THE CSP DICHOTOMY HOLDS FOR DIGRAPHS WITH NO SOURCES AND NO SINKS
(A POSITIVE ANSWER TO A CONJECTURE OF BANG-JENSEN AND HELL)

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ABSTRACT. Bang-Jensen and Hell conjectured in 1990 (using the language of graph homomorphisms) a CSP dichotomy for digraphs with no sources or sinks. The conjecture states that the constraint satisfaction problem for such a digraph is tractable if each component of its core is a circle and is NP-complete otherwise. In this paper we prove this conjecture, and, as a consequence, a conjecture of Bang-Jensen, Hell and MacGillivray from 1995 classifying hereditarily hard digraphs. Further, we show that the CSP dichotomy for digraphs with no sources or sinks agrees with the algebraic characterization conjectured by Bulatov, Jeavons and Krokhin in 2005.

1. INTRODUCTION

The history of the Constraint Satisfaction Problem (CSP) goes back more than thirty years and begins with the work of Montanari [Mon74] and Mackworth [Mac77]. Since that time many combinatorial problems in artificial intelligence and other areas of computer science have been formulated in the language of CSPs. The study of such problems, under this common framework, has applications in database theory [Var00], machine vision recognition [Mon74], temporal and spatial reasoning [SV98], truth maintenance [DD96], technical design [NL], scheduling [LALW98], natural language comprehension [All94] and programming language comprehension [Nad]. Numerous attempts to understand the structure of different CSPs has been undertaken and a wide variety of tools ranging from statistical physics (e.g. [ANP05, KMRT+07]) to universal algebra (e.g. [JCG97]) has been employed. Methods and results developed in seemingly disconnected branches of mathematics transformed the area. The conjecture proved in this paper resisted the approaches based in combinatorics and theoretical computer science for nearly twenty years. Only recent developments in the structural theory of finite algebras provided tools, strong enough, to solve this problem.

For the last ten years the study of CSP has also been a driving force in theoretical computer science. The dichotomy conjecture of Feder and Vardi, published in [FV99], has origins going back to 1993. The conjecture states that a constraint satisfaction problem, for any fixed language, is either NP-complete or solvable in polynomial time. Therefore the class of CSPs would be a subclass of NP avoiding problems of intermediate difficulty. The logical characterization of the class of constraint satisfaction problems (see [FV99] and [Kun]) provides arguments in support of the dichotomy, nevertheless the conjecture remains open.

One of the results of the above mentioned [FV99] shows that the CSP dichotomy conjecture is equivalent to the CSP dichotomy conjecture restricted to digraphs. Therefore the constraint satisfaction problems can be defined in terms of (di)graph homomorphisms studied in graph theory for over thirty years (e.g. [Lev73]). It adds a new dimension to a well established problem and shows the importance of solving CSPs for digraphs. The classification of the complexities of the **H**-coloring problems for undirected graphs, discovered by Hell and Nešetřil [HN90], is an important step and provides a starting point towards proving, or refuting, the CSP dichotomy conjecture. There has since appeared many papers on the complexity of digraph coloring problems (see, e.g. [BJH90, BJHM95, Fed01, GWW92, HNZ96a, HNZ96b, HNZ96c, HZZ93, Mac91, Zhu95]), but as yet, no plausible conjecture on a graph theoretical classification has been proposed. Bang-Jensen and Hell [BJH90] did, however, conjecture a classification (implying the dichotomy) for the class of digraphs with no sources or sinks. Their conjecture significantly generalizes the result of Hell and Nešetřil.

In 1995 Bang-Jensen, Hell and MacGillivray (in [BJHM95]) introduced the notion of hereditarily hard digraphs and conjectured their classification. Surprisingly, they were able to show that this conjecture and the one given in [BJH90] are equivalent. In this paper we prove the conjecture of Bang-Jensen and Hell and, as a consequence, the conjecture of Bang-Jensen, Hell and MacGillivray.

Our paper relies on the interconnection between CSP and algebra as first discovered by Jeavons, Cohen and Gyssens in [JCG97] and refined by Bulatov, Jeavons and Krokhin in [BJK05]. Using this connection Bulatov, Jeavons and Krokhin conjectured a full classification of the NP-complete constraint satisfaction problems. For a small taste of results in the direction of proving this classification see [BIM$^+$06, Bul06, Dal05, Dal06, KV07]. A particularly interesting example, demonstrating the potency of the algebraic approach, is Bulatov's proof of the result of Hell and Nešetřil (see [Bul05]). A recent, purely algebraic result of Maróti and McKenzie [MM07] is one of the key ingredients in the proof of conjecture of Bang-Jensen and Hell. This provides further evidence supporting the extremely strong bond between the Constraint Satisfaction Problem and universal algebra.

2. Preliminaries

We assume the reader possesses a basic knowledge of universal algebra and graph theory. For an easy introduction to the notions of universal algebra that are not defined in this paper (e.g. terms, powers, subalgebras), we invite the reader to consult the monographs [BS81] and [MMT87]. Further information, concerning the structural theory of finite algebras (called tame congruence theory) can be found in [HM88]. For an explanation of the basic terms in graph theory and graph homomorphisms, we recommend [HN04]. Finally, for an introduction to the connections between universal algebra and CSP we recommend [BJK05].

Throughout the paper we deviate from the standard definition of the constraint satisfaction problem, with respect to a fixed language (found in e.g. [BKJ00]), in favor of an equivalent definition (from [FV99, LZ06]). A relational structure $\mathcal{T} = (T, R)$ is an ordered pair where T is a finite non-empty set and R is a finite set of finitary relations on T indexed by a set J. Let d_j denote the arity of the relation $r_j \in R$. The indexed set of all the d_j constitutes the *signature* of $\mathcal{T}$. For two relational structures of the same signature, say $\mathcal{T} = (T, R)$ and $\mathcal{U} = (U, S)$, a map $h : T \to U$ is a homomorphism if $h(t_j) \subseteq s_j$ for all $j \in J$ (where $h(t_j)$ is computed pointwise). A *polymorphism* of a relational structure $\mathcal{T}$ is a homomorphism from a finite cartesian power, say $\mathcal{T}^n$, to $\mathcal{T}$. Precisely, a polymorphism h of $\mathcal{T}$, is an

operation $h : T^n \to T$, such that, for all relations $r \in R$, of arity m, if

$$(a_{i,0}, a_{i,1}, \ldots, a_{i,m-1}) \in r \text{ for all } i < n,$$

then

$$(h(a_{0,0}, a_{1,0}, \ldots, a_{n-1,0}), \ldots, h(a_{0,m-1}, a_{1,m-1}, \ldots, a_{n-1,m-1})) \in r.$$

A digraph is a pair $\mathbf{G} = (V, E)$ where V is a set of vertices and $E \subseteq V \times V$ is a set of edges (note that a digraph is not, by definition, a relational structure, however the difference is only a technicality). A vertex of a digraph is called a *source* (resp. a *sink*), if it has no incoming (resp. outgoing) edges. A closed directed path in a digraph is called a *cycle*, while a *circle* is a cycle with no proper subcycles. A *loop* is an edge from a vertex to itself (i.e. a cycle of length one). Given a digraph $\mathbf{G}$, we sometimes denote the set of vertices of $\mathbf{G}$ by $V(\mathbf{G})$ and similarly the edges of $\mathbf{G}$ by $E(\mathbf{G})$. A homomorphism between two digraphs is a map between the sets of vertices that preserves the edges.

A graph is 3-colorable if and only if it maps homomorphically into the complete graph on three vertices (without loops). The notion of *colorability* is generalized using graph homomorphisms: a digraph, say $\mathbf{G}$, is $\mathbf{H}$-colorable if there exists a homomorphism mapping $\mathbf{G}$ to $\mathbf{H}$.

A digraph $\mathbf{G} = (V, E)$ *retracts* to an induced subgraph $\mathbf{H} = (W, F)$ if there is an endomorphism $h : V \to V$ such that $h(V) = W$ and $h(a) = a$ for all $a \in W$. Such a map h is called a *retraction*. A *core* of a digraph is a minimal induced subgraph to which the digraph retracts (the definition of retraction and core clearly generalize to relational structures). It is a trivial fact that, for any digraph $\mathbf{H}$, and for a core of $\mathbf{H}$, say $\mathbf{H}'$, the set of $\mathbf{H}$-colorable digraphs coincides with the set of $\mathbf{H}'$-colorable digraphs.

In this paper all relational structures, digraphs and algebras are assumed to be finite.

3. The main result

For a relational structure $\mathcal{T} = (T, R)$ we define the language $\mathrm{CSP}(\mathcal{T})$, of relational structures with the same signature as $\mathcal{T}$, to be

$$\mathrm{CSP}(\mathcal{T}) = \{\, \mathcal{U} \mid \text{there is a homomorphism from } \mathcal{U} \text{ to } \mathcal{T} \,\}.$$

Alternatively we can view $\mathrm{CSP}(\mathcal{T})$ as a decision problem:

INPUT: a relational structure $\mathcal{U}$ with the same signature as $\mathcal{T}$
QUESTION: does there exists a homomorphism from $\mathcal{U}$ to $\mathcal{T}$?

