The basic CSP reductions revisited

Libor Barto, joint with Michael Pinsker

Charles University in Prague

Banff workshop
November 2014

Outline and notation

Outline

- Basic CSP reductions – 3 views
- Questions
- Basic CSP reductions revisited

Notation

- $\mathcal{A}$ . . . finite set of relations on A
- $\mathbf{A}$ . . . the clone of polymorphisms of $\mathcal{A}$

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$

- $\mathcal{B}$ is a "pp-power" of $\mathcal{A}$
  ie. $B = A^n$ and
  each l-ary $R \in \mathcal{B}$ is pp-def from $\mathcal{A}$ as an nl-ary relation

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$
- $\mathcal{B}$ is a "pp-power" of $\mathcal{A}$
  ie. $B = A^n$ and
  each l-ary $R \in \mathcal{B}$ is pp-def from $\mathcal{A}$ as an nl-ary relation
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ (induced substructure), where S is pp-def from $\mathcal{A}$

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$
- $\mathcal{B}$ is a "pp-power" of $\mathcal{A}$
  ie. $B = A^n$ and
  each l-ary $R \in \mathcal{B}$ is pp-def from $\mathcal{A}$ as an nl-ary relation
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ (induced substructure), where S is pp-def from $\mathcal{A}$
- $\mathcal{B} \cong \mathcal{A}/\sim$, where $\sim$ is pp-def from $\mathcal{A}$

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$

- $\mathcal{B}$ is a "pp-power" of $\mathcal{A}$
  ie. $B = A^n$ and
  each l-ary $R \in \mathcal{B}$ is pp-def from $\mathcal{A}$ as an nl-ary relation
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ (induced substructure), where S is pp-def from $\mathcal{A}$
- $\mathcal{B} \cong \mathcal{A}/\sim$, where $\sim$ is pp-def from $\mathcal{A}$
- last three together $\Leftrightarrow$ $\mathcal{A}$ pp-interprets $\mathcal{B}$

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$

- $\mathcal{B}$ is a "pp-power" of $\mathcal{A}$
  ie. $B = A^n$ and
  each l-ary $R \in \mathcal{B}$ is pp-def from $\mathcal{A}$ as an nl-ary relation
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ (induced substructure), where S is pp-def from $\mathcal{A}$
- $\mathcal{B} \cong \mathcal{A}/\sim$, where $\sim$ is pp-def from $\mathcal{A}$
- last three together $\Leftrightarrow$ $\mathcal{A}$ pp-interprets $\mathcal{B}$

- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$ . . . WLOG core

Basic reductions – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$

- $\mathcal{B}$ is a "pp-power" of $\mathcal{A}$
  ie. $B = A^n$ and
  each l-ary $R \in \mathcal{B}$ is pp-def from $\mathcal{A}$ as an nl-ary relation
- $\mathcal{B} = \mathcal{A}|_S$ (induced substructure), where S is pp-def from $\mathcal{A}$
- $\mathcal{B} \cong \mathcal{A}/\sim$, where $\sim$ is pp-def from $\mathcal{A}$
- last three together $\Leftrightarrow$ $\mathcal{A}$ pp-interprets $\mathcal{B}$

- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$ ... WLOG core
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core

Basic reductions – via algebraic constructions

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathbf{B}$ is an expansion of $\mathbf{A}$ (ie. $\mathbf{A} \subseteq \mathbf{B}$)

- $\mathbf{B} = \mathbf{A}^n$
- $\mathbf{B}$ is a subalgebra of $\mathbf{A}$
- $\mathbf{B} \cong \mathbf{A}/\sim$, where $\sim$ is a congruence of $\mathbf{A}$
- last three $\Leftrightarrow$ $\mathbf{B} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$

- Finite: WLOG $\mathbf{A}$ idempotent
- Infinite: A bit different

Basic reductions – via clone homomorphisms

Definition

$\xi : \mathbf{A} \to \mathbf{B}$ is a clone homomorphism, if it

- preserves arities
- sends projections to projections $\xi(\pi_i^n) = \pi_i^n$
- preserves composition:
  $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$,
  where $f \in \mathbf{A}$ is n-ary, $g_i \in \mathbf{A}$ is m-ary

Alternatively: ξ-images satisfy the same identities.

