Equationally **non**trivial algebras

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Charles University in Prague

BLAST
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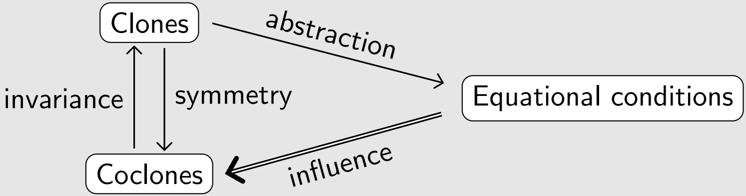
Our playground



Goal: Understand $\Leftarrow$ in "the most general" case
equationally nontrivial clones, finite, idempotent

Q: Why understand $\Leftarrow$?
 A: to understand clones and coclones
Q: Why understand clones or coclones?
 A: to understand algebras
 A: to understand symmetries
Q: Why the most general setting?
 A: sometimes exactly what's needed
 A: nontrivial results possible

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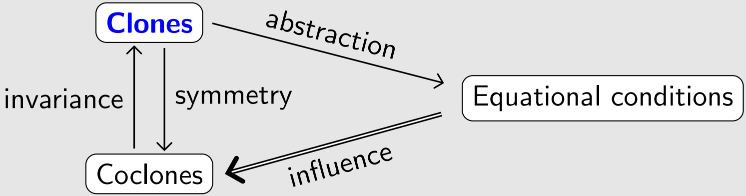

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Clone

Definition

Operation on A = function $A^n \to A$, $n \geq 1$
Clone on A = set of operations on A closed under forming term operations

For each clone $\mathbf{A}$ on A:

- for each $i \leq n$
$$(x_1, \ldots, x_n) \mapsto x_i$$
is in $\mathbf{A}$
- if f,g are binary operations from $\mathbf{A}$, then
$$(x, y, z) \mapsto f(g(f(z, x), y), g(x, x))$$
is in $\mathbf{A}$

Notation: For algebra $\mathbf{A}$, $\mathrm{Clo}(\mathbf{A})$ = all term operations of $\mathbf{A}$

Compute Clo(**A**)

- $(\{0,1\}; \vee)$
- $(\{0,1\}; \vee, \wedge)$
- $(\{0,1\}; \mathrm{majority})$
- $(\{0,1\}; \vee, \wedge, \neg)$
- $(\mathbb{Z}_p; x + y)$
- $(\mathbb{Z}_p; x - y + z)$

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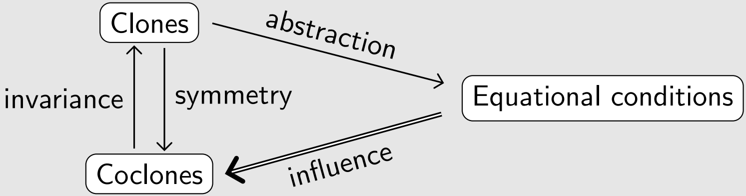

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Q: Why clones? A: Abstraction of algebra

- carries important information – subuniverses, congruences
- for some purposes carries all necessary information (sometimes term equivalent algebras are essentially the same)

Q: Why understand algebras?

Too popular viewpoint

Group theory, Semigroup theory

- group: algebraic structure $\mathbf{G} = (G; \cdot, {}^{-1}, 1)$ satisfying ...
- permutation group: when G happens to be a set of bijections, $\cdot$ is composition, ...
- monoid: algebraic structure $\mathbf{M} = (M; \cdot, 1)$ satisfying ...
- transformation monoid: ...

Universal algebra

- algebra: any algebraic structure $\mathbf{Z} = (Z;$ some operations $)$

Rants

- Model theorist: models of purely algebraic signature, why do you avoid relations?
- Algebraist: groups are complicated enough, nothing interesting can be said about general algebras
- All: have you ever seen a 37-ary operation? You shouldn't study such a nonsense

Alternative viewpoint

	concrete	abstract
unary invert. symmetries	permutation group	group
unary symmetries	transformation monoid	monoid
higher arity symmetries	clone	abstract clone

- permutation group: Subset of $\{f : A \to A\}$ closed under composition and id_A and inverses...