In either approach we are concerned with the computational complexity (of membership of the language, or of the decision problem respectively) for a given relational structure. The CSP dichotomy conjecture proposed in [FV99] states:

The CSP dichotomy conjecture. *For a relational structure $\mathcal{T}$ the problem $\mathrm{CSP}(\mathcal{T})$ is either NP-complete or solvable in polynomial time.*

The (di)graph coloring problems can be viewed as special cases of CSP. Although a digraph $\mathbf{H} = (W, F)$ is technically different from a relational structure, the set of $\mathbf{H}$-colorable digraphs is obviously polynomially equivalent to the CSP for an appropriate relational structure and therefore we denote the class of all $\mathbf{H}$-colorable digraphs by $\mathrm{CSP}(\mathbf{H})$. Due to the reduction presented in [FV99] every constraint satisfaction problem is polynomially equivalent to a digraph homomorphism problem. Thus we can restate the CSP dichotomy conjecture in the following way:

The CSP dichotomy conjecture. *For a fixed digraph $\mathbf{H}$, deciding whether a give digraph is $\mathbf{H}$-colorable is either NP-complete or solvable in polynomial time.*

This brings us to the main problem of the paper, a conjecture nearly ten years older than the CSP dichotomy conjecture, and a special case of it. It deals with digraphs with no sources or sinks and was first formulated by Bang-Jensen and Hell in [BJH90]:

The conjecture of Bang-Jensen and Hell. *Let* **H** *be a digraph without sources or sinks. If each component of the core of* **H** *is a circle, then* CSP(**H**) *is polynomially decidable. Otherwise* CSP(**H**) *is NP-complete.*

Note that the above conjecture is a substantial generalization of the **H**-coloring result of Hell and Nešetřil [HN90].

The notion of hereditarily hard digraphs was introduced by Bang-Jensen, Hell and MacGillivray in [BJHM95]. A digraph **H** is said to be hereditarily hard if the **H**′-coloring problem is NP-complete for all loopless digraphs **H**′ that contain **H** as a subgraph. The following conjecture was posed and shown to be equivalent to the Bang-Jensen and Hell conjecture in [BJHM95]:

The conjecture of Bang-Jensen, Hell and MacGillivray. *Let* **H** *be a digraph. If the digraph* $R(\mathbf{H})$ *(which is obtained by iteratively removing the sources and sinks from* **H** *until none remain) does not admit a homomorphism to a circle of length greater than one, then* **H** *is hereditarily hard. Otherwise* **H** *has a polynomial extension.*

In this section we prove the Bang-Jensen and Hell conjecture and therefore the conjecture of Bang-Jensen, Hell and MacGillivray. In this proof we assume Theorem 3.1 which will be proved in the subsequent sections of the paper. The reasoning uses weak near unanimity operations (defined in Section 4) and Taylor operations (used only to connect Theorems 3.2 and 3.3, and therefore not defined here [HM88, Tay77, LZ06]).

Theorem 3.1. *If a digraph without sources and sinks admits a weak near unanimity polymorphism then it retracts onto the disjoint union of circles.*

It is easy to see that the colorability by digraphs retracting to a disjoint union of circles is tractable (see e.g. [BJH90]). It remains to prove the NP-completeness of the digraphs not retracting to such a union. Before we do so, we recall two fundamental results.

It follows from [HM88, Lemma 9.4 and Theorem 9.6] that a part of the result of Máróti and McKenzie [MM07, Theorem 1.1] can be stated as follows:

Theorem 3.2 ([MM07])**.** *A finite relational structure* $\mathcal{T}$ *admits a Taylor polymorphism if and only if it admits a weak near unanimity polymorphism.*

The following result was originally proved in [BKJ00] and [LZ03] and, as stated below, can be found in [LZ06, Theorem 2.3]:

Theorem 3.3 ([LZ06])**.** *Let* $\mathcal{T}$ *be a relational structure which is a core. If* $\mathcal{T}$ *does not admit a Taylor polymorphism, then* CSP($\mathcal{T}$) *is* NP-*complete.*

If a digraph **H** without sources or sinks does not retract to a disjoint union of circles, then its core **H**′ also does not. Thus, by Theorem 3.1, it follows that **H**′ does not admit a weak near unanimity polymorphism and by Theorem 3.2 and Theorem 3.3 it follows that CSP(**H**′) is NP-complete, completing the proof of the conjecture of Bang-Jensen and Hell.

The conjecture (posed in [BKJ00]), classifying the CSPs from the algebraic point of view, can be stated as follows (see, e.g. [LZ06]).

The algebraic CSP dichotomy conjecture. *Let* $\mathcal{T}$ *be a relational structure that is a core. If* $\mathcal{T}$ *admits a Taylor polymorphism, then* CSP($\mathcal{T}$) *is polynomial time solvable. Otherwise* CSP($\mathcal{T}$) *is* NP-*complete.*

Note that the proof of the conjecture of Bang-Jensen and Hell immediately implies that the structure of the NP-complete digraph coloring problems agrees with the algebraic CSP dichotomy conjecture. The remainder of the paper is dedicated to the proof of the Theorem 3.1 connecting the structural properties of digraphs with no sources or sinks with their polymorphisms.

4. Notation

In this section we introduce the notation required throughout the remainder of the paper.

4.1. Neighborhoods in graphs.
For a fixed digraph $\mathbf{G} = (V, E)$ we denote $(a, b) \in E$ by $a \to b$, and we use $a \xrightarrow{k} b$ to say that there is a directed path from a to b of length precisely k. More generally for any oriented path α, with endpoints c, d, we write $a \xrightarrow{\alpha} b$ if there exists a homomorphism ϕ from α into $\mathbf{G}$ such that $\phi(c) = a$ and $\phi(d) = b$. For any $W \subseteq V$ we define

$$W^{+n} = \{v \in V \mid (\exists w \in W)\ w \xrightarrow{n} v\}$$

and similarly

$$W^{-n} = \{v \in V \mid (\exists w \in W)\ v \xrightarrow{n} w\}.$$

We define $W^0 = W$, and write a^{+n} (or a^{-n}, a^0) instead of $\{a\}^{+n}$ ($\{a\}^{-n}, \{a\}^0$ respectively) for any $a \in V$. More generally, for an oriented path α, we write

$$W^\alpha = \{v \in V \mid (\exists w \in W)\ w \xrightarrow{\alpha} v\}.$$

As before we use a^α for $\{a\}^\alpha$. Sometimes, for ease of presentation, we write $a \xrightarrow{k,n} b$ to denote $a \xrightarrow{k} b$ and $a \xrightarrow{n} b$.

4.2. Digraph path powers.
Let $\mathbf{G} = (V, E)$ be a digraph and α be an oriented path. We define a path power of the digraph $\mathbf{G}$, which we denote by $\mathbf{G}^\alpha$, in the following way: the vertices of the power are the vertices of the digraph $\mathbf{G}$ and a pair $(c, d) \in V^2$ is an edge in $\mathbf{G}^\alpha$ if and only if $c \xrightarrow{\alpha} d$ in $\mathbf{G}$. For a directed path α of length n we define $\mathbf{G}^{+n} = \mathbf{G}^\alpha$. Note that, if $f : V^m \to V$ is a polymorphism of $\mathbf{G}$ then it is also a polymorphism of any path power of this digraph. Path powers are special cases of primitive positive definitions (used in e.g. [Bul05]) or indicator constructions introduced in [HN90] in order to deal with the colorability problem for undirected graphs.

4.3. Components.
A *connected* digraph is a digraph such that there exists an oriented path, consisting of at least one edge, between every choice of two vertices. A *strongly connected* digraph is a digraph such that, for every choice of two vertices, this path can be chosen to be directed. By a *component* (resp. *strong component*) of a digraph $\mathbf{G}$, we mean a maximal (under inclusion) induced subgraph that is connected (resp. strongly connected). Note that, according to this definition, a single vertex with the empty set of edges is not connected and thus not every digraph decomposes into a union of components (or strong components). Given a digraph $\mathbf{G}$ with no sources or sinks, we say that a strong component $\mathbf{H}$ of $\mathbf{G}$ is a *top component* if $V(\mathbf{H})^{+1} = V(\mathbf{H})$. Similarly, we say that a strong component $\mathbf{H}$ of $\mathbf{G}$ is a *bottom component* if $V(\mathbf{H})^{-1} = V(\mathbf{H})$.

4.4. **Algebraic length.** The following definition is taken from [HNZ96b]. For any oriented path α we define the algebraic length $al(\alpha)$ to be

$$al(\alpha) = |\{\text{edges going forward in } \alpha\}| - |\{\text{edges going backward in } \alpha\}|.$$

For a digraph $\mathbf{G} = (V, E)$ we put

$$al(\mathbf{G}) = \min\{i > 0 \,|\, (\exists v \in V) \; (\exists \text{ a path } \alpha) \; v \xrightarrow{\alpha} v \text{ and } al(\alpha) = i\},$$

whenever the set on the right hand side is non-empty and ∞ otherwise. We note that for digraphs with no sources or sinks (or with a cycle) the algebraic length of a non-empty digraph is always a natural number. It is an easy observation that a connected digraph $\mathbf{G}$ retracts to a circle if and only if there exists a circle (or equivalently a cycle) in $\mathbf{G}$ of length $al(\mathbf{G})$.