The 3 together for finite

Theorem (Bodnaruk et el./Geiger; Birkhoff)

TFAE if A, B are finite:

1. $\mathcal{A}$ *pp-interprets* $\mathcal{B}$

$$\mathcal{A} \overset{\textit{pp-power}}{\rightsquigarrow} \mathcal{E} \overset{\textit{substr}}{\rightsquigarrow} \mathcal{F} \overset{\textit{quotient}}{\rightsquigarrow} \mathcal{B}$$

2. $\mathbf{B}$ *is an expansion of a clone in* $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$

$$\mathbf{A} \rightsquigarrow \mathbf{A}^n \overset{\textit{subalg}}{\rightsquigarrow} \mathbf{C} \overset{\textit{quotient}}{\rightsquigarrow} \mathbf{C}/\sim \overset{\textit{expansion}}{\rightsquigarrow} \mathbf{B}$$

3. *There exists a clone homomorphisms* $\xi : \mathbf{A} \to \mathbf{B}$

$(3) \Rightarrow (2)$

Assume $\xi : \mathbf{A} \to \mathbf{B}$ is a clone homomorphism.
Want:

$$\mathbf{A} \rightsquigarrow \mathbf{A}^n \overset{\text{subalg}}{\rightsquigarrow} \mathbf{C} \overset{\text{quotient}}{\rightsquigarrow} \mathbf{C}/\sim \overset{\text{expansion}}{\rightsquigarrow} \mathbf{B}$$

- Say $B = \{b_1, \ldots, b_k\}$
- $n = A^k$
- $C = k$-ary operations in $\mathbf{A}$
  ($\mathbf{C}$ is the free algebra with B generators)
- Define $f : C \to B$ by $t \mapsto \xi(t)(b_1, \ldots, b_k)$
- $\sim = \ker f$ is a congruence of $\mathbf{A}$ and f gives an isomorphism $\mathbf{C}/\sim \to \xi(\mathbf{A})$

The 3 together for infinite

Theorem (Bodirsky, Nešetřil; Bodirsky, Pinsker)

TFAE if $\mathcal{A}, \mathcal{B}$ ω-categorical:

1. *$\mathcal{A}$ pp-interprets $\mathcal{B}$*

$$\mathcal{A} \overset{\textit{pp-power}}{\rightsquigarrow} \mathcal{E} \overset{\textit{substr}}{\rightsquigarrow} \mathcal{F} \overset{\textit{quotient}}{\rightsquigarrow} \mathcal{B}$$

2. *$\mathbf{B}$ is an expansion of a clone in $\mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$.*

$$\mathbf{A} \rightsquigarrow \mathbf{A}^n \overset{\textit{subalg}}{\rightsquigarrow} \mathbf{C} \overset{\textit{quotient}}{\rightsquigarrow} \mathbf{C}/\sim \overset{\textit{expansion}}{\rightsquigarrow} \mathbf{B}$$

3. *There exists a continuous clone homomorphisms $\xi : \mathbf{A} \to \mathbf{B}$ such that $\overline{\xi(\mathbf{A})}$ is oligomorphic*

Algebraic dichotomy conjecture vs. reality

Conjecture (Bulatov, Jeavons, Krokhin; Barto, Kozik)

Assume A finite. TFAE

- $\mathrm{CSP}(\mathcal{A})$ in P
- **A** contains an operation t of arity ≥ 2 such that

$$t(x_1, x_2, \ldots, x_n) = t(x_2, \ldots, x_n, x_1)$$

Reality can be worse:

Algebraic dichotomy conjecture vs. reality

Future theorem (Antonín Barto, Bálint Maróti 2063)

Assume A finite. TFAE

- $\mathrm{CSP}(\mathcal{A})$ *in P*
- **A** *contains an operations* $t_1, t_2, \dots$ *such that*

$$t_1(x_1, t_2(x_{37}, x_2), t_3(x_{123})) = t_3(t_3(t_2(x_{13}, x_2)))$$
$$t_{20}(t_{12}(x_1), x_1, x_2) = t_{13}(x_2, x_1)$$
$$\dots$$

...but certainly the characterization looks like this (for any complexity class)

Questions

- Theorem linking the 3 views does not cover all the easy reductions...