  can be given by a generating unary algebra

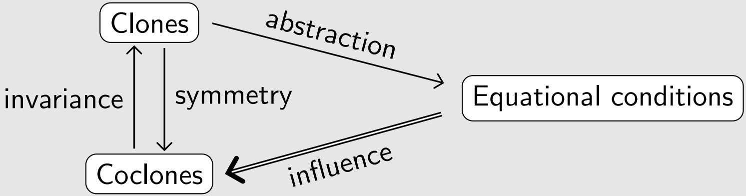
- group: Forget concrete mappings, remember composition
- clone: Subset of $\{f : A^n \to A : n \in \mathbb{N}\}$ closed under composition and projections

  can be given by a generating algebra

- abstract clone: Forget concrete mappings, remember composition

  aka variety, finitary monad over SET, Lawvere theory

Our playground



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Q: Why clones? A: General algebras, **symmetries**

Clones:

- classical algebraic structures $\to$ general algebras $\to$ clones
- permutation group $\to$ clone

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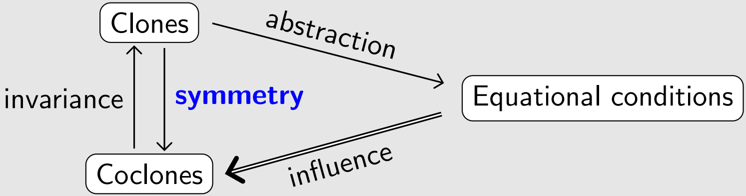

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Symmetries of relational structures

Definition

$f : A^n \to A$ is compatible with $R \subseteq A^k$

(f is a symmetry of R, f is a polymorphism of R,
R is invariant under f)

if $f(\mathbf{a}_1, \dots \mathbf{a}_n) \in R$ whenever $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$

Notation: For a set of relations $\mathbb{A}$,
$\mathrm{Pol}(\mathbb{A})$ = all operations compatible with all relations in $\mathbb{A}$

Fact: $\mathrm{Pol}(\mathbb{A})$ is a clone.

Compute $\mathrm{Pol}(\mathbb{A})$

- $(\{0,1\};\ x \wedge y \to z, x \wedge y \to \neg z)$
- $(\{0,1\};\ \leq)$
- $(\{0,1\};$ all binary relations$)$
- $(\{0,1,2\};\ \neq)$
- $(\mathbb{Z}_p;$ vector subspaces of $\mathbb{Z}_p^3)$
- $(\mathbb{Z}_p;$ affine subspaces of $\mathbb{Z}_p^3)$

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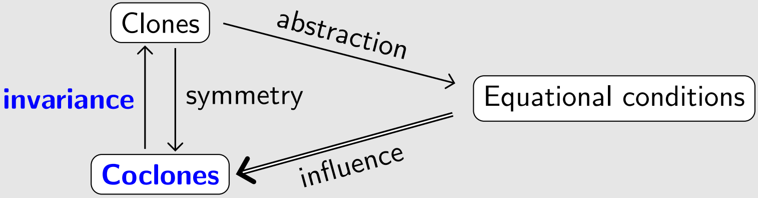

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Invariant relations of a clone

Notation: For a set of relations $\mathbb{A}$,

Inv($\mathbf{A}$) = all relations invariant under all operations in $\mathbf{A}$

Fact: It is closed under pp-definitions =

1st order definitions using $\exists, =,$ *and*

Example: If binary R, S in Inv($\mathbf{A}$), then

$$\{(x, y, z) : (\exists u)(\exists v)\ R(x, u) \text{ and } S(u, v) \text{ and } R(y, y)\}$$

is in Inv($\mathbf{A}$)

Definition

Coclone on A = set of (nonempty) relations on A closed under pp-definitions

Compute Coclo($\mathbb{A}$)

- $(\{0,1\};\ x \wedge y \to z, x \wedge y \to \neg z)$
- $(\{0,1\};\ \leq)$
- $(\{0,1\};$ all binary relations$)$
- $(\{0,1,2\};\ \neq)$
- $(\mathbb{Z}_p;$ vector subspaces of $\mathbb{Z}_p^3)$
- $(\mathbb{Z}_p;$ affine subspaces of $\mathbb{Z}_p^3)$

Compute Inv($\mathbf{A}$)

- $(\{0, 1\}; \vee)$
- $(\{0, 1\}; \vee, \wedge)$
- $(\{0, 1\}; \mathrm{majority})$
- $(\{0, 1\}; \vee, \wedge, \neg)$
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Clones $\leftrightarrow$ Coclones

Theorem ([Geiger]; [Bodnarchuk et al.])