4.5. **Algebraic notation.** By $\overline{a}$ we denote the tuple $(a, a, \ldots, a)$ (the arity will always be clear by the context) and by $\overrightarrow{a}$ we denote the tuple $(a_0, a_1, \ldots, a_n)$. Further, we extend the notation $\overline{a}$ to the sets in the following way. For a set W let $\overline{W}$ be an appropriate cartesian power of W. Thus for example, given a vertex a of a digraph $\mathbf{G}$, the set $\overline{a^{+n}}$ is the collection of all tuples whose coordinates are vertices reachable by an n-path from a.

An idempotent operation on a set A is an operation, say $f : A^n \to A$, such that $f(\overline{a}) = a$, for all $a \in A$. In accordance with [MM07], by a weak near unanimity operation we understand an idempotent operation $w(x_0, \ldots, x_{n-1})$ that satisfies

$$w(y, x, \ldots, x) = w(x, y, \ldots, x) = \cdots = w(x, x, \ldots, y),$$

for any choice of x and y in the underlying set. Moreover, for a term t of arity n, we define

$$t^{(i)}(x_0, x_1, \ldots, x_{n-1}) = t(x_{n-i}, x_{n-i+1}, \ldots, x_0, x_1, \ldots, x_{n-i-1}),$$

for each $0 \leq i < n$, where addition on the indices is performed modulo n.

5. Preliminary results on digraphs

We start with a number of basic results describing the connection between digraphs and its path powers. The following lemma reveals the behavior of the algebraic lengths of paths in powers of a digraph.

Lemma 5.1. *Let $\mathbf{G}$ be a digraph without sources or sinks. Let α be an oriented path of algebraic length k and let $a \xrightarrow{\alpha} b$ in $\mathbf{G}$. Then $a \xrightarrow{\beta} b$ in $\mathbf{G}^{+k}$ for some oriented path β of algebraic length one.*

Proof. For a fixed, large enough number j, consider all the oriented paths of the form $a \xrightarrow{l_1} a_1 \xleftarrow{l_2} a_2 \xrightarrow{l_3} \cdots \leftrightarrows a_{l_j} = b$ where $l_1 - l_2 + \cdots \pm l_j = k$. Choose such an oriented path in which k divides a maximal initial segment of the l_i's. Let l_{i_0} be the last element of this segment. If $i_0 + 1 < j$ then (assuming without loss of generality that i_0 is odd) the path

$$a \xrightarrow{l_1} \cdots \xrightarrow{l_{i_0}} a_{i_0} \xleftarrow{l_{i_0+1}} a_{i_0+1} \xrightarrow{l_{i_0+2}} a_{i_0+2} \cdots$$

can be altered, using the fact that a_{i_0+1} (and possibly other vertices) is not a source, to obtain

$$a \xrightarrow{l_1} \cdots \xrightarrow{l_{i_0}} a_{i_0} \xleftarrow{l'_{i_0+1}} a'_{i_0+1} \xrightarrow{l'_{i_0+2}} a_{i_0+2} \cdots$$

where l'_{i_0+1} is greater than l_{i_0+1} and is divisible by k. This contradicts the choice of i_0.

If, on the other hand, $i_0 + 1 = j$ the number k divides $l_1 - l_2 + \cdots \pm l_{i_0}$ and, using the fact that $l_1 - l_2 + \cdots \pm l_{i_0} \mp l_{i_0+1} = k$, we infer that k divides l_{i_0+1}, again

contradicting the choice of i_0. Thus $i_0 = j$ and we can find a path $a \xrightarrow{l_1} a_1 \xleftarrow{l_2} a_2 \xrightarrow{l_3} \ldots a_{l_j} = b$ with $l_1 - l_2 + \cdots \pm l_j = k$ where each l_i is divisible by k. This gives an oriented path of algebraic length one in $\mathbf{G}^{+k}$. $\square$

As a consequence we get

Corollary 5.2. *Let $\mathbf{G}$ be a digraph, without sources or sinks, such that $al(\mathbf{G}) = 1$. Then $al(\mathbf{G}^{+k}) = 1$ for any natural number k.*

Proof. Let $a \xrightarrow{\alpha} a$, where α is an oriented path of algebraic length one. Then, by following the path α k-many times, we obtain $a \xrightarrow{\beta} a$ for an oriented path β of algebraic length k. Now the statement follows from the previous lemma. $\square$

Theorem 3.1 is proved in Section 7 for strongly connected digraphs first and therefore we need some preliminary results on such digraphs. The following very simple lemma is needed to prove some of the further corollaries in this section.

Lemma 5.3. *Let c be a vertex in a strongly connected digraph. Then the greatest common divisor (GCD) of the lengths of the cycles in this digraph is equal to the GCD of the lengths of the cycles containing c.*

Proof. Suppose, for contradiction, that the GCD, say n', of the lengths of the cycles containing c is bigger than the GCD of the lengths of the cycles for the entire digraph. Then there exists a cycle $d \xrightarrow{l} d$ such that n' does not divide l. On the other hand, since the digraph is strongly connected, $c \xrightarrow{l'} d$ and $d \xrightarrow{l''} c$ for some numbers l', l''. The number n', by definition, divides $l' + l''$ and $l' + l + l''$ and thus divides l, a contradiction. $\square$

Moreover the following easy proposition holds.

Proposition 5.4. *For any connected digraph $\mathbf{G}$ and any oriented cycle α in $\mathbf{G}$, the number $al(\mathbf{G})$ divides $al(\alpha)$.*

Proof. Let $\mathbf{G}$ and $a \xrightarrow{\alpha} a$ be as in the statement of the proposition. Let b be a vertex in $\mathbf{G}$ such that $b \xrightarrow{\beta} b$ is an oriented path satisfying $al(\beta) = al(\mathbf{G})$. Since $\mathbf{G}$ is connected there is an oriented path γ such that $b \xrightarrow{\gamma} a$ and thus $b \xrightarrow{\gamma} a \xrightarrow{\alpha} a \xrightarrow{\gamma'} b$ with $al(\gamma') = -al(\gamma)$. Following appropriate paths we can obtain an oriented cycle, from b to b, of algebraic length $al(\alpha) - k \cdot al(\mathbf{G})$, for any number k. The minimality of $al(\mathbf{G})$ implies that $al(\mathbf{G})$ divides $al(\alpha)$. $\square$

The following lemma is heavily used in the proof of Theorem 3.1 for strongly connected digraphs in Section 7.

Lemma 5.5. *If, for a strongly connected digraph $\mathbf{G} = (V, E)$, the GCD of the lengths of the cycles in $\mathbf{G}$ is equal to one, then*

$$(\exists m) \ (\forall a, b \in V) \ (\forall n) \ \text{if } n \geq m \text{ then } a \xrightarrow{n} b.$$

Proof. Fix an arbitrary element $c \in V$. By Lemma 5.3 we find some cycles containing c such that their lengths $k_1, \ldots, k_i$ satisfy $GCD(k_1, \ldots, k_i) = 1$. Thus c is contained in an l-cycle whenever l is a linear combination of $k_1, \ldots, k_i$ with nonnegative integer coefficients. It is easy to see that there is a natural number m' such that, for every $n' \geq m'$, n' can be expressed as such a linear combination, hence c is in a n'-cycle for each such n'. Now it suffices to put $m = m' + 2|V|$ since, for arbitrary vertices $a, b \in V$, there are directed paths of length at most $|V|$ from a to c and from c to b. $\square$

The following easy corollary follows.

Corollary 5.6. *For a strongly connected digraph* $\mathbf{G}$ *with GCD of the lengths of the cycles equal to one, and for any number* n, *the digraph* $\mathbf{G}^{+n}$ *is strongly connected.*

For strongly connected digraphs, the greatest common divisor of the lengths of the cycles and the algebraic length of the digraph coincide.

Corollary 5.7. *For a strongly connected digraph, the GCD of the lengths of the cycles is equal to the algebraic length of the digraph.*

Proof. Let us fix a digraph $\mathbf{G} = (V, E)$ and denote by n the greatest common divisor of the lengths of the cycles in $\mathbf{G}$. Since, by Proposition 5.4, the algebraic length of $\mathbf{G}$ divides the length of every cycle in $\mathbf{G}$ then $al(\mathbf{G}) \le n$.

Conversely, let $a = a_0 \xrightarrow{l_0} b_0 \xleftarrow{k_0} a_1 \xrightarrow{l_1} \cdots \xleftarrow{k_{m-1}} a_m = a$ be an oriented path of algebraic length $al(\mathbf{G})$. Let k'_i be such that $b_i \xleftarrow{k_i} a_{i+1} \xleftarrow{k'_i} b_i$ for all i. Note that n divides $k_i + k'_i$ and $\sum_{i<m} l_i + \sum_{i<m} k'_i$. Thus n divides $\sum_{i<m} l_i - \sum_{i<m} k_i = al(\mathbf{G})$ which shows that $n \le al(\mathbf{G})$ and the lemma is proved. $\square$

Finally we remark that if α is an oriented path of algebraic length one and $\mathbf{G}$ has no sources and no sinks, then $E(\mathbf{G}^\alpha) \supseteq E(\mathbf{G})$. In particular, if $al(\mathbf{G}) = 1$, then $al(\mathbf{G}^\alpha) = 1$.