Questions

- Theorem linking the 3 views does not cover all the easy reductions...
- ...and by adding homomorphic equivalence we get an essentially coarser ordering. What is this ordering?

Questions

- Theorem linking the 3 views does not cover all the easy reductions...
- ...and by adding homomorphic equivalence we get an essentially coarser ordering. What is this ordering?
- In most identities relevant in CSP, there are no nested terms. Is it possible to prove that nesting is not necessary?

Basic reductions revisited – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$
- $\mathcal{B}$ is a pp-power of $\mathcal{A}$
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$
- $\mathcal{B} = \mathcal{A}/\sim$
- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core

Basic reductions revisited – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$ redundant
- $\mathcal{B}$ is a pp-power of $\mathcal{A}$
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$
- $\mathcal{B} = \mathcal{A}/\sim$
- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core

Basic reductions revisited – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$ redundant
- $\mathcal{B}$ is a pp-power of $\mathcal{A}$
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ redundant
- $\mathcal{B} = \mathcal{A}/\sim$
- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core

Basic reductions revisited – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$ redundant
- $\mathcal{B}$ is a pp-power of $\mathcal{A}$
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ redundant
- $\mathcal{B} = \mathcal{A}/\sim$ redundant
- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core

Basic reductions revisited – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$ redundant
- $\mathcal{B}$ is a pp-power of $\mathcal{A}$
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ redundant
- $\mathcal{B} = \mathcal{A}/\sim$ redundant
- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core redundant (less obvious)

Basic reductions revisited – via relations

$\mathrm{CSP}(\mathcal{B})$ is log-space reducible to $\mathrm{CSP}(\mathcal{A})$ if

- $\mathcal{B}$ is pp-definable from $\mathcal{A}$ redundant
- $\mathcal{B}$ is a pp-power of $\mathcal{A}$
- $\mathcal{B} = \mathcal{A}|_{\mathcal{S}}$ redundant
- $\mathcal{B} = \mathcal{A}/\sim$ redundant
- $\mathcal{B}$ is homomorphically equivalent to $\mathcal{A}$
- $\mathcal{B} = \mathcal{A} \cup \{a\}$, if $\mathcal{A}$ is a core redundant (less obvious)

Observation

TFAE

- *$\mathcal{B}$ can be obtained from $\mathcal{A}$ using the above constructions*
- *$\mathcal{B}$ is homomorphically equivalent to a pp-power of $\mathcal{A}$*

Basic reductions revisited – via algebraic constructions

- $\mathbf{B}$ is an expansion of $\mathbf{A}$ (ie. $\mathbf{A} \subseteq \mathbf{B}$)
- $\mathbf{B} = \mathbf{A}^n$
- $\mathbf{B}$ is a subalgebra of $\mathbf{A}$
- $\mathbf{B} = \mathbf{A}/\sim$, where $\sim$ is a congruence of $\mathbf{A}$

Basic reductions revisited – via algebraic constructions

- $\mathbf{B}$ is an expansion of $\mathbf{A}$ (ie. $\mathbf{A} \subseteq \mathbf{B}$)
- $\mathbf{B} = \mathbf{A}^n$
- $\mathbf{B}$ is a subalgebra of $\mathbf{A}$
- $\mathbf{B} = \mathbf{A}/\sim$, where $\sim$ is a congruence of $\mathbf{A}$

Definition

$\mathbf{B}$ is a unary modification of $\mathbf{A}$ if $\exists\, f : A \to B$, $\exists\, g : B \to A$:
$\mathbf{B} = \langle \bar{t} : t \in \mathbf{A} \rangle, \quad \bar{t}(x_1, \ldots, x_k) = f(t(g(x_1), \ldots, g(x_k)))$

