The category of varieties is alg-universal

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$\mathcal{L}$ - The lattice of interpretability types of varieties

W. D. Neumann (74), O. C. García, W. Taylor(84)

"DEFINITION" A variety $\mathbb{V}$ is interpretable in a variety $\mathbb{W}$, if we can assign to every operational symbol in $\mathbb{V}$ a term in $\mathbb{W}$ (with the same arity) in such a way that the terms satisfy all equations valid in $\mathbb{V}$.

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- A variety $\mathbb{V}$ with one ternary operation m satisfying $m(x, y, y) \approx m(y, y, x) \approx x$ is is interpretable in $\mathbb{W}$, iff there exists a Mal'cev term in $\mathbb{W}$ (ternary term t such that $t(x, y, y) \approx t(y, y, x) \approx x$).

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This gives a quasiorder $\longrightarrow$ a poset, in fact a lattice – the lattice $\mathcal{L}$

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They investigated the category of varieties instead of the lattice $\mathcal{L}$.

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Var-objects: varieties (finitary, monosorted), signature is NOT fixed

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The following categories are equivalent

- Varieties and interpretations
- The dual of the category of varieties and concrete functors
- (Abstract) clones and clone homomorphisms
- (Finitary) monads over **Set** and monad homomorphisms

Alg-universality

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EXAMPLES **Graph**, $\mathbf{Alg}(2)$, $\mathbf{Alg}(1,1)$ Hedrlín, Pultr 66

The result

THEOREM The category **IdempVar** of idempotent varieties is alg-universal.

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  - Every poset can be embedded into the lattice $\mathcal{L}$ (known before).
- Class many pairwise incomparable varieties in $\mathcal{L}$ exists, iff the Vopěnka principle doesn't hold.
- Every concrete category can be fully embedded into **IdempVar**, iff the class of measurable cardinals is a set. Hedrlín, Kučera, Pultr 73

Proof

Suffices to embed any alg-universal category.

- Full embedding $\mathbf{Alg}(1,1) \to \mathbf{Alg}_*(1,1)$ – full subcategory formed by algebras (A, α, β) such that $a, \alpha(a), \beta(a)$ are pairwise distinct for all $a \in A$.

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(D1) $c_a(x, 18y) \approx d_a(x, y)$

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$f : (A, \alpha, \beta) \to (A', \alpha', \beta')$

$\Phi(f)$: $c_a \to c_{f(a)}$, $d_a \to d_{f(a)}$

Thank you for your attention!