6. A CONNECTION BETWEEN GRAPHS AND ALGEBRA

In this section we present basic definitions and results concerning the connection between digraphs and algebras. Let $\mathbf{G} = (V, E)$ be a digraph admiting a weak near unanimity polymorphism $w(x_0, x_1, \ldots, x_{h-1})$. We associate with $\mathbf{G}$ an algebra $\mathbf{A} = (V, w)$ for which E is a subuniverse of $\mathbf{A}^2$. Note that for any subuniverse of $\mathbf{A}$, say W, we can define the digraph $\mathbf{G}_{|W} = (W, E \cap W \times W)$ (or $(W, E_{|W})$) which admits the weak near unanimity polymorphism $w|_{W^h}$ and the algebra $(W, w|_{W^h})$ is a subalgebra of $\mathbf{A}$. For the remainder of this section we assume that $\mathbf{G}$ and $\mathbf{A}$ are as above.

The first lemma describes the influence of the structure of the digraph on the subuniverses of the algebra.

Lemma 6.1. *For any subuniverse* W *of* $\mathbf{A}$ *the sets* W^{+1} *and* W^{-1} *are subuniverses of* $\mathbf{A}$.

Proof. Take any elements $a_0, \ldots, a_{h-1}$ from W^{+1} and choose $b_0, \ldots, b_{h-1} \in W$ such that $b_i \to a_i$ for all i. Then $w(b_0, \ldots, b_{h-1}) \to w(a_0, \ldots, a_{h-1})$ showing that $w(a_0, \ldots, a_{h-1}) \in W^{+1}$ and the claim is proved. The proof for W^{-1} is similar. $\square$

Since the weak near unanimity operation is idempotent, all the one element subsets of V are subuniverses of $\mathbf{A}$. Using the previous lemma, the following result follows trivially.

Corollary 6.2. *For all* $a \in V$, *for all oriented paths* α *and every number* n, *the sets* a^{+n}, a^{-n} *and* a^α *are subuniverses of* $\mathbf{A}$.

Subuniverses of $\mathbf{A}$ can also be obtained in another way.

Lemma 6.3. *Let* $\mathbf{H}$ *be a strong component of* $\mathbf{G}$. *Assume that the GCD of the lengths of the cycles in* $\mathbf{H}$ *is equal to one. Then* $V(\mathbf{H})$ *is a subuniverse of* $\mathbf{A}$.

Proof. Using Lemma 5.5 we find a number m such that there is a directed path from $b \xrightarrow{m} c$ in $\mathbf{H}$, for all $b, c \in V(\mathbf{H})$. Fix a vertex $a \in V(\mathbf{H})$. There is a directed path of length $a \xrightarrow{m} b$, for all $b \in V(\mathbf{H})$ and a directed path $c \xrightarrow{m} a$, for all $c \in V(\mathbf{H})$. Thus, $V(\mathbf{H}) = a^{+m} \cap a^{-m}$ is a subuniverse. $\square$

We present a second construction leading to a subuniverse of the algebra.

Lemma 6.4. *If $\mathbf{H} = (W, F)$ is the largest induced subgraph of $\mathbf{G}$ without sources or sinks, then W is a subuniverse of $\mathbf{A}$.*

Proof. Clearly, the vertices of $\mathbf{H}$ can be described as those having arbitrarily long directed paths to them and from them. Since $\mathbf{G}$ is finite, there exists a natural number k such that

$$W = \{w \mid (\exists v, v' \in V) \; v \xrightarrow{k} w \;\text{ and }\; w \xrightarrow{k} v'\}.$$

Thus $W = V^{+k} \cap V^{-k}$ and we are done, since both sets on the right hand side are subuniverses. $\square$

7. Strongly connected digraphs

In this section we present a proof Theorem 3.1 in the case of strongly connected digraphs. The reasoning uses directed paths only and thus, in this section, by a path we always mean a directed path.

Theorem 7.1. *If a strongly connected digraph of algebraic length k admits a weak near unanimity polymorphism, then it contains a cycle (and circle) of length k (and thus retracts onto it).*

Using Corollary 5.7, the result can be restated in terms of the GCD of the lengths of cycles in $\mathbf{G}$ and we will freely use this duality. Theorem 7.1 is a consequence of the following result:

Theorem 7.2. *If a strongly connected digraph $\mathbf{G}$ of algebraic length one admits a weak near unanimity polymorphism then it contains a loop.*

We present a proof of Theorem 7.1, assuming Theorem 7.2, and devote the remainder of this section to proving Theorem 7.2.

Proof of Theorem 7.1. Fix an arbitrary vertex c in a strongly connected digraph of algebraic length k. Using Lemma 5.3 and Corollary 5.7 we obtain cycles containing c with the GCD of their lengths equal to k. Thus, in the path power $\mathbf{G}^{+k}$, the GCD of lengths of cycles containing c is equal to one. Let $\mathbf{H}$ be the strong component of $\mathbf{G}^{+k}$ containing c. Using Lemma 6.3 we infer that $V(\mathbf{H})$ is a subuniverse of the algebra $(V(\mathbf{G}^{+k}), w)$ and thus it admits a weak near unanimity polymorphism. The algebraic length of $\mathbf{H}$ (again by Corollary 5.7) is one and therefore, by Theorem 7.2 it follows that there is a loop in $\mathbf{G}^{+k}$. This trivially implies a k-cycle in $\mathbf{G}$ which, by Proposition 5.4, is a circle and the theorem is proved. $\square$

The remaining part of this section is devoted to the proof of Theorem 7.2. We start by choosing a digraph $\mathbf{G} = (V, E)$ to be a minimal (with respect to the number of vertices) counterexample to Theorem 7.2. We fix a weak near unanimity polymorphism $w(x_0, \ldots, x_{h-1})$ of this digraph and associate with it the algebra $\mathbf{A} = (V, w)$. The proof will proceed by a number of claims.

Claim 7.3. *The digraph $\mathbf{G}$ can be chosen to contain a 2-cycle.*

Proof. Using Lemma 5.5 we find a minimal k such that a 2^k-cycle is contained in $\mathbf{G}$. Consider the path power $\mathbf{G}^{+2^{k-1}}$. It contains a 2-cycle and admits a weak near unanimity polymorphism. Moreover, since k was chosen to be minimal, and $\mathbf{G}$ did not contain a loop, the path power $\mathbf{G}^{+2^{k-1}}$ does not contain a loop either. By Corollary 5.6 the path power is strongly connected and by Corollary 5.2 it has algebraic length equal to one. Thus, the digraph $\mathbf{G}^{+2^{k-1}}$ is also a counterexample to Theorem 7.2 (with the same number of vertices as $\mathbf{G}$) and therefore we can use it as a substitute for $\mathbf{G}$. $\square$

From this point on we assume that $\mathbf{G}$ contains a 2-cycle (an undirected edge). The next claim allows us to choose and fix an undirected edge with special properties.

Claim 7.4. *There are vertices $a, b \in V$, forming an undirected edge in $\mathbf{G}$, and a binary term t of $\mathbf{A}$ such that $a = t(w(\bar{a}, b), w(\bar{b}, a))$.*

Proof. Let $M \subseteq V$ be a minimal (under inclusion) subuniverse of $\mathbf{A}$ containing an undirected edge and let $a, b \in M$ be vertices in such an edge. Since vertices $w(\bar{a}, b)$, $w(\bar{b}, a) \in M$ form an undirected edge in $\mathbf{G}$, the set $\{w(\bar{a}, b), w(\bar{b}, a)\}$ generates, in the algebraic sense, the set M (by the minimality of M). Therefore there exists a term t such that $t(w(\bar{a}, b), w(\bar{b}, a)) = a$. $\square$

In the following claims we fix vertices a, b and a term $t(x, y)$ such that $a \to b \to a$ and $a = t(w(\bar{a}, b), w(\bar{b}, a))$ (provided by the previous claim). Note that, by the definition of the operation $w(x_0, \ldots, x_{h-1})$, for any numbers $i, j < h$, we obtain $a = t(w^{(i)}(\bar{a}, b), w^{(j)}(\bar{b}, a))$.

Using Lemma 5.5 we find and fix a minimal number n such that $a^{+(n+1)} = V$. We put $W = a^{+n}$ and $F = (W \times W) \cap E$ so that $\mathbf{H} = (W, F)$ is an induced subgraph of the digraph $\mathbf{G}$. Using Corollary 6.2 we infer that W is a subuniverse of $\mathbf{A}$ and thus $\mathbf{H}$ admits a weak near unanimity polymorphism. In the following claims we will show that the algebraic length of some strong component of $\mathbf{H}$ is one which will contradict the minimality of $\mathbf{G}$.

