Constraint Satisfaction Problems of Bounded Width II

Libor Barto

joint work with Marcin Kozik

Department of Algebra
Faculty of Mathematics and Physics
Charles University in Prague
Czech Republic

CanaDAM 2009

Reminder from Marcin's talk

To prove the LZ conjecture, it is enough to prove that
every $(2,3)$-system compatible with an $SD(\wedge)$ algebra has a solution.

Reminder from Marcin's talk

To prove the LZ conjecture, it is enough to prove that
every $(2,3)$-system compatible with an $SD(\wedge)$ algebra has a solution.

- $\mathbf{A}$: $SD(\wedge)$ algebra, i.e. $\mathbf{A}$ has WNUs of all but finitely many arities
- $\mathbf{B}_i, i < n$: subalgebras of $\mathbf{A}$ (Potatoes, draw them disjoint)
- $\mathbf{B}_{ij}, i, j < n$: subalgebras of $\mathbf{B}_i \times \mathbf{B}_j$ (Edges between potatoes)
  - $B_{ij} = B_{ji}^{-1}$ (Edges are undirected)
  - B_{ii} is the diagonal (for formal reasons)

Reminder from Marcin's talk

To prove the LZ conjecture, it is enough to prove that every $(2,3)$-system compatible with an $SD(\wedge)$ algebra has a solution.

- $\mathbf{A}$: $SD(\wedge)$ algebra, i.e. $\mathbf{A}$ has WNUs of all but finitely many arities
- $\mathbf{B}_i, i < n$: subalgebras of $\mathbf{A}$ (Potatoes, draw them disjoint)
- $\mathbf{B}_{ij}, i,j < n$: subalgebras of $\mathbf{B}_i \times \mathbf{B}_j$ (Edges between potatoes)
  - $B_{ij} = B_{ji}^{-1}$ (Edges are undirected)
  - B_{ii} is the diagonal (for formal reasons)
- $(1,2)$-system Every point extends to an edge
  - i.e. for all $i,j < n$, $b \in B_i$ there exists $c \in B_j$ s.t. $b - c$ (that is $(b,c) \in B_{ij}$)

Reminder from Marcin's talk

To prove the LZ conjecture, it is enough to prove that
every $(2,3)$-system compatible with an $SD(\wedge)$ algebra has a solution.

- $\mathbf{A}$: $SD(\wedge)$ algebra, i.e. $\mathbf{A}$ has WNUs of all but finitely many arities
- $\mathbf{B}_i, i < n$: subalgebras of $\mathbf{A}$ (Potatoes, draw them disjoint)
- $\mathbf{B}_{ij}, i, j < n$: subalgebras of $\mathbf{B}_i \times \mathbf{B}_j$ (Edges between potatoes)
  - $B_{ij} = B_{ji}^{-1}$ (Edges are undirected)
  - B_{ii} is the diagonal (for formal reasons)
- $(1,2)$-system Every point extends to an edge
  - i.e. for all $i, j < n$, $b \in B_i$ there exists $c \in B_j$ s.t. $b - c$ (that is $(b,c) \in B_{ij}$)
- $(2,3)$-system Every edge extends to a triangle
  - i.e. for all $i, j, k < n$, $b \in B_i, c \in B_j$, $b - c$ there exists $d \in B_k$ s.t. $b - d, c - d$

Reminder from Marcin's talk

To prove the LZ conjecture, it is enough to prove that
every $(2,3)$-system compatible with an $SD(\wedge)$ algebra has a solution.

- $\mathbf{A}$: $SD(\wedge)$ algebra, i.e. $\mathbf{A}$ has WNUs of all but finitely many arities
- $\mathbf{B}_i, i < n$: subalgebras of $\mathbf{A}$ (Potatoes, draw them disjoint)
- $\mathbf{B}_{ij}, i, j < n$: subalgebras of $\mathbf{B}_i \times \mathbf{B}_j$ (Edges between potatoes)
  - $B_{ij} = B_{ji}^{-1}$ (Edges are undirected)
  - B_{ii} is the diagonal (for formal reasons)
- $(1,2)$-system Every point extends to an edge
  - i.e. for all $i, j < n$, $b \in B_i$ there exists $c \in B_j$ s.t. $b - c$ (that is $(b,c) \in B_{ij}$)
- $(2,3)$-system Every edge extends to a triangle
  - i.e. for all $i, j, k < n$, $b \in B_i, c \in B_j$, $b - c$ there exists $d \in B_k$ s.t. $b - d, c - d$

Solution $=$ $(1,2)$-subsystem with one-element potatoes $=$ clique

How to prove LZ conjecture?

