dichotomies in constraint satisfaction complexity," in *Automata, Languages and Programming*, ser. Lecture Notes in Computer Science, L. Aceto, I. Damgard, L. A. Goldberg, M. M. Halldórsson, A. Ingólfsdóttir, and I. Walukiewicz, Eds. Springer Verlag, 2008, pp. 184 –196.

[20] M. Bodirsky, "Constraint satisfaction problems with infinite templates," in *Complexity of Constraints*, ser. Lecture Notes in Computer Science, N. Creignou, P. G. Kolaitis, and H. Vollmer, Eds., vol. 5250. Springer, 2008, pp. 196–228.

[21] M. Pinsker, "Algebraic and model theoretic methods in constraint satisfaction," *arXiv e-prints*, p. arXiv:1507.00931, 2015.

[22] L. Barto and M. Pinsker, "The algebraic dichotomy conjecture for infinite domain constraint satisfaction problems," in *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, ser. LICS '16. New York, NY, USA: ACM, 2016, pp. 615–622.

[23] M. Bodirsky and M. Mamino, "Constraint Satisfaction Problems over Numeric Domains," in *The Constraint Satisfaction Problem: Complexity and Approximability*, ser. Dagstuhl Follow-Ups, A. Krokhin and S. Zivny, Eds. Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017, vol. 7, pp. 79–111.

[24] M. H. Siggers, "A strong Mal'cev condition for locally finite varieties omitting the unary type," *Algebra universalis*, vol. 64, no. 1-2, pp. 15–20, 2010.

[25] K. Kearnes, P. Marković, and R. McKenzie, "Optimal strong Mal'cev conditions for omitting type 1 in locally finite varieties," *Algebra universalis*, vol. 72, no. 1, pp. 91–100, 2014.

[26] M. Maróti and R. McKenzie, "Existence theorems for weakly symmetric operations," *Algebra Universalis*, vol. 59, no. 3-4, pp. 463–489, 2008.