Claim 7.5. *For any element in W there exists a cycle in $\mathbf{H}$ and a path (also in $\mathbf{H}$) connecting the cycle to this element.*

Proof. Let d_0 denote an arbitrary element of W. Since $a^{+(n+1)} = W^{+1} = V$ there is $d_1 \in W$ such that $d_1 \to d_0$. Similarly, there exists $d_2 \in W$ such that $d_2 \to d_1$. By repeating this procedure, we get both statements of the claim. $\square$

The next claim will allow us to fix some more elements necessary for further contruction:

Claim 7.6. *There exist element $c, c' \in W$ and a number k such that:*

(1) $c' \to a$,
(2) $c \xrightarrow{k} c$ in $\mathbf{H}$, and
(3) $c \xrightarrow{k-n-1} c'$ in $\mathbf{H}$.

Proof. Since $W^{+1} = V$ there exists $c' \in W$ such that $c' \to a$. Let l be the length of a cycle provided by Claim 7.5 for $c' \in W$. For a sufficiently large multiple k of l there is a path in $\mathbf{H}$ of length $k - n - 1$ from some element of the cycle to c', we call this element c. This finishes the proof. $\square$

From this point on we fix vertices c and c' in W and a number k to satisfy the conditions of the last claim. The following claims focus on uncovering the structure of the strong component containing c in $\mathbf{H}$.

Claim 7.7. *For any $m \leq n$ either $a^{+m} \subseteq a^{+n}$ or $a^{+m} \subseteq b^{+n}$.*

Proof. Since a is in a 2-cycle, we obviously have $a^{+n} \supseteq a^{+(n-2)} \supseteq a^{+(n-4)} \ldots$ which proves the claim for even m's. If, on the other hand, m is odd we have $b^{+n} \supseteq a^{+(n-1)} \supseteq a^{+(n-3)} \ldots$ completing the proof. $\square$

The next two claims are of major importance for the proof of Theorem 7.2. They are used to show that the algebraic length of the strong component of $\mathbf{H}$ containing c is one.

Claim 7.8. *For any $m \leq n$ and for any $0 \leq i, j < h$ the following inclusion holds*

$$t(w^{(i)}(\overline{a^{+n}}, a^{+m}), w^{(j)}(\overline{a^{+m}}, a^{+n})) \subseteq a^{+n}$$

Proof. Since $a = t(w^{(i)}(\overline{a}, b), w^{(j)}(\overline{b}, a))$, we have

$$a^{+n} \supseteq t(w^{(i)}(\overline{a^{+n}}, b^{+n}), w^{(j)}(\overline{b^{+n}}, a^{+n})).$$

By idempotency we have $a = t(w^{(i)}(\overline{a}, a), w^{(j)}(\overline{a}, a))$ and therefore

$$a^{+n} \supseteq t(w^{(i)}(\overline{a^{+n}}, a^{+n}), w^{(j)}(\overline{a^{+n}}, a^{+n})).$$

Now the claim follows directly from Claim 7.7. $\qquad\square$

The following technical lemma will allow us to find directed paths in the strong component of $\mathbf{H}$ containing c.

Claim 7.9. *The following implication holds in $\mathbf{H}$ (i.e. all the paths and vertices lie inside $\mathbf{H}$). For any numbers $0 \leq i, j < h$ and all $e, e', f \in W$ and $\overrightarrow{d}, \overrightarrow{d'}, \overrightarrow{g} \in \overline{W}$,*

$$\text{if} \quad \begin{matrix} e \\ \downarrow k \\ e' \end{matrix} \quad \text{and} \quad \begin{matrix} d_l \\ \downarrow k \\ d'_l \end{matrix} \quad \text{for all } l, \quad \text{then} \quad \begin{matrix} t(w^{(i)}(\overrightarrow{d}, c), w^{(j)}(\overline{c}, e)) \\ \downarrow k \\ t(w^{(i)}(\overrightarrow{d'}, f), w^{(j)}(\overrightarrow{g}, e')) \end{matrix}$$

Proof. Note that, by looking at the tuples of elements pointwise, we can find the following paths in $\mathbf{G}$:

where the paths from c to c' are provided by Claim 7.6 and lie entirely in $\mathbf{H}$. Applying the appropriate term to the consecutive elements of the paths (rows in the diagram above) we obtain a path of length k connecting $t(w^{(i)}(\overrightarrow{d}, c), w^{(j)}(\overline{c}, e))$ to $t(w^{(i)}(\overrightarrow{d'}, f), w^{(j)}(\overrightarrow{g}, e'))$. It remains to prove that all the elements of this path are in W. The first $k - n - 1$ elements of the path are in W, since W is a subuniverse and they are results of an application of a term to elements of the subuniverse. For $m \geq 0$, the $(k - n + m)$-th element of the path is a member of $t(w^{(i)}(\overline{a^{+n}}, a^{+m}), w^{(j)}(\overline{a^{+m}}, a^{+n}))$ and thus in W by Claim 7.8. $\qquad\square$

We now construct a cycle in $\mathbf{H}$, that contains c, of length coprime to k.

Claim 7.10. *There exists a path $c \xrightarrow{(h+1)k-1} c$ in digraph $\mathbf{H}$.*

Proof. In the proof of this claim we only use elements and paths that lie inside $\mathbf{H}$. Fix $d \in W$ (provided by Claim 7.6) such that $c \to d \xrightarrow{k-1} c$ in $\mathbf{H}$. By repeatedly

applying Claim 7.9 we obtain:

$$t(w(c,\ldots,c,c,c), w(c,c\ldots,c))$$
$$\downarrow k$$
$$t(w(c,\ldots,c,c,d), w(d,c,\ldots,c)) \quad = \quad t(w^{(1)}(c,\ldots,c,d,c), w^{(1)}(c,\ldots,c,d))$$
$$\downarrow k$$
$$t(w^{(2)}(c,\ldots,d,d,c), w^{(1)}(c,\ldots,c,d)) \quad = \quad t(w^{(1)}(c,\ldots,c,d,d), w^{(1)}(c,\ldots,c,d))$$
$$\vdots$$
$$= \quad t(w^{(h-1)}(d,\ldots,d,d,c), w^{(1)}(c,\ldots,c,d))$$
$$\downarrow k$$
$$t(w^{(h-1)}(d,\ldots,d,d,d), w^{(1)}(d,\ldots,d,d))$$

and since the algebra is idempotent the starting point of this path is c and the ending point is d. Thus $c \xrightarrow{hk} d$ (for h the arity of the operation $w(x_0,\ldots,x_{h-1})$) which immediately gives us the claim. $\qquad\square$

By Claims 7.6 and 7.10, the strong component of $\mathbf{H}$ containing c has GCD of the lengths of its cycles equal to one and thus, by Lemma 6.3, its vertex set forms a subuniverse of the algebra $\mathbf{A}$. As a digraph it admits a weak near unanimity polymorphism. By Corollary 5.7 it has algebraic length one and (as an induced subgraph of $\mathbf{G}$) it has no loops. Since $\mathbf{H}$ was chosen to be strictly smaller than $\mathbf{G}$ we obtain a contradiction with the minimality of $\mathbf{G}$ and the proof of Theorem 7.2 is completed.

8. THE GENERAL CASE

In this section we prove Theorem 3.1 in its full generality. Nevertheless the majority of this section is devoted to the proof of the following result.

Theorem 8.1. *If a digraph with no sources or sinks has algebraic length one and admits a weak near unanimity polymorphism then it contains a loop.*

Using the above result we prove the core theorem of the paper.

Theorem 3.1. *If a digraph with no sources or sinks admits a weak near-unanimity polymorphism then it retracts onto the disjoint union of circles.*

Proof. Let $\mathbf{G}$ be a digraph with no sources or sinks which admits a weak near unanimity polymorphism. Let n be the algebraic length of some component of $\mathbf{G}$. The path power $\mathbf{G}^{+n}$ admits a weak near unanimity polymorphism, has no sources or sinks and, by Lemma 5.1, has algebraic length equal to one. Thus, Theorem 8.1 applied to $\mathbf{G}^{+n}$ provides a loop in the path power and therefore an n-cycle in $\mathbf{G}$.

Let n be minimal, under divisibility, in the set of algebraic lengths of components of $\mathbf{G}$. Since the algebraic length of a component divides (by Proposition 5.4) the length of any cycle in it, every n-cycle (for such a minimal n) is a circle. Moreover, by the same reasoning, circles obtained for two different minimal n's cannot belong to the same component. Thus each component of $\mathbf{G}$ maps homomorphically onto an n-cycle (for any minimal n dividing the algebraic length of this component) and it is not difficult to see that these homomorphisms can be chosen so that their union is a retraction. This proves the theorem. $\qquad\square$

Therefore the only missing piece of the proof to the conjecture of Bang-Jensen and Hell is Theorem 8.1. We prove this result by way of contradiction. Suppose
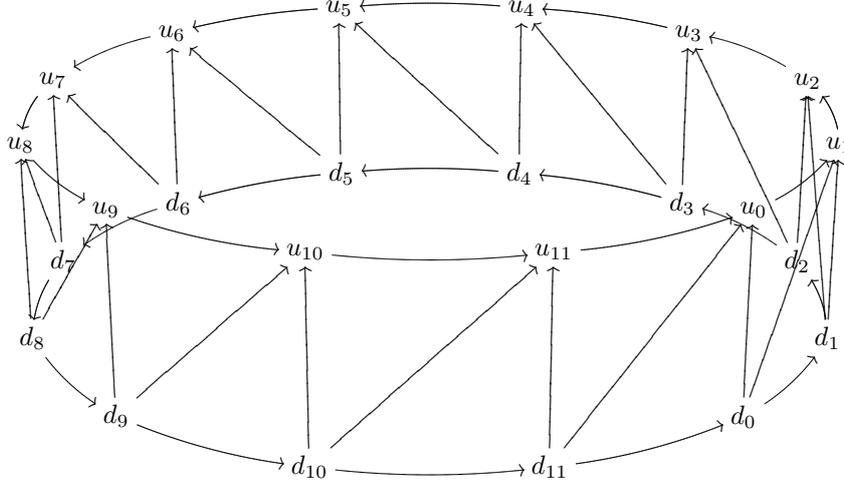
FIGURE 1. The 12-tambourine.

that $\mathbf{G} = (V, E)$ is a minimal (with respect to the number of vertices) counterexample to Theorem 8.1 and let $\mathbf{A} = (V, w(x_0, \ldots, x_{h-1}))$ be the algebra associated with $\mathbf{G}$, in the sense of Section 6, for some weak near unanimity polymorphism $w(x_0, \ldots, x_{h-1})$.