Start with a $(2,3)$-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller $(2,3)$-subsystem compatible with $\mathbf{A}$!

How to prove LZ conjecture?

Start with a $(2, 3)$-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller $(2, 3)$-subsystem compatible with $\mathbf{A}$!

- Start with any proper subset and walk around potatoes

How to prove LZ conjecture?

Start with a $(2,3)$-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller $(2,3)$-subsystem compatible with $\mathbf{A}$!

- Start with any proper subset and walk around potatoes
  - We get a proper $(1,2)$-subsystem (the $(2,3)$-property is needed)

How to prove LZ conjecture?

Start with a $(2,3)$-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller $(2,3)$-subsystem compatible with $\mathbf{A}$!

- Start with any proper subset and walk around potatoes
  - We get a proper $(1,2)$-subsystem (the $(2,3)$-property is needed)
  - If we start with a subalgebra, it will be compatible with $\mathbf{A}$

How to prove LZ conjecture?

Start with a $(2,3)$-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller $(2,3)$-subsystem compatible with $\mathbf{A}$!

- Start with any proper subset and walk around potatoes
  - We get a proper $(1,2)$-subsystem (the $(2,3)$-property is needed)
  - If we start with a subalgebra, it will be compatible with $\mathbf{A}$
- We want more than a $(1,2)$-subsystem – we start with an absorbing subalgebra

How to prove LZ conjecture?

Start with a $(2,3)$-system compatible with **A**.

If one of the potatoes is more than 1-element, find smaller $(2,3)$-subsystem compatible with **A**!

- Start with any proper subset and walk around potatoes
  - We get a proper $(1,2)$-subsystem (the $(2,3)$-property is needed)
  - If we start with a subalgebra, it will be compatible with **A**
- We want more than a $(1,2)$-subsystem – we start with an absorbing subalgebra

Definition

A nonempty subalgebra **C** *of an algebra* **B** *is absorbing, if there is an operation* t *of* **B** *such that*
$t(C, C, \ldots, C, B) \cup t(C, C, \ldots, C, B, C) \cup \cdots \cup t(B, C, C \ldots, C) \subseteq C$

How to prove LZ conjecture?

Start with a $(2,3)$-system compatible with $\mathbf{A}$.

If one of the potatoes is more than 1-element, find smaller $(2,3)$-subsystem compatible with $\mathbf{A}$!

- Start with any proper subset and walk around potatoes
  - We get a proper $(1,2)$-subsystem (the $(2,3)$-property is needed)
  - If we start with a subalgebra, it will be compatible with $\mathbf{A}$
- We want more than a $(1,2)$-subsystem – we start with an absorbing subalgebra

Definition

A nonempty subalgebra $\mathbf{C}$ *of an algebra* $\mathbf{B}$ *is absorbing, if there is an operation* t *of* $\mathbf{B}$ *such that*
$t(C,C,\dots,C,B) \cup t(C,C,\dots,C,B,C) \cup \dots \cup t(B,C,C\dots,C) \subseteq C$

- We are still not able to get triangles inside...

Prague strategy

pattern = sequence of indices of potatoes, say $w = 0, 5, 2, 5, 10$

For $a \in B_0$, $b \in B_{10}$ write

$a \xrightarrow{w} b$, if $a - c - d - e - b$ for some $c \in B_5$, $d \in B_2$, $e \in B_5$.

Prague strategy

pattern = sequence of indices of potatoes, say $w = 0, 5, 2, 5, 10$

For $a \in B_0$, $b \in B_{10}$ write

$a \xrightarrow{w} b$, if $a - c - d - e - b$ for some $c \in B_5$, $d \in B_2$, $e \in B_5$.

