Set functors determined by values on objects

Libor Barto

Charles University in Prague,
Czech Republic

OAL Nashville 2007

Set functors

DEFINITION Set functor = a functor **Set** $\to$ **Set**.

EXAMPLE Let $\mathbb{V}$ be a variety.

- $F_{\mathbb{V}}$ – Free functor: $F_{\mathbb{V}}(X)$ is the (underlying set of the) free algebra = terms over X modulo identities in $\mathbb{V}$.

Set functors

DEFINITION Set functor = a functor **Set** $\to$ **Set**.

EXAMPLE Let $\mathbb{V}$ be a variety.

- $F_{\mathbb{V}}$ – Free functor: $F_{\mathbb{V}}(X)$ is the (underlying set of the) free algebra = terms over X modulo identities in $\mathbb{V}$.
- $F^1_{\mathbb{V}}$ – Subfunctor of $F_{\mathbb{V}}$: $F^1_{\mathbb{V}}$ is the set of all terms over X of height at most 1 modulo identities in $\mathbb{V}$.

DVO functors

DEFINITION A set functor F is DVO (Determined by Values on Objects), if $F \cong G$ for every set functor G with $|F(X)| = |G(X)|$ for all X.

DVO functors

DEFINITION A set functor F is DVO (Determined by Values on Objects), if $F \cong G$ for every set functor G with $|F(X)| = |G(X)|$ for all X.

OPEN PROBLEM Find all DVO functors!

DVO functors

DEFINITION A set functor F is DVO (Determined by Values on Objects), if $F \cong G$ for every set functor G with $|F(X)| = |G(X)|$ for all X.

OPEN PROBLEM Find all DVO functors!

EXAMPLES OF DVO FUNCTORS

- Subfunctors of $F_{\mathbb{V}}$, where $\mathbb{V}$ = semilattices. (i.e. $F_{\mathbb{V}}$ = power set functor)

DVO functors

DEFINITION A set functor F is DVO (Determined by Values on Objects), if $F \cong G$ for every set functor G with $|F(X)| = |G(X)|$ for all X.

OPEN PROBLEM Find all DVO functors!

EXAMPLES OF DVO FUNCTORS

- Subfunctors of $F_{\mathbb{V}}$, where $\mathbb{V}$ = semilattices. (i.e. $F_{\mathbb{V}}$ = power set functor)
- Subfunctors of idempotent reduct of $F_{\mathbb{V}}$, where $\mathbb{V}$ = boolean groups. ("power set functor avoiding even numbers")

Known results

Known results $\Rightarrow$ suffices to study finitary faithful connected functors = functors $F^1_{\mathbb{V}}$ for (finitary) idempotent varieties $\mathbb{V}$

Known results

Known results $\Rightarrow$ suffices to study finitary faithful connected functors = functors $F^1_{\mathbb{V}}$ for (finitary) idempotent varieties $\mathbb{V}$

- Every DVO functor is finitary. A. Barkhudaryan, R. El Bashir, V.Koubek, V. Trnková 03, 07

Known results

Known results $\Rightarrow$ suffices to study finitary faithful connected functors = functors $F^1_{\mathbb{V}}$ for (finitary) idempotent varieties $\mathbb{V}$

- Every DVO functor is finitary. A. Barkhudaryan, R. El Bashir, V.Koubek, V. Trnková 03, 07
- nonfaithful DVO functors are characterized. The same authors

Known results

Known results $\Rightarrow$ suffices to study finitary faithful connected functors = functors $F^1_{\mathbb{V}}$ for (finitary) idempotent varieties $\mathbb{V}$

- Every DVO functor is finitary. A. Barkhudaryan, R. El Bashir, V.Koubek, V. Trnková 03, 07
- nonfaithful DVO functors are characterized. The same authors
- result for nonconnected functors can be obtained from the connected case

Increases

DEFINITION Let F be a set functor (connected finitary faithful), $X_n = \{x_1, \dots, x_n\}$ be a set. Increase of F on X_n (notation $F'(X_n)$) is the set of all essentially n-ary elements of $F(X_n)$.

Increases

DEFINITION Let F be a set functor (connected finitary faithful), $X_n = \{x_1, \dots, x_n\}$ be a set. Increase of F on X_n (notation $F'(X_n)$) is the set of all essentially n-ary elements of $F(X_n)$.