The first part of the proof is dedicated to finding a particular counterexample satisfying more restrictive conditions than $\mathbf{G}$. To do so we need to define a special family of digraphs called *tambourines*. The *n*-tambourine is the digraph $(\{d_0, \ldots, d_{n-1}, u_0, \ldots, u_{n-1}\}, F_n)$ such that

$$ F_n = \bigcup_i \{(d_i, d_{i+1}), (d_i, u_i), (d_i, u_{i+1}), (u_i, u_{i+1})\}. $$

where the addition on the indices is computed modulo n. The 12-tambourine can be found in Figure 1. We begin the proof of the theorem with the following claim:

Claim 8.2. *We can choose a digraph $\mathbf{G}$ and a number n such that*

- *the n-tambourine maps homomorphicaly into $\mathbf{G}$,*
- *every element of $\mathbf{G}$ is in an n-cycle and*
- *$\mathbf{G}^{+(mn+1)} = \mathbf{G}$ for any number m.*

To prove this claim, we begin with an easy subclaim and work towards replacing $\mathbf{G}$ with a particular path power of $\mathbf{G}$ which satisfies the additional conditions. Note that, for any oriented path α, the path power G^α admits $w(x_0, \ldots, x_{h-1})$ as a polymorphism and has no sources or sinks. If such a path power has algebraic length one and does not contain a loop, then it can be taken as a substitute for $\mathbf{G}$.

Subclaim 8.2.1. *The digraph $\mathbf{G}$ contains vertices d and u such that $d \xrightarrow{|V|,|V|+1} u$.*

Proof. Let α be the oriented path

$$ \underbrace{\rightarrow \cdots \rightarrow}_{|V|+1} \underbrace{\leftarrow \cdots \leftarrow}_{|V|} $$

Using the fact that $al(\alpha) = 1$ and that $\mathbf{G}$ has no sources or sinks, it follows that $E(\mathbf{G}) \subseteq E(\mathbf{G}^\alpha)$. Moreover, let a, b be vertices in $\mathbf{G}$ such that b is contained in a cycle and $a \xrightarrow{k} b$, for some k. Then $a \xrightarrow{k'} b$ for some $k' \leq |V|$ and choosing b' (from

the cycle containing b) such that $b' \xrightarrow{k'+1} b$ we obtain

$$b' \xrightarrow{k'+1} b \xrightarrow{(|V|+1)-(k'+1)} c \xleftarrow{(|V|+1)-(k'+1)} b \xleftarrow{k'} a \text{ for some } c.$$

Thus $b' \xrightarrow{\alpha} a$ and this implies that every component of $\mathbf{G}$ becomes a strong component of $\mathbf{G}^{\alpha}$.

Let $\mathbf{H} = (W, F)$ be a component of $\mathbf{G}$ with a path of algebraic length one. Then, there exists $F' \supseteq F$ such that the digraph $\mathbf{H}' = (W, F')$ is a strong component of $\mathbf{G}^{\alpha}$. The digraph $\mathbf{H}'$ contains $\mathbf{H}$ as a subgraph and therefore its algebraic length is one. The path power $\mathbf{G}^{\alpha}$ admits $w(x_0, \ldots, x_{h-1})$ as a polymorphism and thus, by Lemma 6.3, the digraph $\mathbf{H}'$ admits an appropriate restriction of $w(x_0, \ldots, x_{h-1})$. Theorem 7.2 provides a loop in $\mathbf{H}'$ which in turn implies the existence of vertices $d, u \in W$ such that $d \xrightarrow{|V|,|V|+1} u$ in $\mathbf{G}$. $\qquad\square$

Proof of Claim 8.2. We fix $n = |V|!$ and argue that, for some k, the path power $\mathbf{G}_k = \mathbf{G}^{+(kn+1)}$ satisfies the assertions of the claim and therefore can be taken as a substitute for $\mathbf{G}$. Note that, for any number k, the digraph $\mathbf{G}_k$ admits $w(x_0, \ldots, x_{h-1})$ as a polymorphism, it has no sources or sinks and, by Corollary 5.2, it has algebraic length one.

We first prove that, for all k, the digraph $\mathbf{G}_k$ does not contain a loop. If $\mathbf{G}_k$ does contain a loop, then there exists a cycle of length $kn+1$ in some strong component of $\mathbf{G}$. The length of any circle in $\mathbf{G}$ is coprime to $kn+1$, therefore the GCD of the lengths of cycles in this strong component is one, and, using Corollary 5.7, Lemma 6.3 and Theorem 7.2, we obtain a loop in this strong component and therefore also in $\mathbf{G}$, a contradiction. Thus, to prove the claim, it remains to verify the additional required properties.

We now show that, for the fixed number n, the n-tambourine maps homomorphically into $\mathbf{G}_k$ for $k \geq 4$. Let d, u be vertices of $\mathbf{G}$ provided by Sublaim 8.2.1. Since $\mathbf{G}$ has no sources or sinks we can find vertices d', u', each contained in a cycle, such that d' is connected by a directed path to d and u is connected by a directed path to u'. By following the cycles containing d' and u' multiple times we get d'_0, u'_0, each contained in a cycle, such that $d'_0 \xrightarrow{3n,3n+1} u'_0$. Moreover, again following the cycles multiple times, we obtain

$$d'_0 \xrightarrow{n} d'_0 \xrightarrow{3n,3n+1} u'_0 \xrightarrow{n} u'_0.$$

Let d'_i denote the i-th element of the cycle $d'_0 \xrightarrow{n} d'_0$ and, similarly u'_i the i-th element of the cycle $u'_0 \xrightarrow{n} u'_0$. Then, for any number $k \geq 4$ and any $i < n$, we have $d'_i \xrightarrow{kn+1} u'_i$ and $d'_i \xrightarrow{kn+1} u'_{(i+1) \bmod n}$. On the other hand $d'_i \xrightarrow{kn+1} d'_{(i+1) \bmod n}$ and $u'_i \xrightarrow{kn+1} u'_{(i+1) \bmod n}$. Thus, for any $k \geq 4$, the map $d_i \mapsto d'_i, u_i \mapsto u'_i$ is a homomorphism from the n-tambourine in the path power $\mathbf{G}_k$.

To prove the second assertion of the claim we need to show that, if $k \geq 4$, than any vertex of $\mathbf{G}_k$ is in an n-cycle. We fix such a number k and let $W \subset V$ be the subuniverse of $\mathbf{A}$ generated by $\{d'_0, \ldots, d'_{n-1}, u'_0, \ldots, u'_{n-1}\}$. Let $\mathbf{G}'_k$ be the subgraph induced by $\mathbf{G}_k$ on W. The digraph $\mathbf{G}'_k$ obviously admits a restriction of $w(x_0, \ldots, x_{h-1})$ and (since the n-tambourine maps homorphically into it) has algebraic length one. Choose an arbitrary $a \in W$. Then, by the definition of W, we have a term $t(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1})$ such that $a = t(d'_0, \ldots, d'_{n-1}, u'_0, \ldots, u'_{n-1})$.

Therefore,

$$\left.\begin{array}{c}
t(d'_0, \ldots, d'_{n-2}, d'_{n-1}, u'_0, \ldots, u'_{n-2}, u'_{n-1}) \\
\downarrow \\
t(d'_1, \ldots, d'_{n-1}, d'_0, u'_1, \ldots, u'_{n-1}, u'_0) \\
\vdots \\
t(d'_{n-1}, \ldots, d'_{n-3}, d'_{n-2}, u'_{n-1}, \ldots, u'_{n-3}, u'_{n-2}) \\
\downarrow \\
t(d'_0, \ldots, d'_{n-2}, d'_{n-1}, u'_0, \ldots, u'_{n-2}, u'_{n-1})
\end{array}\right\} n$$

and thus a is in an n-cycle. This proves that $\mathbf{G}'_k$ has no sources and no sinks and since it cannot be a counterexample smaller than $\mathbf{G}$ we infer that $W = V$. Therefore the second assertion holds for all the digraphs $\mathbf{G}_k$ with $k \geq 4$.

