Applications of the Constraint Satisfaction Problem to Universal Algebra

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CSP for universal algebraist

Definition

$\mathbf{A}$ . . . finite idempotent algebra (always)

Instance of $\mathrm{CSP}(\mathbf{A})$ = finite set V + set $\mathcal{C}$ of constraints

Constraint = subalgebra of $\mathbf{A}^I$, where $I \subseteq V$ is the scope

Solution of the instance = mapping $f : V \to A$ such that $f_{|I} \in R_{|I}$ for every constraint $R \leq \mathbf{A}^I$

Example

every scope is equal to V

$\Downarrow$

set of solutions = intersection of constraints

to study CSP = to study intersection properties of subpowers

Application 1: $\mathrm{SD}(\wedge)$ algebras 1/3

A is $\mathrm{SD}(\wedge)$, if $\mathrm{HSP}(\mathbf{A})$ is meet semi-distributive
$\Leftrightarrow$ $\mathrm{HSP}(\mathbf{A})$ omits $\mathbf{1}, \mathbf{2}$
$\Leftrightarrow$ **A** has Willard terms
$\Leftrightarrow$ **A** not interpretable into module $\Leftrightarrow \dots$

Definition

An instance of $\mathrm{CSP}(\mathbf{A})$ is $(2,3)$-minimal, if

- $\forall$ three-element $I \subseteq V$ $\exists R \subseteq \mathbf{A}^K$ in $\mathcal{C}$ such that $I \subseteq K$
- $\forall$ at most two-element $I \subseteq V$ $\forall R \subseteq \mathbf{A}^K$, $R' \subseteq \mathbf{A}^{K'}$ in $\mathcal{C}$ such that $I \subseteq K, K'$ we have $R_{|I} = R'_{|I}$

Application 1: $\mathrm{SD}(\wedge)$ algebras 2/3

Theorem (Barto, Kozik 09)

$\mathbf{A}$ is $\mathrm{SD}(\wedge)$ iff every $(2,3)$-minimal instance of $\mathrm{CSP}(\mathbf{A})$ has a solution

Corollary

If $\mathbf{A}$ is $\mathrm{SD}(\wedge)$ and $R_1, \ldots, R_n \leq \mathbf{A}^n$ have the same binary projections, then $\cap R_i \neq \emptyset$

Application 1: $\mathrm{SD}(\wedge)$ algebras

WNU = operation f satisfying
$f(x,\dots,x,y) = f(x,\dots,x,y,x) = \dots = f(y,x,\dots,x)$

Corollary (Kozik, Valeriote)

***A** is $\mathrm{SD}(\wedge)$ iff **A** has WNUs of all arities ≥ 3.*

Proof.

Take V big enough. Let $\mathbf{F}$ = free algebra in HSP($\mathbf{A}$) over $\{x,y\}$.
for every three element I we include one constraint $R_I \leq \mathbf{F}^I$, where

$$R_I = \langle (x,x,y),(x,y,x),(y,x,x)\rangle$$

It is $(2,3)$-minimal instance of $\mathrm{CSP}(\mathbf{F}) \Rightarrow \exists$ solution $f : V \to F$.
V is big $\Rightarrow \exists i,j,k$ $f(i) = f(j) = f(k) = b$
$b \in R_{\{i,j,k\}} \Rightarrow \mathbf{A}$ has WNU of arity 3 $\square$

Application 2: CD $\Rightarrow$ NU

$\mathbb{A}$ ... relational structure (on a finite set A)
Pol($\mathbb{A}$) ... clone of all operations compatible with all relations in $\mathbb{A}$

Theorem (Geiger, Bodnarchuk, Kaluznin, Kotov, Romov 68)

$\forall$ *finite algebra* $\mathbf{A}$ $\exists$ $\mathbb{A}$ *such that* $\mathrm{Pol}(\mathbb{A}) = \mathrm{Clo}(\mathbf{A})$