Basic reductions revisited – via algebraic constructions

- $\mathbf{B}$ is an expansion of $\mathbf{A}$ (ie. $\mathbf{A} \subseteq \mathbf{B}$)
- $\mathbf{B} = \mathbf{A}^n$
- $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ redundant
- $\mathbf{B} = \mathbf{A}/\sim$, where $\sim$ is a congruence of $\mathbf{A}$

Definition

$\mathbf{B}$ is a unary modification of $\mathbf{A}$ if $\exists\ f : A \to B$, $\exists\ g : B \to A$:
$\mathbf{B} = \langle \bar{t} : t \in \mathbf{A} \rangle, \quad \bar{t}(x_1, \dots, x_k) = f(t(g(x_1), \dots, g(x_k)))$

Basic reductions revisited – via algebraic constructions

- $\mathbf{B}$ is an expansion of $\mathbf{A}$ (ie. $\mathbf{A} \subseteq \mathbf{B}$)
- $\mathbf{B} = \mathbf{A}^n$
- $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ redundant
- $\mathbf{B} = \mathbf{A}/\sim$, where $\sim$ is a congruence of $\mathbf{A}$ redundant

Definition

$\mathbf{B}$ is a unary modification of $\mathbf{A}$ if $\exists\, f : A \to B$, $\exists\, g : B \to A$:
$\mathbf{B} = \langle \bar{t} : t \in \mathbf{A} \rangle, \quad \bar{t}(x_1, \ldots, x_k) = f(t(g(x_1), \ldots, g(x_k)))$

Basic reductions revisited – via algebraic constructions

- $\mathbf{B}$ is an expansion of $\mathbf{A}$ (ie. $\mathbf{A} \subseteq \mathbf{B}$)
- $\mathbf{B} = \mathbf{A}^n$
- $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ redundant
- $\mathbf{B} = \mathbf{A}/\sim$, where $\sim$ is a congruence of $\mathbf{A}$ redundant

Definition

$\mathbf{B}$ is a unary modification of $\mathbf{A}$ if $\exists\ f : A \to B$, $\exists\ g : B \to A$:
$\mathbf{B} = \langle \overline{t} : t \in \mathbf{A} \rangle, \quad \overline{t}(x_1, \ldots, x_k) = f(t(g(x_1), \ldots, g(x_k)))$

Observation

TFAE for ω-categorical

- *$\mathcal{B}$ is homomorphically equivalent to a pp-power of $\mathcal{A}$*
- $\mathbf{B}$ *is an expansion of a unary modification of a power of* $\mathbf{A}$ *(finite: equivalently g can be taken injective)*

Basic reductions revisited – via weak clone homomorphisms

Definition

$\xi : \mathbf{A} \to \mathbf{B}$ is a weak clone homomorphism, if it

- preserves arities
- preserves composition with projections:
  $\xi(f(\pi_{I_1}, \ldots, \pi_{I_n})) = \xi(f)(\pi_{I_1}, \ldots, \pi_{I_n})$, where $f \in \mathbf{A}$ is n-ary

Alternatively: ξ-images satisfy the same strongly linear identities (=height 1 terms on both sides)

The 3 together revisited for finite

Theorem

TFAE if A, B are finite:

1. *$\mathcal{B}$ is homo equivalent to a pp-power of $\mathcal{A}$*

$$\mathcal{A} \overset{\text{pp-power}}{\rightsquigarrow} \mathcal{E} \overset{\text{homo-eq}}{\rightsquigarrow} \mathcal{B}$$

2. *$\mathbf{B}$ is expansion of a unary modification of a power of $\mathbf{A}$*

$$\mathbf{A} \rightsquigarrow \mathbf{A}^n \overset{\text{unary mod}}{\rightsquigarrow} \mathbf{D} \overset{\text{expansion}}{\rightsquigarrow} \mathbf{B}$$

3. *There exists a weak clone homomorphisms $\xi : \mathbf{A} \to \mathbf{B}$*

$(3) \Rightarrow (2)$

Assume $\xi : \mathbf{A} \to \mathbf{B}$ is a clone homomorphism.
Want:

$$\mathbf{A} \rightsquigarrow \mathbf{A}^n \overset{\text{shrink}}{\rightsquigarrow} \mathbf{D} \overset{\text{expansion}}{\rightsquigarrow} \mathbf{B}$$