For finite A, Pol, Inv *are (mutually inverse) bijections*

$$\textit{Clones on } A \quad \leftrightarrow \quad \textit{Coclones on } A$$

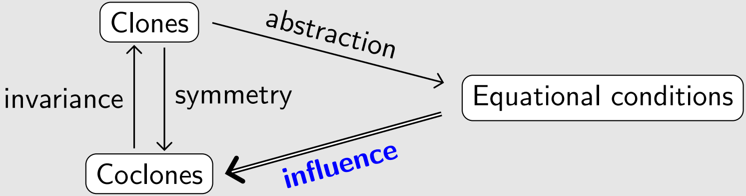
Remarks:

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathrm{Inv}(\mathbf{A}))$, $\mathrm{Coclo}(\mathbb{A}) = \mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))$
- Clones determined by invariant relations
- Coclones determined by symmetries
- Understanding clones = understanding coclones

Proof: Regard the set of n-ary operations in $\mathbf{A}$ as $|A|^n$-ary relation

From now on: $\mathbf{A}$ clone, $\mathbb{A}$ corresponding coclone

Our playground



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Influence: equational conditions $\to$ coclones

Theorem ([Maltsev])

A *contains a Maltsev operation* $(m(x, y, y) = m(y, y, x) = x)$
iff
Each R *in* $\mathbb{A}$ *is rectangular* $(\mathbf{ab}, \mathbf{ab}', \mathbf{a}'\mathbf{b} \in R \Rightarrow \mathbf{ab}' \in R)$

Theorem ([Baker,Pixley])

A *contains a majority operation*
$(m(x, x, y) = m(x, y, x) = m(y, x, x) = x)$
iff
Each R *is determined by its projections to pairs of coordinates.*

Does Clo(**A**) have Maltsev or majority operation?

- $(\{0,1\}; \vee)$
- $(\{0,1\}; \vee, \wedge)$
- $(\{0,1\}; \mathrm{majority})$
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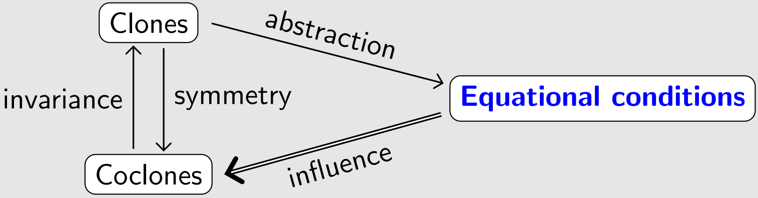

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Equational condition

Definition

Equational condition = condition of the form
"there exists operations … satisfying equations …."
(infinitely many operations or equations allowed)

Examples: the existence of a Maltsev term, the existence of a majority term

Remarks:

- Equational conditions are ordered by strength
- Equational condition is nontrivial if it is not satisfied in some clone
- Clone is equationally nontrivial if it satisfies some nontrivial equational condition

Is Clo($\mathbf{A}$) equationally trivial?

- $(\{0,1\}; \vee)$
- $(\{0,1\}; \vee, \wedge)$
- $(\{0,1\}; \mathrm{majority})$
- $(\{0,1\}; \vee, \wedge, \neg)$
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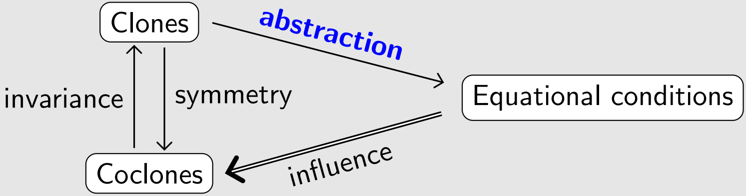

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Abstraction: clones $\to$ equational conditions

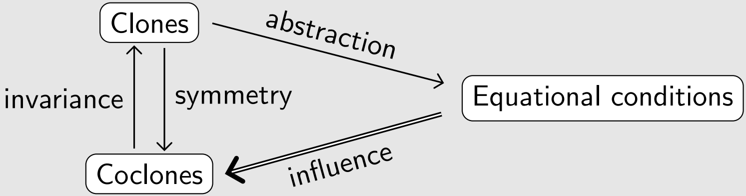
Consider two clones equivalent if they satisfy the same equational conditions

Abstraction: clone $\to$ its equivalence class

Remarks:

- The set of equivalence classes is lattice ordered
- Simple formalization: It is the order induced by clone homomorphisms

Our playground



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Idempotent clones

Definition

Clone **A** is idempotent

if $f(x, x, \ldots, x) = x$ for each f in **A**

$\Leftrightarrow$ unary part of **A** is trivial

Why this assumption?