PROPOSITION $|F(X_n)| = \sum_{i=1}^{n} \binom{n}{i} |F'(X_i)|$

Increases

DEFINITION Let F be a set functor (connected finitary faithful), $X_n = \{x_1, \dots, x_n\}$ be a set. Increase of F on X_n (notation $F'(X_n)$) is the set of all essentially n-ary elements of $F(X_n)$.

PROPOSITION $|F(X_n)| = \sum_{i=1}^{n} \binom{n}{i} |F'(X_i)|$

The sequence

$$\langle \left|F'(X_2)\right|, \left|F'(X_3)\right|, \left|F'(X_4)\right|, \dots \rangle$$

is called increase sequence of F.

It follows that, for all n, $|F(X_n)|$ is determined by the increase sequence.

Increase sequences of some set functors I

$\langle 0,0,0,0,0,0,0,\dots\rangle$ is the increase sequence of a DVO functor

$\langle 1,0,0,0,0,0,0,\dots\rangle$ DVO

$\langle 1,1,0,0,0,0,0,\dots\rangle$ not DVO

$\langle 1,1,1,0,0,0,0,\dots\rangle$ DVO

$\langle 1,1,1,1,0,0,0,\dots\rangle$ DVO

$\dots$

$\langle 1,1,1,1,\dots,\rangle$ not DVO

Increase sequences of some set functors I

$\langle 0,0,0,0,0,0,0,\dots\rangle$ is the increase sequence of a DVO functor
$\langle 1,0,0,0,0,0,0,\dots\rangle$ DVO
$\langle 1,1,0,0,0,0,0,\dots\rangle$ not DVO
$\langle 1,1,1,0,0,0,0,\dots\rangle$ DVO
$\langle 1,1,1,1,0,0,0,\dots\rangle$ DVO
$\dots$
$\langle 1,1,1,1,\dots,\rangle$ not DVO

$\langle 0,1,0,0,0,0,0,\dots\rangle$ not DVO
$\langle 0,1,0,1,0,0,0,\dots\rangle$ DVO
$\langle 0,1,0,1,0,1,0,\dots\rangle$ DVO
$\dots$
$\langle 0,1,0,1,0,1,\dots\rangle$ not DVO

Increase sequences of some set functors II

$\langle 1, 1, 1, 1, 1, 0, 0, \dots \rangle$ DVO
$\langle 2, 1, 1, 1, 1, 0, 0, \dots \rangle$ DVO
$\langle 1, 2, 1, 1, 1, 0, 0, \dots \rangle$ not DVO
$\langle 1, 1, 2, 1, 1, 0, 0, \dots \rangle$ DVO
$\langle 1, 1, 1, 2, 1, 0, 0, \dots \rangle$ DVO
$\langle 1, 1, 1, 1, 2, 0, 0, \dots \rangle$ not DVO

Increase sequences of some set functors II

$\langle 1,1,1,1,1,0,0,\dots\rangle$ DVO
$\langle 2,1,1,1,1,0,0,\dots\rangle$ DVO
$\langle 1,2,1,1,1,0,0,\dots\rangle$ not DVO
$\langle 1,1,2,1,1,0,0,\dots\rangle$ DVO
$\langle 1,1,1,2,1,0,0,\dots\rangle$ DVO
$\langle 1,1,1,1,2,0,0,\dots\rangle$ not DVO

$\langle 3,1,1,1,1,0,0,\dots\rangle$ not DVO
$\langle 1,3,1,1,1,0,0,\dots\rangle$ not DVO
$\langle 1,1,3,1,1,0,0,\dots\rangle$ not DVO
$\langle 1,1,1,3,1,0,0,\dots\rangle$ not DVO
$\langle 1,1,1,1,3,0,0,\dots\rangle$ not DVO

Increase sequences of some set functors III

$\langle 1, 1, 1, 0, 0, 0, \binom{8}{1}, 0, 0, \dots \rangle$ DVO

$\langle 1, 1, 1, 0, 0, 0, \binom{8}{2}, 0, 0, \dots \rangle$ DVO

$\langle 1, 1, 1, 0, 0, 0, \binom{8}{1} + \binom{8}{2}, 0, 0, \dots \rangle$ DVO

$\langle 1, 1, 1, 0, 0, 0, \binom{8}{3}, 0, 0, \dots \rangle$ not DVO

$\langle 1, 1, 1, 0, 0, \binom{7}{1}, 0, 0, \dots \rangle$ DVO

$\langle 1, 1, 1, 0, 0, \binom{7}{2}, 0, 0, \dots \rangle$ not DVO

Increase sequences of some functors IV - 2^ω

...

