CSP and NU(4)

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Everything is finite
CSP for digraphs

**DEFINITION**  Fix a directed graph (digraph) $H - \textit{template}$. $CSP(H)$ is the following decision problem:

**INPUT:** Digraph $G$

**OUTPUT:** Is there a homomorphism $G \rightarrow H$?
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**EXAMPLES** $k$-Colorability, $k$-Sat, SysLinEq
Universal algebra in CSP

Bulatov, Cohen, Gyssens, Jeavons, Krokhin 97, 05
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template digraph $\mathbb{H} = (H, E)$

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Algebraic dichotomy conjecture

**THEOREM** If $\text{HSP}(H)$ contains a trivial algebra (every operation is a projection), then $\text{CSP}(H)$ is NP-complete.

**CONJECTURE** Otherwise $\text{CSP}(H)$ is in $P$.

**FACT** TFAE

- $\text{HSP}(H)$ doesn’t contain a trivial algebra
- $\text{HSP}(H)$ omits type 1 (Hobby, McKenzie)
- $\text{HSP}(H)$ satisfies some nontrivial idempotent Malcev condition
(2,3)-Consistency checking

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Initialization

- For an edge $A \rightarrow B$ in $G$, put $M_{AB} =$ edges of $H$
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**Repeat the following until nothing can be deleted**

- For all pairs $A \neq B \in G$, $(x, y) \in M_{AB}$ and $C \in G$:
  - If there is no $z \in H$ such that $(x, z) \in M_{AC}$ and $(y, z) \in M_{BC}$, then delete $(x, y)$ from $M_{AB}$
OBSERVATION  This can be done in polynomial time
Definition of bounded width

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- or each $M_{AB}$ is non-empty. Then ???.
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**DEFINITION** $H$ has width $(2, 3)$. If for every $G$

After the $(2, 3)$-consistency checking all $M_{AB}$ non-empty

$\Rightarrow G \rightarrow H$. 
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Similarly width $(k, l)$, bounded width.
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Bounded width $\Rightarrow$ Polynomially solvable

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CSP and NU(4)
Bounded width conjecture

THEOREM Larose, Zádori If $H$ has bounded width, then $\text{HSP}(H)$ doesn’t contain a reduct of a module (over finite ring)
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**Theorem** Larose, Zádori If $H$ has bounded width, then $HSP(H)$ doesn’t contain a reduct of a module (over finite ring).

**Conjecture** The converse is also true.

**Fact** TFAE

- $HSP(H)$ doesn’t contain a reduct of a module
- $HSP(H)$ omits types 1, 2
- All algebras in $HSP(H)$ are $SD(\wedge)$ (meet semidistributive congr. lat.).
- Malcev conditions...
Known results

**THEOREMS**

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- \( H \) has an \( NU(3) \) term (majority) \( \Rightarrow \) \( H \) has width \((2, 3)\) (in a very strong sense - \textit{strict width})
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- \( H \) has an \( NU(4) \) term \( \Rightarrow H \) has width \( (3, 4), NU(5) \Rightarrow (4, 5), \ldots \)
Known results

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- $H$ has an $NU(4)$ term $\Rightarrow H$ has width $(3, 4)$, $NU(5) \Rightarrow (4, 5)$, ...
- Kiss, Valeriote $H$ has $CD(3)$ terms $\Rightarrow H$ has bounded width
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- Carvalho, Dalmau, Marković, Maróti $H$ has $CD(4)$ terms $\Rightarrow \overline{H}$ has bounded width
- Bulatov $H$ has 2-semilattices term $\Rightarrow \overline{H}$ has bounded width
- Bulatov $HSP(H)$ omits 1, 2, 3 $\Rightarrow \overline{H}$ has bounded width
Our result

**OPEN PROBLEM** Is there a graph $\mathcal{H}$, which has width $(3, 4)$, but not $(2, 3)$?
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THEOREM Barto, Kozik If $H$ has $NU(4)$ term, then $H$ has width $(2, 3)$ in a stronger sense:
OPEN PROBLEM  Is there a graph $H$, which has width $(3, 4)$, but not $(2, 3)$?

THEOREM  Barto, Kozik  If $H$ has NU(4) term, then $H$ has width $(2, 3)$ in a stronger sense:

Let $G$ be a graph, $M_{AB}$ the sets of possible values for pairs as before. For all $A, B$ vertices of $G$ and $(x, y) \in M_{AB}$ there is a homomorphism $f : G \rightarrow H$ such that $f(A) = x, f(B) = y$. 
web: http://www.karlin.mff.cuni.cz/~barto

Thank you for your attention!