- Complementary to group/semigroup theory
- Many useful equational conditions are idempotent
- Gives some information about general clones

Our playground

$$\begin{tikzcd}
\boxed{\text{Clones}} \arrow[r, "\text{abstraction}"] \arrow[d, shift left, "\text{symmetry}"] & \boxed{\text{Equational conditions}} \arrow[dl, Rightarrow, "\text{influence}"] \\
\boxed{\text{Coclones}} \arrow[u, shift left, "\text{invariance}"] &
\end{tikzcd}$$

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Equationally nontrivial clones

Theorem ([Birkhoff]; [Bodirsky]; [Bulatov])

For finite idempotent clone $\mathbf{A}$ *TFAE*

(i) *Some part of the domain is purely combinatorial:*

Formally $\mathbf{TRIV} \in \mathsf{HSP}(\mathbf{A})$ *(*$\mathbf{TRIV}$ *is the clone of projections on a 2-element set)*

Equivalently $\mathbf{TRIV} \in \mathsf{HS}(\mathbf{A})$

(ii) $\mathbb{A}$ *has the highest expressive power*

Formally, $\mathbb{A}$ *pp-interprets all finite relational structures*

(iii) $\mathbf{A}$ *is equationally trivial*

Definition (Just for this talk)

$\mathbf{A}$ is a Taylor clone if it is finite, idempotent, and equationally **non**trivial

Taylor term

Theorem ([Taylor])

For an idempotent clone **A** *TFAE*

- **A** *is equationally nontrivial*
- **A** *satisfies nontrivial height 1 equational condition involving a single operation symbol:*

$$
\begin{aligned}
t(x, \_, \_, \dots) &= t(y, \_, \_, \dots) \\
t(\_, x, \_, \dots) &= t(\_, y, \_, \dots) \\
&\vdots \\
t(\dots, \_, \_, x) &= t(\dots, \_, \_, y)
\end{aligned}
$$

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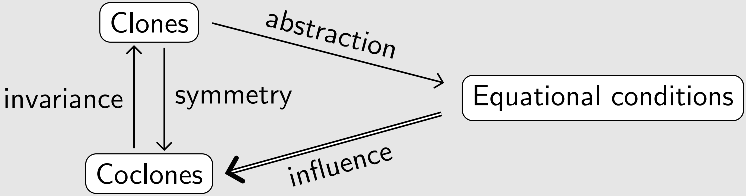

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Sometimes it's exactly what's needed

Complexity of constraint satisfaction problems (CSPs)
...**large** part of computational complexity

Fixed finite template CSP

- a class of computational problems, one for each finite relational structure $\mathbb{A}$

  $\mathrm{CSP}(\mathbb{A})$ = membership in $\{\mathbb{X} : \mathbb{X} \to \mathbb{A}\}$
- tiny fraction of computational complexity (from global perspective)
- very broad class, base case for more optimistic goals

Theorem: [Bulatov, Jeavons, Krokhin] The complexity depends only on the position of **A** in the order.
Consequence: If **A** is not Taylor, the $\mathrm{CSP}(\mathbb{A})$ is NP–complete.

THEOREM ([Bulatov]; [Zhuk])

If **A** *is Taylor, then* $\mathrm{CSP}(\mathbb{A})$ *is solvable in polynomial time.*

Our playground

$$\begin{tikzcd}
\boxed{\text{Clones}} \arrow[dr, "\text{abstraction}"] \arrow[dd, shift left, "\text{symmetry}"] & \\
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- **A:** **nontrivial results possible**

Tools

Classic:

- Commutator theory [Smith], ...
- Tame congruence theory [Hobby, McKenzie], ...