Definition

A $(1, 2)$-system is called a Prague strategy, if

- *for any pattern starting and ending at the same potato, say $w = 1, 2, 4, 2, 8, 1$*
- *for any $a, b \in B_1$*
- *if a, b are connected in $B_1 \cup B_2 \cup B_4 \cup B_8$, then there exists a number k such that $a \xrightarrow{w^k} b$*

How to really prove LZ conjecture

Observation

$(2,3)$-system is a Prague strategy.

How to really prove LZ conjecture

Observation

$(2,3)$-system is a Prague strategy.

Start with a Prague strategy compatible with $\mathbf{A}$

If one of the potatoes is more than 1-element, find a smaller Prague substrategy compatible with $\mathbf{A}$!

How to really prove LZ conjecture

Observation

$(2,3)$-system is a Prague strategy.

Start with a Prague strategy compatible with $\mathbf{A}$

If one of the potatoes is more than 1-element, find a smaller Prague substrategy compatible with $\mathbf{A}$!

- If some potato has a proper absorbing subalgebra, we can do it!

How to really prove LZ conjecture

Observation

$(2,3)$-system is a Prague strategy.

Start with a Prague strategy compatible with **A**

If one of the potatoes is more than 1-element, find a smaller Prague substrategy compatible with **A**!

- If some potato has a proper absorbing subalgebra, we can do it!
- Proper absorbing subalgebras are not rare!!!

How to really prove LZ conjecture

Observation

$(2,3)$-system is a Prague strategy.

Start with a Prague strategy compatible with $\mathbf{A}$

If one of the potatoes is more than 1-element, find a smaller Prague substrategy compatible with $\mathbf{A}$!

- If some potato has a proper absorbing subalgebra, we can do it!
- Proper absorbing subalgebras are not rare!!!

Theorem (Absorption Theorem)

Let $\mathbf{C}, \mathbf{D}$ be $SD(\wedge)$ algebras. If $\mathbf{R}$ is a connected subalgebra of $\mathbf{C} \times \mathbf{D}$, then either $R = C \times D$, or $\mathbf{C}$ or $\mathbf{D}$ has a proper absorbing subalgebra.

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)
  Let $C_1, \dots, C_m$ be the α-blocks

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)
  Let $C_1, \dots, C_m$ be the α-blocks
- Two kinds of potatoes
  - Good B_i: B_{0i} "respects" α (the images of C_j's in B_i are disjoint)

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)
  Let $C_1, \ldots, C_m$ be the α-blocks
- Two kinds of potatoes
  - Good B_i: B_{0i} "respects" α (the images of C_j's in B_i are disjoint)
  - Bad B_i: B_{0i} doesn't respect α

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)

  Let $C_1, \ldots, C_m$ be the α-blocks
- Two kinds of potatoes
  - Good B_i: B_{0i} "respects" α (the images of C_j's in B_i are disjoint)
  - Bad B_i: B_{0i} doesn't respect α

    The image of each C_j is B_i! (from maximality of α and Absorption Theorem)

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)
  Let $C_1, \ldots, C_m$ be the α-blocks
- Two kinds of potatoes
  - Good B_i: B_{0i} "respects" α (the images of C_j's in B_i are disjoint)
  - Bad B_i: B_{0i} doesn't respect α
    The image of each C_j is B_i! (from maximality of α and Absorption Theorem)
- For $J \subseteq \{1, \ldots, m\}$ we consider subsystem $\mathcal{B}^J$
  - In B_0 we take the subset $B_0^J = \cup_{j \in J} C_j$
  - In other potatoes we take the image of B_0^J

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)

  Let $C_1, \ldots, C_m$ be the α-blocks
- Two kinds of potatoes
  - Good B_i: B_{0i} "respects" α (the images of C_j's in B_i are disjoint)
  - Bad B_i: B_{0i} doesn't respect α

    The image of each C_j is B_i! (from maximality of α and Absorption Theorem)
- For $J \subseteq \{1, \ldots, m\}$ we consider subsystem $\mathcal{B}^J$
  - In B_0 we take the subset $B_0^J = \cup_{j \in J} C_j$
  - In other potatoes we take the image of B_0^J
- $\mathcal{B}^J$ is a $(1,2)$-system (for any J)