- Say $B = \{b_1, \ldots, b_k\}$
- $n = A^k$
- $C =$ k-ary operations in $\mathbf{A}$
- $\mathbf{D} =$ the clone generated by $\xi(\mathbf{A})$
- Define $f : A^k \to B$ by $t \mapsto \xi(t)(b_1, \ldots, b_k)$ on C, otherwise arbitrary
- Define $g : B \to A^k$ by $b_i \mapsto \pi_i$
- $\mathbf{D}$ is the unary modification of $\mathbf{A}^k$ given by f, g since $\overline{t} = \xi(t)$

Algebraic dichotomy conjecture vs. reality revisited

Conjecture (Bulatov, Jeavons, Krokhin; Barto, Kozik)

Assume A finite. TFAE

- $\mathrm{CSP}(\mathcal{A})$ in P
- **A** contains an operation t of arity ≥ 2 such that

$$t(x_1, x_2, \ldots, x_n) = t(x_2, \ldots, x_n, x_1)$$

Reality can be worse

Algebraic dichotomy conjecture vs. reality revisited

Future theorem (Anna Kozik, Hermenegilda Pinsker 2059)

Assume A finite. TFAE

- $\mathrm{CSP}(\mathcal{A})$ *in P*
- **A** *contains an operations* $t_1, t_2, \dots$ *such that*

$$t_1(x_3, x_1, x_2, x_2) = t_2(x_3, x_3, x_1)$$
$$t_{20}(x_1, x_1, x_2) = t_{13}(x_2, x_1)$$

...but certainly the characterization looks like this
(for any complexity class)

The 3 together revisited for infinite

Theorem

TFAE if $\mathcal{A}, \mathcal{B}$ ω-categorical.

1. *$\mathcal{B}$ is homo equivalent to a pp-power of $\mathcal{A}$*
2. ***B** is expansion of a unary modification of a power of **A***

And these conditions are implied by

3 *There exists continuous $\xi : \mathbf{A} \to \mathbf{B}$ which preserves arities and*

$$\xi(\alpha(t(\beta_1, \ldots \beta_n))) = \xi(\alpha)\xi(t)(\xi(\beta_1), \ldots, \xi(\beta_n))$$

where $f \in \mathbf{A}$ is n-ary, and $\alpha, \beta_i \in \mathbf{A}$ are unary bijections.

Summary

Remember: The complexity of $\mathrm{CSP}(\mathcal{A})$ for finite A depends only on linear identities satisfied by $\mathbf{A}$

Summary

Remember: The complexity of $\mathrm{CSP}(\mathcal{A})$ for finite A depends only on linear identities satisfied by **A**

To do:

- Can we use it?

Summary

Remember: The complexity of $\mathrm{CSP}(\mathcal{A})$ for finite A depends only on linear identities satisfied by $\mathbf{A}$

To do:

- Can we use it?
- Improve the 3 views theorem for infinite domains

Summary

Remember: The complexity of $\mathrm{CSP}(\mathcal{A})$ for finite A depends only on linear identities satisfied by **A**

To do:

- Can we use it?
- Improve the 3 views theorem for infinite domains
- Basic reductions $\rightarrow$ preoder such that
  $\mathcal{A} \leq \mathcal{B}$ then $\mathrm{CSP}(\mathcal{A})$ is easier then $\mathrm{CSP}(\mathcal{B})$
  Can we find some more reductions?
  Optimally characterizing log-space reduction (optimistic)

Summary

Remember: The complexity of $\mathrm{CSP}(\mathcal{A})$ for finite A depends only on linear identities satisfied by **A**

To do:

- Can we use it?
- Improve the 3 views theorem for infinite domains
- Basic reductions $\rightarrow$ preoder such that
  $\mathcal{A} \leq \mathcal{B}$ then $\mathrm{CSP}(\mathcal{A})$ is easier then $\mathrm{CSP}(\mathcal{B})$
  Can we find some more reductions?
  Optimally characterizing log-space reduction (optimistic)

Thank you!