...

...

...

...

...

...

...

...

...

Is the list complete?

QUESTION Let $\mathbb{V}$ be an idempotent nontrivial variety. Let

- $t(x_1, \ldots, x_n)$ be a term in $\mathbb{V}$,
- G be the stabilizer of t in $\mathbb{V}$:

$$G = \{g \in S_n \mid t(x_1, x_2, \ldots, x_n) \approx_{\mathbb{V}} t(x_{g(1)}, x_{g(2)}, \ldots, x_{g(n)})\}$$

- $2 \leq k$ be an odd integer $\leq$ the size of the smallest orbit of G.

Must there be a mapping $r : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ such that the term $t(x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)})$ depends on all of the variables $x_1, \ldots x_k$?

Is the list complete?

QUESTION Let $\mathbb{V}$ be an idempotent nontrivial variety. Let

- $t(x_1, \ldots, x_n)$ be a term in $\mathbb{V}$,
- G be the stabilizer of t in $\mathbb{V}$:

$$G = \{g \in S_n \mid t(x_1, x_2, \ldots, x_n) \approx_{\mathbb{V}} t(x_{g(1)}, x_{g(2)}, \ldots, x_{g(n)})\}$$

- $2 \leq k$ be an odd integer $\leq$ the size of the smallest orbit of G.

Must there be a mapping $r : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ such that the term $t(x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)})$ depends on all of the variables $x_1, \ldots x_k$?

PROPOSITION Yes for $k = 3$.

In particular

QUESTION Let $\mathbb{V}$ be an idempotent nontrivial variety, let

- $t(x_1, \ldots, x_n)$ be a term in $\mathbb{V}$,
- G be the stabilizer of t in $\mathbb{V}$:

$$G = \{g \in S_n \mid t(x_1, x_2, \ldots, x_n) \approx_{\mathbb{V}} t(x_{g(1)}, x_{g(2)}, \ldots, x_{g(n)})\}$$

- $2 \leq k \leq n$ be an odd integer

Assume that G is transitive.
Must there be a mapping $r : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ such that the term $t(x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)})$ depends on all of the variables $x_1, \ldots x_k$?

In particular

QUESTION Let $\mathbb{V}$ be an idempotent nontrivial variety, let

- $t(x_1, \ldots, x_n)$ be a term in $\mathbb{V}$,
- G be the stabilizer of t in $\mathbb{V}$:

$$G = \{g \in S_n \mid t(x_1, x_2, \ldots, x_n) \approx_{\mathbb{V}} t(x_{g(1)}, x_{g(2)}, \ldots, x_{g(n)})\}$$

- $2 \leq k \leq n$ be an odd integer

Assume that G is transitive.
Must there be a mapping $r : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ such that the term $t(x_{r(1)}, x_{r(2)}, \ldots, x_{r(n)})$ depends on all of the variables $x_1, \ldots x_k$?

PROPOSITION Yes, if G contains n-cycle.

What can be said about even ks

QUESTION Let $\mathbb{V}$ be an idempotent nontrivial variety. Let

- $t(x_1, \dots, x_n)$ be a term in $\mathbb{V}$,
- G be the stabilizer of t in $\mathbb{V}$,
- $2 \leq k$ be an odd integer $\leq$ the size of the smallest orbit of G.

Must there be a mapping $r : \{1, \dots, n\} \to \{1, 2, \dots, k\}$ such that the term $t(x_{r(1)}, x_{r(2)}, \dots, x_{r(n)})$ depends on all of the variables $x_1, \dots x_k$?

The main step in "Characterization of p_n-Sequences for Nonidempotent Algebras" is

THEOREM Kisielewicz 87 For even n and $k = 2$, if the answer is negative, then $\left|F'_{\mathbb{V}}(X_n)\right|$ is a linear combination of $\binom{n}{1}, \binom{n}{3}, \dots$ with nonnegative integer coefficients.

web: http://www.karlin.mff.cuni.cz/~barto

Thank you for your attention!