Constraint Satisfaction Problems of Bounded Width II

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joint work with Marcin Kozik

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To prove the LZ conjecture, it is enough to prove that every \((2,3)\)-system compatible with an \(SD(\wedge)\) algebra has a solution.
Reminder from Marcin’s talk

To prove the LZ conjecture, it is enough to prove that every $(2,3)$-system compatible with an $SD(\wedge)$ algebra has a solution.

- **A**: $SD(\wedge)$ algebra, i.e. $A$ has WNUs of all but finitely many arities
- **$B_i$, $i < n$**: subalgebras of $A$ (Potatoes, draw them disjoint)
- **$B_{ij}$, $i, j < n$**: subalgebras of $B_i \times B_j$ (Edges between potatoes)
  - $B_{ij} = B_{ji}^{-1}$ (Edges are undirected)
  - $B_{ii}$ is the diagonal (for formal reasons)
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Solution = $(1, 2)$-subsystem with one-element potatoes = clique
How to prove LZ conjecture?

Start with a (2, 3)-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller (2, 3)-subsystem compatible with $\mathbf{A}$!
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- Start with any proper subset and walk around potatoes

Definition
A nonempty subalgebra $C$ of an algebra $B$ is absorbing, if there is an operation $t$ of $B$ such that $t(C, C, \ldots, C) \cup t(C, C, \ldots, C, B) \cup \cdots \cup t(B, C, C, \ldots, C) \subseteq C$
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Prague strategy

pattern = sequence of indices of potatoes, say \( w = 0, 5, 2, 5, 10 \)

For \( a \in B_0, b \in B_{10} \) write

\[ a \xrightarrow{w} b, \quad \text{if} \quad a - c - d - e - b \text{ for some } c \in B_5, d \in B_2, e \in B_5. \]
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Definition

A $(1,2)$-system is called a Prague strategy, if

- for any pattern starting and ending at the same potato, say $w = 1, 2, 4, 2, 8, 1$
- for any $a, b \in B_1$
- if $a, b$ are connected in $B_1 \cup B_2 \cup B_4 \cup B_8$, then there exists a number $k$ such that $a \xrightarrow{w^k} b$
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Theorem (Absorption Theorem)

Let $C, D$ be $SD(\wedge)$ algebras. If $R$ is a connected subalgebra of $C \times D$, then either $R = C \times D$, or $C$ or $D$ has a proper absorbing subalgebra.
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- Two kinds of potatoes
  - Good $B_i$: $B_0i$ “respects” $\alpha$ (the images of $C_j$’s in $B_i$ are disjoint)
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- For $J \subseteq \{1, \ldots, m\}$ we consider subsystem $B^J$
  - In $B_0$ we take the subset $B_0^J = \bigcup_{j \in J} C_j$
  - In other potatoes we take the image of $B_0^J$
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- $B^J$ is a $(1, 2)$-system (for any $J$)
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- $B^{\{j\}}$ is compatible with $A$
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then $\mathcal{B}^K$ is a Prague strategy! (cheating a bit...)
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Theorem (Ugly)

Let $M$ be an $SD(\land)$ algebra. Let $\mathcal{R}$ be a family of subsets of $M$ such that
- $M \in \mathcal{R}$
- if $J \in \mathcal{R}$, $k \in J$ and $K = w(k, k, \ldots, k, J)$ for some WNU $w$ of $M$, then $K \in \mathcal{R}$

Then $\mathcal{R}$ contains a singleton.
Summary

Main ingredients of the proof:

- Absorption and Prague strategies
- Absorption Theorem
- Ugly Theorem
Cheers

Thanks for your attention!