In the digraph $\mathbf{G}_4$ every element is in an n-cycle and therefore $E(\mathbf{G}_4^{+(nm+1)}) \subseteq E(\mathbf{G}_4^{+(n(m+1)+1)})$ for any number m. Thus, there is a number l such that for any $m \geq l$ we have $\mathbf{G}_4^{+(nm+1)} = \mathbf{G}_4^{+(nl+1)}$. Take $\mathbf{G}' = \mathbf{G}_4^{+(nl+1)} = \mathbf{G}_{(4nl+l+4)n+1}$ and note that, according to the previous paragraphs of this proof, such a digraph satisfies all but the last assertion of the claim. Let m be arbitrary. Then $(\mathbf{G}')^{+(mn+1)} = \mathbf{G}_4^{+((mnl+l+m)n+1)} = \mathbf{G}_4^{+(nl+1)} = \mathbf{G}'$ and thus $\mathbf{G}'$ can be taken to substitute $\mathbf{G}$ and the claim is proved. $\qquad\square$

From this point on we substitute $\mathbf{G}$ with a digraph provided by the previous claim and fix it together with the number n. For ease of notation we denote the number modulo n using brackets (e.g $[n+1] = 1$). We already know that the n-tambourine maps homomorphically into $\mathbf{G}$, but we must choose such a homomorphism carefully.

Claim 8.3. *The n-tambourine can be mapped homomorphically into $\mathbf{G}$ in such a way that*

$$d'_i = t^{(i)}(w(\overline{d'_0}, d'_1), w(\overline{d'_1}, d'_2), \ldots, w(\overline{d'_{n-1}}, d'_0)) \text{ for all } i < n,$$

where d'_i is the image of d_i.

Proof. Let $d_i \mapsto d'_i, u_i \mapsto u'_i$ be a homomorphism from the n-tambourine into $\mathbf{G}$. Then, for any i, we have

$$\begin{array}{ccc}
w(\overline{u'_i}, u'_{[i+1]}) & \longrightarrow & w(\overline{u'_{[i+1]}}, u'_{[i+2]}) \longrightarrow \cdots \\
\uparrow & \nearrow & \uparrow \qquad \cdots \\
w(\overline{d'_i}, d'_{[i+1]}) & \longrightarrow & w(\overline{d'_{[i+1]}}, d'_{[i+2]}) \longrightarrow \cdots
\end{array}$$

and thus $d_i \mapsto w(\overline{d'_i}, d'_{[i+1]}), u_i \mapsto w(\overline{u'_i}, u'_{[i+1]})$ is also a homomorphism from the n-tambourine into $\mathbf{G}$. By repeating this procedure we obtain an infinite sequence of homomorphisms from the n-tambourine into $\mathbf{G}$ and thus some homomorphism has to appear twice in this sequence. This homomorphism satisfies the claim, since the term $t(x_0, \ldots, x_{n-1})$ can be easily obtained as a composition of the polymorphism $w(x_0, \ldots, x_{h-1})$ used in the construction of the sequence. $\qquad\square$

In the remaining part of the proof we fix vertices $d'_0, \ldots, d'_{n-1}, u'_0, \ldots, u'_{n-1}$ provided by the previous claim and a term $t(x_0, \ldots, x_{n-1})$ associated with them. Let φ_k be the oriented path $\triangle \xrightarrow{\varphi_k} \circ$
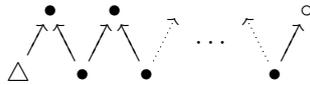
with exactly k edges (the last edge of the path is pointing forward for odd k, as in the above picture, and backward for even k).

Claim 8.4. *The neighborhood $(d_0')^{\varphi_n}$ contains all vertices of* $\mathbf{G}$.

Proof. Note that, in the n-tambourine, we have

$$(d_0)^{\varphi_n} = \{d_0, \ldots, d_{n-1}, u_0, \ldots, u_{n-1}\}$$

and thus in the digraph $\mathbf{G}$ we have

$$(d_0')^{\varphi_n} \supseteq \{d_0', \ldots, d_{n-1}', u_0', \ldots, d_{n-1}'\}.$$

Let $\mathbf{G}'$ denote the subgraph of $\mathbf{G}$ induced on the set $(d_0')^{\varphi_n}$. Then, by Corollary 6.2, $\mathbf{G}'$ admits a restriction of $w(x_0, \ldots, x_{h-1})$ as a polymorphism and has algebraic length one. Further restricting the digraph $\mathbf{G}'$, denote the largest induced subgraph of $\mathbf{G}'$ without sources or sinks by $\mathbf{G}''$. By Lemma 6.4 it admits a weak near unanimity polymorphism. Moreover the elements $\{d_0', \ldots, d_{n-1}', u_0', \ldots, d_{n-1}'\}$ are among the vertices of $\mathbf{G}''$. Thus $\mathbf{G}''$ is a counterexample to Theorem 8.1 and therefore has to be equal to $\mathbf{G}$. This proves the claim. $\qquad\square$

We choose (and fix) k to be a minimal number such that $(d_0')^{\varphi_{k+1}} = V$. Define $W_i = (d_i')^{\varphi_k}$, for each $i < n$. We set

$$W = \bigcap_{i<n} W_i$$

and since W is an intersection of subuniverses of $\mathbf{A}$, by Corollary 6.2, it is itself a subuniverse of $\mathbf{A}$. We denote by $\mathbf{H}$ the subgraph of $\mathbf{G}$ induced by W and prove that $\mathbf{H}$ is a counterexample to Theorem 8.1 contradicting the minimality of $\mathbf{G}$.

The most involved part of the proof deals with constructing an oriented cycle of algebraic length one in $\mathbf{H}$. Two following claims introduce tools for "projecting" certain paths from $\mathbf{G}$ into $\mathbf{H}$.

Claim 8.5. *There exists a term $s(x_0, \ldots, x_{p-1})$ such that for every coordinate $q < p$ there exists i such that*

$$s^{(q)}(W_l, W, \ldots, W) \subseteq W_{[i-l]} \cap W_{[i-l+1]} \text{ for any } l < n.$$

Proof. Let $p = hn$ and let $s(x_0, \ldots, x_{p-1})$ be defined by

$$t\left(w(x_0, \ldots, x_{h-1}), w(x_h, \ldots, x_{2h-1}), \ldots, w(x_{(n-1)h}, \ldots, x_{hn-1})\right).$$

For all $q < p$, let i be maximal such that $q = ih + q''$ for some non-negative q''. Then, for all $l < n$

$$\begin{aligned}
s^{(q)}(W_l, \overline{W}) &\subseteq t^{(i)}\left(w^{(q'')}(W_l, \overline{W}), w(\overline{W}), \ldots, w(\overline{W})\right) \\
&\subseteq t^{(i)}\left(w^{(q'')}(\overline{W_l}, W_{[l+1]}), w(\overline{W_{[l+1]}}, W_{[l+2]}), \ldots, w(\overline{W_{[l+n-1]}}, W_l)\right) \\
&= t^{([i-l])}\left(w(\overline{W_0}, W_1), \ldots, w^{(q'')}(\overline{W_l}, W_{[l+1]}), \ldots, w(\overline{W_{n-1}}, W_0)\right) \\
&\subseteq W_{[i-l]}
\end{aligned}$$

where the last inclusion follows from Claim 8.3 and the fact that

$$\begin{aligned}
d_{[i-l]}' &= t^{([i-l])}(w(\overline{d_0'}, d_1'), \ldots, w(\overline{d_l'}, d_{[l+1]}'), \ldots, w(\overline{d_{n-1}'}, d_0')) \\
&= t^{([i-l])}(w(\overline{d_0'}, d_1'), \ldots, w^{(q'')}(\overline{d_l'}, d_{[l+1]}'), \ldots, w(\overline{d_{n-1}'}, d_0')).
\end{aligned}$$

Similar reasoning shows that

$$s^{(q)}(W_l, \overline{W}) \subseteq t^{(i)}\left(w^{(q'')}(W_l, \overline{W}), w(\overline{W}), \ldots, w(\overline{W})\right)$$

$$\subseteq t^{(i)}\left(w^{(q'')}(W_l, \overline{W_{[l-1]}}), w(W_{[l+1]}, \overline{W_l}), \ldots, w(W_{[l+n-1]}, \overline{W_{[l+n-2]}})\right)$$

$$= t^{[i-l+1]}\left(w(W_1, \overline{W_0}), \ldots, w^{(q'')}(W_l, \overline{W_{[l-1]}}), \ldots, w(W_0, \overline{W_{n-1}})\right)$$

$$\subseteq W_{[i-l+1]}$$

and the proof is finished. $\qquad\square$

Further, using the term constructed in the last claim, we can construct a term satisfying stronger conditions.

