CSP and NU(4)

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Everything is finite

CSP for digraphs

DEFINITION Fix a directed graph (digraph) $\mathbb{H}$ - *template*.
CSP($\mathbb{H}$) is the following decision problem:

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EXAMPLES k-Colorability, k-Sat, SysLinEq

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Bulatov, Cohen, Gyssens, Jeavons, Krokhin 97, 05

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FACT The complexity of $CSP(\mathbb{H})$ depends only on $\mathbf{H}$

SUSPISION The complexity of $CSP(\mathbb{H})$ depends only on $\mathrm{HSP}(\mathbf{H})$

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FACT TFAE

- $\mathrm{HSP}(\mathbf{H})$ doesn't contain a trivial algebra
- $\mathrm{HSP}(\mathbf{H})$ omits type 1 (Hobby, McKenzie)
- $\mathrm{HSP}(\mathbf{H})$ satisfies some nontrivial idempotent Malcev condition

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Repeat the following until nothing can be deleted

- For all pairs $A \neq B \in G$, $(x, y) \in M_{AB}$ and $C \in G$:

  If there is no $z \in H$ such that $(x, z) \in M_{AC}$ and $(y, z) \in M_{BC}$, then delete (x, y) from M_{AB}

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Bounded width $\Rightarrow$ Polynomially solvable

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FACT TFAE

- $\mathrm{HSP}(\mathbf{H})$ doesn't contain a reduct of a module
- $\mathrm{HSP}(\mathbf{H})$ omits types $1, 2$
- All algebras in $\mathrm{HSP}(\mathbf{H})$ are $SD(\wedge)$ (meet semidistributive cong. lat.).
- Malcev conditions...

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- Bulatov **H** has 2-semilattices term $\Rightarrow \mathbb{H}$ has bounded width
- Bulatov $\mathrm{HSP}(\mathbf{H})$ omits $1,2,3 \Rightarrow \mathbb{H}$ has bounded width

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Let $\mathbb{G}$ be a graph, $M_{\dots}$ the sets of possible values for pairs as before. For all A, B vertices of $\mathbb{G}$ and $(x,y) \in M_{AB}$ there is a homomorphism $f : \mathbb{G} \to \mathbb{H}$ such that $f(A) = x, f(B) = y$.

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Thank you for your attention!