A loop lemma for nonidempotent cores

Libor Barto and Marcin Kozik

Department of Algebra, Charles University, Prague

AAA 97, Wien, 1–3 March 2019

European Research Council
Established by the European Commission

CoCoSym: Symmetry in Computational Complexity

This project has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 771005)

loop lemma = theorem of the form

Theorem (a loop lemma)

Let $\mathbf{A}$ *be an algebra such that [...(weak) algebraic assumptions...].*
Let $R \leq \mathbf{A}^2$ *be such that [...(weak) structural assumptions...].*

Then R *has a loop, ie.* $(\exists a)\ (a, a) \in R$

known: many

this talk: a new loop lemma

Typical algebraic assumptions on **A**

- finiteness, e.g. **A** is finite
- tame unaries, e.g.
  - **A** is idempotent = only unary term operation is id
  - **A** is core = every unary term operation is a permutation
- some equational condition, e.g.
  A satisfies some nontrivial Maltsev condition

strong Maltsev condition = there exist term operations $t_1, \ldots, t_n$ satisfying [...fixed set of identities...]

Maltsev condition = countable disjunction of strong ones

Example: $(\exists m)$ such that $m(x, x, y) \approx y \approx m(y, x, x)$

Recall: Maltsev conditions $\leftrightarrow$ properties of compatible relations

Fact: **A** satisfies some nontrivial (strong) Maltsev condition
$\Leftrightarrow$ HSP(**A**) does not contain a naked set
= each operation is a projection

Typical structural assumptions on $R \le \mathbf{A}^2$

Viewpoint: digraph $(A; R)$, where $A =$ vertices, $R =$ edges

- $R \le_{subdirect} \mathbf{A}^2$ (no sources or sinks)
- $(A; R)$ is connected
- R is symmetric $= (A; R)$ is an undirected graph
- $(A; R)$ is far from a directed cycle
  - $(A; R)$ is not bipartite (in symmetric case)
  - $(A; R)$ is linked: $(\forall a, b \in A)$ $a \nearrow\!\nwarrow \nearrow\!\nwarrow \cdots \nearrow\!\nwarrow b$
  - $(A; R)$ has no homomorphism to a non-trivial directed cycle

The first loop lemma

Theorem ([Hell, Nešetřil'90]; [Bulatov'05])

Let $\mathbf{A}$ *be an algebra such that*

- A *is finite*
- $\mathbf{A}$ *is idempotent*
- $\mathbf{A}$ *satisfies some nontrivial Maltsev condition*

Let $R \leq \mathbf{A}^2$ *be such that*

- $R \leq_{subdirect} \mathbf{A}^2$
- $(A; R)$ *is connected*
- R *is symmetric*
- $(A; R)$ *is not bipartite*

Then R *has a loop.*

A loop lemma called "the loop lemma"

Theorem ([Barto, Kozik, Niven'08])

Let $\mathbf{A}$ *be an algebra such that*

- A *is finite*
- $\mathbf{A}$ *is idempotent*
- $\mathbf{A}$ *satisfies some nontrivial Maltsev condition*

Let $R \leq \mathbf{A}^2$ *be such that*

- $R \leq_{subdirect} \mathbf{A}^2$
- $(A; R)$ *is connected*
- $(A; R)$ *has no homomorphism to a non-trivial directed cycle*

Then R *has a loop.*

- it tells us something nontrivial about binary relations compatible with **A**
  not congruences; large class of finite algebras
  - NP-hardness results for some constraint satisfaction problems (CSPs)
  - Thm: **A** finite, idempotent
    satisfies a nontrivial Malsev condition
    $\Rightarrow$ **A** has a term operation $s(r, a, r, e) \approx s(a, r, e, a)$
    [Kearnes, Marković, McKenzie'14]
- it has led to new useful concepts and theorems (e.g. absorption theorem)

Why aren't we satisfied?

- only finite algebras
- only idempotent algebras

Motivation for generalizations

- universal algebra
- infinite domain CSP

A new loop lemma

Theorem ([Barto, Kozik])

Let $\mathbf{A}$ *be an algebra such that*

- A *is finite*
- $\mathbf{A}$ *is a core*
- $\mathbf{A}$ *satisfies some nontrivial Maltsev condition*

Let $R \leq \mathbf{A}^2$ *be such that*

- $R \leq_{subdirect} \mathbf{A}^2$
- $(A; R)$ *is linked*

Then R *has a loop.*

Corollary

If **A** *is a finite core satisfying a nontrivial Maltsev condition, then* **A** *has term operations such that*

$$\begin{aligned}
&t(\alpha_1 x, \dots, \alpha_k x, \beta_1 y, \dots, \beta_k y, \gamma_1 x, \dots, \gamma_k x, \delta_1 z, \dots, \delta_k z,\\
&\quad \epsilon_1 y, \dots, \epsilon_k y, \zeta_1 z, \dots, \zeta_k z)\\
\approx\ &t(\eta_1 y, \dots, \eta_k y, \theta_1 x, \dots, \theta_k x, \iota_1 z, \dots, \iota_k z, \kappa_1 x, \dots, \kappa_k x,\\
&\quad \lambda_1 z, \dots, \lambda_k z, \mu_1 y, \dots, \mu_k y)
\end{aligned}$$

We would like to

- have a nicer corollary (at least k fixed)
- prove an infinite version
- get rid of the the coreness assumption
- weaken the structural assumption

Definition

B absorbs **A**, written $B \lhd \mathbf{A}$ if $(\forall i)\ t(B, \dots, B, \underset{i}{A}, B, \dots, B) \subseteq B$

By induction on $|A|$. The induction step is:

- find a proper subalgebra $\mathbf{B} \lhd \mathbf{A}$
  - find either proper absorption or a transitive term operation
    $$(\forall i)\ (\forall a \in A) t(A, \dots, A, \underset{i}{\{a\}}, A, \dots, A) = A$$
  - get absorption from linked R + transitive term operation
- improve $\mathbf{B}$ so that $R' := R \cap B^2 \leq_{subdirect} \mathbf{B}^2$
- prove that R' is still linked
- use induction hypothesis for $R' \leq \mathbf{B}^2$

2 new ingredients

- getting transitive operation without idempotency
- getting absorption without idempotency

getting absorption **with** idempotency

- from $R <_{subdirect} \mathbf{A}^2$ linked one can pp–define $S <_{subdirect} \mathbf{A}^2$ with a central element a: $(\forall b)(a, b) \in S$
- such S + transitive operation gives absorption

getting absorption **without** idempotency

- Zhuk: from Rosenberg's classification it should follow that linked $R <_{subdirect} \mathbf{A}^2$ gives absorption
  ... it is enough to go through xxx cases
- "ingenious" idea: look at Rosenberg's proof
  ...or Pinsker's master thesis
- it is there

Thank you!