Claim 8.6. *There exists a term* $r(x_0, \ldots, x_{m-1})$ *such that for every coordinate* $q < m$

$$r^{(q)}\left(\bigcup_{l<n} W_l, W, \ldots, W\right) \subseteq W.$$

Proof. Let $s(x_0, \ldots, x_{p-1})$ be the p-ary term provided by the previous claim. Note that the term

$$s_2(x_0, x_1, \cdots, x_{p^2-1}) = s(s(x_0, \ldots, x_{p-1}), \ldots, s(x_{p^2-p}, \ldots, x_{p^2-1}))$$

has the property that for every coordinate $q < p^2 - 1$ there exists an i such that

$$s_2^{(q)}(W_l, \overline{W}) \subseteq W_{[i-l]} \cap W_{[i-l+1]} \cap W_{[i-l+2]}.$$

To prove a more general statement we recursively define a sequence of terms

- $s_1(x_0, \ldots, x_{p-1}) = s(x_0, \ldots, x_{p-1})$ and
- $s_{j+1}(x_0, \ldots, x_{p^j-1}) = s(s_j(x_0, \ldots, x_{p^{j-1}-1}), \ldots, s_j(x_{(p-1)p^{j-1}}, \ldots, x_{p^j-1}))$.

We claim that for any j, any $q < p^j$ and any $l < n$ there is an i such that

$$s_j^{(q)}(W_l, W, \ldots, W) \subseteq W_{[i-l]} \cap \ldots \cap W_{[i-l+j]}.$$

We prove this fact by induction. The first step of the induction holds via Claim 8.5. Assume that the fact holds for j, then for any l (setting q' to be the result of integer division of q by p and q'' the remainder of this division) there exist i and i' such that

$$s_{j+1}^{(q)}(W_l, \overline{W}) \subseteq s^{(q')}\left(s_j^{(q'')}(W_l, \overline{W}), s_j(\overline{W}), \ldots, s_j(\overline{W})\right)$$

$$\subseteq s^{(q')}\left(W_{[i-l]} \cap \ldots \cap W_{[i-l+j]}, \overline{W}\right)$$

$$\subseteq W_{[i'+i-l]} \cap \ldots \cap W_{[i'+i-l+(j+1)]}$$

where the second inclusion follows from the induction step and the last one from Claim 8.5. Setting $r(x_0, \ldots, x_{m-1})$ equal to $s_{n-1}(x_0, \ldots, x_{p^n-1})$ proves the claim. $\qquad\square$

From this point on we fix a term $r(x_0, \ldots, x_{m-1})$ (of arity m) provided by the previous claim. To prove additional properties of the set W (e.g. the fact that it is not empty) we require the following easy claim.

Claim 8.7. *Let* α *be an oriented path and let* $a_0 \to a_1$ *and* $b_0 \to b_1$ *be edges that belong to cycles. If* $a_0 \xrightarrow{\alpha} b_0$, *then* $a_1 \xrightarrow{\alpha} b_1$.

Proof. We prove the claim by induction with respect to the number of edges in α. Let the vertices a_0, a_1, b_0, b_1 be as in the statement of the claim. Assume that $a_0 \to b_0$. Since there is a number i such that $a_1 \xrightarrow{in-1} a_0 \to b_0 \to b_1$ then, by Claim 8.2, $a_1 \to b_1$. The same reasoning can be applied to the case of $a_0 \leftarrow b_0$ and the first step of the induction is proved.

For a path α consisting of more than one edge we can assume, without loss of generality, that the last edge is going forward. Then $a_0 \xrightarrow{\alpha'} a'_0 \to b_0$ for some element a'_0 (where α' is the oriented path obtained by removing the last edge of α). By Claim 8.2, it follows that a'_0 is in an n-cycle and therefore $a'_0 \to a'_1 \xrightarrow{n-1} a'_0$ for some a'_1. By the induction hypothesis $a_1 \xrightarrow{\alpha'} a'_1$ and, by the first step of the induction, $a'_1 \to b_1$ which proves the claim. $\qquad\square$

We recall the definition of the top and bottom components of the graph from Subsection 4.3 and prove some basic properties of W.

Claim 8.8. *The digraph* $\mathbf{H}$ *has no sources and no sinks and*

- *if k is even, then every bottom component is contained in W, and*
- *if k is odd, then every top component is contained in W.*

Proof. First we show that, for any vertices a, b such that $a \xrightarrow{i} b \xrightarrow{j} a$ for some i, j,

$$\text{if } a \in W_l \text{ then } b \in W_{[l+i]}.$$

To see this note that if $d'_l \xrightarrow{\varphi_k} a$ and $a \to b \xrightarrow{j} a$ then, using Claim 8.7 and the edge $d'_l \to d'_{[l+1]}$, we infer that $d'_{[l+1]} \xrightarrow{\varphi_k} b$. The same procedure repeated i-many times provides the result for arbitrary i.

Let $a \in W$ be arbitrary and b be such that $a \xrightarrow{i} b \xrightarrow{j} a$ for some numbers i, j. Since $a \in W$ it follows, using the note above, that $b \in \bigcap_{l<n} W_{[l+i]} = W$ and this implies that W is a union of strong components. Since, by Claim 8.2, every vertex in $\mathbf{G}$ belongs to an n-cycle, the digraph $\mathbf{H}$ has no sources or sinks.

Let k be even and let a be a member of a bottom component. Since every element of the graph, by Claim 8.2, belongs to a cycle, there exists b, in the bottom component containing a, such that $a \to b$. Since $(d'_0)^{\varphi_{k+1}} = V$, we have $d'_0 \xrightarrow{\varphi_{k-1}} c \leftarrow a' \to b$ for some a' and c. The vertex a is in a bottom component and therefore a' must be a member of the same bottom component. This implies that $a' \to b \xrightarrow{i} a'$, for some i, and further that $a \to b \xrightarrow{ni-1} a' \to c$. Thus, by Claim 8.2, we have $a \to c$ and $a \in W_0$. Therefore every bottom component is contained in W_0. To see that every a from a bottom component is contained in an arbitrary W_l we find a b satisfying $a \xrightarrow{l} b \xrightarrow{i} a$ for some i and apply the note from the beginning of the proof of the claim. The claim is proved for even k's and the same reasoning provides a proof for odd k and top components. $\qquad\square$

Now we are ready to prove the final claim of this section.

Claim 8.9. *The algebraic length of* $\mathbf{H}$ *is one.*

Proof. In the case where k is odd, we want to find $a, b, c \in W$ and $e \in W_0$ such that

$$a \longrightarrow b \longrightarrow c$$
$$\diagdown \quad \diagup$$
$$e$$

To find such elements we put $e = d'_1$ and find, using Claim 8.8, $b \in W$ from a top component such that $u'_{[2]} \xrightarrow{in-1} b$ for some i. There exist a and c in the same component (and thus in W by Claim 8.8) such that $a \to b \to c$. Since $d'_1 \xrightarrow{1,2} u'_{[2]}$, we have $e \xrightarrow{in+1} b$ and $e \xrightarrow{in+1} c$ and therefore, by Claim 8.2, the elements a, b, c and e satisfy the required properties. Then, using the term $s(x_0, \ldots, x_{m-1})$, we

produce the following oriented path

$$
\begin{array}{c}
b = s(\bar{b}) \quad s(c,\bar{b}) = s^{(1)}(\bar{b},c) \quad s^{(1)}(c,\bar{b},c) = s^{(2)}(\bar{b},c,c) \quad s^{(n-1)}(\bar{c}) = c \\
\bullet \qquad\qquad \bullet \qquad\qquad\qquad \bullet \qquad\qquad\qquad\qquad \bullet \\
\\
\bullet \qquad\qquad\qquad \bullet \qquad\qquad \cdots \\
s(e,\bar{a}) \qquad\qquad s^{(1)}(e,\bar{a},b)
\end{array}
$$
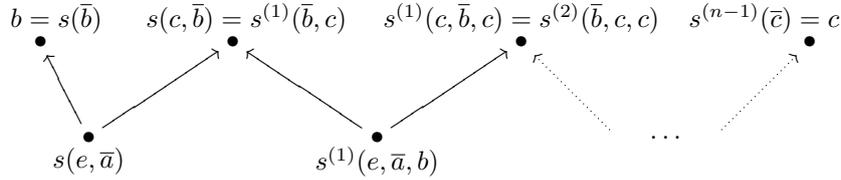
By Claim 8.6, all the elements of this path belong to W. Thus we have constructed a path in $\mathbf{H}$ of algebraic length zero connecting b to c. Since $b \to c$ we immediately obtain that the algebraic length of $\mathbf{H}$ is one.

In the case where k is even, we similarly find $a, b, c \in W$ and $e \in W_0$ (using u_1' for e) such that

$$
\begin{array}{c}
e \\
\nearrow\nwarrow \\
a \longleftarrow b \longleftarrow c
\end{array}
$$
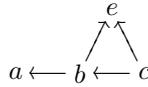
The construction of a path of algebraic length one is the same as it is for odd k, with the exception that the direction of the edges is reversed. $\qquad\square$

Thus $\mathbf{H}$ is a digraph without sources or sinks (by Claim 8.8), admitting a weak near unanimity polymorphism and, by the last claim, having algebraic length equal to one. Since, by the definition of W, the number of vertices in $\mathbf{H}$ is strictly smaller than the number of vertices in $\mathbf{G}$, we obtain a contradiction with the minimality of $\mathbf{G}$ and Theorem 8.1 is proved.

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