No absorbing sets $\Rightarrow$ some structure

Now we can assume "no proper absorbing set in any potato"

- Assume $|B_0| > 1$
- Take a maximal congruence α of $\mathbf{B}_0$ (congruence = compatible equivalence)
  Let $C_1, \dots, C_m$ be the α-blocks
- Two kinds of potatoes
  - Good B_i: B_{0i} "respects" α (the images of C_j's in B_i are disjoint)
  - Bad B_i: B_{0i} doesn't respect α

    The image of each C_j is B_i! (from maximality of α and Absorption Theorem)
- For $J \subseteq \{1, \dots, m\}$ we consider subsystem $\mathcal{B}^J$
  - In B_0 we take the subset $B_0^J = \cup_{j \in J} C_j$
  - In other potatoes we take the image of B_0^J
- $\mathcal{B}^J$ is a $(1,2)$-system (for any J)
- $\mathcal{B}^{\{j\}}$ is compatible with $\mathbf{A}$

Smaller Prague substrategy in “no absorption” case

- Every operation of **A** yields an operation on $\{1, \ldots, m\}$

Smaller Prague substrategy in “no absorption” case

- Every operation of $\mathbf{A}$ yields an operation on $\{1, \ldots, m\}$
- Let $J \subseteq \{1, 2, \ldots, m\}$. If

Smaller Prague substrategy in "no absorption" case

- Every operation of $\mathbf{A}$ yields an operation on $\{1, \dots, m\}$
- Let $J \subseteq \{1, 2, \dots, m\}$. If
  - $\mathcal{B}^J$ is a Prague strategy

Smaller Prague substrategy in "no absorption" case

- Every operation of $\mathbf{A}$ yields an operation on $\{1, \ldots, m\}$
- Let $J \subseteq \{1, 2, \ldots, m\}$. If
  - $\mathcal{B}^J$ is a Prague strategy
  - w is a WNU of $\mathbf{A}$

Smaller Prague substrategy in "no absorption" case

- Every operation of **A** yields an operation on $\{1, \ldots, m\}$
- Let $J \subseteq \{1, 2, \ldots, m\}$. If
  - $\mathcal{B}^J$ is a Prague strategy
  - w is a WNU of **A**
  - $k \in J$ and $K = w(k, k, \ldots, k, J)$ $(= \{w(k, k, \ldots, k, j) : j \in J\})$

Smaller Prague substrategy in "no absorption" case

- Every operation of $\mathbf{A}$ yields an operation on $\{1, \ldots, m\}$
- Let $J \subseteq \{1, 2, \ldots, m\}$. If
  - $\mathcal{B}^J$ is a Prague strategy
  - w is a WNU of $\mathbf{A}$
  - $k \in J$ and $K = w(k, k, \ldots, k, J)$ $(= \{w(k, k, \ldots, k, j) : j \in J\})$

then $\mathcal{B}^K$ is a Prague strategy! (cheating a bit...)

Smaller Prague substrategy in "no absorption" case

- Every operation of **A** yields an operation on $\{1, \ldots, m\}$
- Let $J \subseteq \{1, 2, \ldots, m\}$. If
  - $\mathcal{B}^J$ is a Prague strategy
  - w is a WNU of **A**
  - $k \in J$ and $K = w(k, k, \ldots, k, J)$ $\;(= \{w(k, k, \ldots, k, j) : j \in J\})$

  then $\mathcal{B}^K$ is a Prague strategy! (cheating a bit...)

Theorem (Ugly)

Let **M** *be an* $SD(\wedge)$ *algebra. Let* $\mathcal{R}$ *be a family of subsets of* M *such that*

- $M \in \mathcal{R}$
- *if* $J \in \mathcal{R}$, $k \in J$ *and* $K = w(k, k, \ldots, k, J)$ *for some WNU* w *of* **M**, *then* $K \in \mathcal{R}$

Then $\mathcal{R}$ *contains a singleton.*

Summary

Main ingredients of the proof:

- Absorption and Prague strategies
- Absorption Theorem
- Ugly Theorem

Cheers

Thanks for your attention!