Definition

Finite $\mathbf{A}$ is finitely related, if $\exists$ $\mathbb{A}$ with finitely many relations such that $\mathrm{Pol}(\mathbb{A}) = \mathrm{Clo}(\mathbf{A})$

Example

Algebras with near-unanimity term (by Baker-Pixley)
(Recall: near-unanimity = operation f satisfying
$x = f(x, \ldots, x, y) = f(x, \ldots, x, y, x) = \cdots = f(y, x, \ldots, x)$)

Application 2: CD $\Rightarrow$ NU

Theorem (Barto 09)

If $\mathbf{A}$ *is finitely related and* $\mathrm{HSP}(\mathbf{A})$ *is congruence distributive, then* $\mathbf{A}$ *has a near-unanimity term.*

Proof.

Say $\mathbb{A}$ has at most k-ary relations, $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$.
n … big enough natural number
$V = A^n$
$F \leq \mathbf{A}^V$ … free algebra on n-generators (=n-ary operations)
For every at most k-element $I \subseteq V$ we include the constraint $F_{|I}$
Solutions of this instance = n-ary operations of $\mathbf{A}$
………. $\square$

More collapses of Maltsev conditions for finitely related algebras

Application 3: few subpowers $\Rightarrow$ finitely related 1/2

$\mathbf{A}$ has few subpowers, if $|\{R \leq \mathbf{A}^n\}| \leq 2^{polynomial(n)}$
$\Leftrightarrow$ subpowers of $\mathbf{A}$ have small generating sets
$\Leftrightarrow$ $\mathbf{A}$ has a cube term $\Leftrightarrow$ ...

Example

Maltsev algebras, algebras with near-unanimity operation
Few subpowers $\Rightarrow$ HSP($\mathbf{A}$) is congruence modular

Theorem (Aichinger, Mayr, McKenzie 09)

Every finite algebra with few subpowers is finitely related.

Proof.

Use compact representations of subpowers developed for CSP by Dalmau, Bulatov; Berman, Idziak, Markovic, McKenzie, Valeriote, Willard

. . . . $\square$

Application 3: few subpowers $\Rightarrow$ finitely related 2/2

Corollary

On a finite set, there is countably many clones with few subpowers (in particular, there is countably many Maltsev clones on a finite set).

(2 years ago open for expansions of $\mathbb{Z}_8$!)

Conjecture (Valeriote's conjecture, Edinburgh conjecture)

If $\mathbf{A}$ is finitely related and $\mathrm{HSP}(\mathbf{A})$ is congruence modular, then $\mathbf{A}$ has few subpowers.

Application 4: Taylor algebras

A is Taylor, if HSP(**A**) doesn't contain a G-set
$\Leftrightarrow$ HSP(**A**) omits **1**
$\Leftrightarrow$ HSP(**A**) satisfies a nontrivial Maltsev condition
$\Leftrightarrow$ **A** has a Taylor term $\Leftrightarrow$ ...

Theorem (Barto, Kozik, Niven 08)

Let $\mathbf{A}$ *be a Taylor algebra,* $R \leq \mathbf{A} \times \mathbf{A}$ *subdirect and assume that* $\exists\ k, l$ *such that* $(R^k \circ R^{-k})^l = A^2$. *Then* $\exists\ a \in A \quad (a, a) \in R$.

Corollary (Siggers, Markovic, McKenzie 10)

$\mathbf{A}$ *is Taylor iff* $\mathbf{A}$ *has a term* t *satisfying* $t(xyyz) = t(yzxx)$.

Proof.

$\mathbf{F}$ ... free algebra on $\{x, y, z\}$
R ... subalgebra of $\mathbf{F}^2$ generated by $(x, y), (y, z), (y, x), (z, x)$.
Apply the theorem $\square$

CSP $\Rightarrow$ it is a good idea to study algebras via relations

CSP $\Rightarrow$ better understanding of intersection properties $\Rightarrow$ Maltsev conditions

CSP $\Rightarrow$ new important classes of algebras

CSP $\Rightarrow$ absorption rocks

Thank you!