More recent:

- Absorption theory [Barto, Kozik], ...
- Bulatov's theory
- Zhuk's theory

Results on structure of relations

Theorem ([Maróti, McKenzie]; [Barto, Kozik, Niven]; [BK])

If $\mathbf{A}$ *is Taylor, then:*

- *There exists* $n > 1$ *such that every symmetric* n*-ary relation in* $\mathbb{A}$ *contains a constant tuple*
- *Every linked binary relation in* $\mathbb{A}$ *contains a loop*
- *For every prime* $p > |A|$*, every cyclically-symmetric* p*-ary relation in* $\mathbf{A}$ *contains a constat tuple*

Consequences for equations

Theorem ([MM], [Kearnes,Marković,McKenzie], [BK])

For an idempotent clone **A** *on finite set TFAE*

(i) **A** *is Taylor*

(ii) **A** *has a weak near unanimity operation of some arity* $n > 1$

$$w(x, \ldots, x, y) = w(x, \ldots, x, y, x) = \cdots = w(y, x, \ldots x)$$

(iii) **A** *has a 4-ary Sigger's operation*

$$t(r, a, r, e) = t(a, r, e, a)$$

(iv) **A** *has a cyclic operation of each prime arity* $p > |A|$

$$t(x_1, x_2, \ldots, x_p) = t(x_2, \ldots, x_p, x_1)$$

Note: (iii): A weakest nontrivial equation!

Digression to infinite

Weakest nontrivial equation for finite idempotent clones

Recall: From any set of idempotent operations $f_1, \dots$ on a finite set A satisfying nontrivial equations

one can build a term operation s such that
$s(r,a,r,e) = s(a,r,e,a)$

Intuition: A is finite $\Rightarrow$ composition is not sufficiently free

"Obviously": It is impossible to find a weakest equations without the restriction to finite sets

Weakest nontrivial equation for idempotent algebras

Wrong!

Theorem ([Olšák])

For an idempotent clone $\mathbf{A}$ *TFAE*

- $\mathbf{A}$ *is equationally nontrivial*
- $\mathbf{A}$ *contains a 6-ary* t *such that*

$$t(x,x,y,y,y,x) = t(x,y,x,y,x,y) = t(y,x,x,x,y,y)$$

End of digression

Absorption theory

Definition

Let $B \subseteq A$ be in $\mathbb{A}$.

B absorbs $\mathbf{A}$ if $\exists$ n-ary t in $\mathbf{A}$ such that
$t(A, B, B, \ldots, B) \subseteq B$, $t(B, A, B, \ldots, B) \subseteq B$, $\ldots$

Theorem

If binary R in $\mathbb{A}$ is subdirect and linked, then $B = A^2$ or
$\mathbf{A}$ has a proper absorbing set.

Theorem

If B, C are minimal absorbing sets of $\mathbf{A}$,
binary R in $\mathbb{A}$ is subdirect and linked, and
$R \cap (B \times C) \neq \emptyset$,
then $B \times C \subseteq R$.

Bulatov's theory

$\mathbf{A} \to$ digraph on A, 3 types of edges: semilattice, majority, affine

Definition

(a, b) is a semilattice edge if $\exists$ binary s in $\mathbf{A}$ such that

$$s(a, b) = s(b, a) = b$$

(a, b) is a majority edge if $\ldots$ [a more complex condition] $\ldots$

(a, b) is an affine edge if $\ldots$ [even more complex condition] $\ldots$

Theorem

$\mathbf{A}$ *is Taylor iff the digraph of* $\mathbf{B}$ *is connected for each subalgebra* $\mathbf{B}$.

Theorem

If B, C are minimal affine & semilattice upward-closed subsets of A, binary R in $\mathbb{A}$ is subdirect and linked, and
$R \cap (B \times C) \neq \emptyset$,
then $B \times C \subseteq R$.

Zhuk's theory

Strong structure theorems on "indecomposable" relations.

Crucial concepts: binary absorption, center, ...

Definition

A subset B of A is a center of $\mathbf{A}$ if [such and such relation] is compatible with [something weird] and [some other condition].

Theorem

If B, C are minimal centers of $\mathbf{A}$,
binary R in $\mathbb{A}$ is subdirect and linked, and
$R \cap (B \times C) \neq \emptyset$,
then $R \cap (B \times C)$ is subdirect and linked.

Comparison

Methods

- absorption: heavily relational, lightly algebraic
- Bulatov: extremely algebraic, heavily relational
- Zhuk: heavily relational, lightly algebraic

Common: Some results look **very** similar
(different concepts, same assumptions and conclusions)

But: There is no clear connection, e.g. adding operations to a clone does not destroy absorption, can change colors, or centers

Also: Bulatov and Zhuk sometimes need to remove some operations (while remaining Taylor)

Taylor minimal clones

Work of:

- Zarathustra Brady (great write-up on his website)
- B + Bulatov + Kozik + Zhuk (last 2 weeks)

Definition

An idempotent clone is Taylor minimal if

- it is Taylor
- no proper subclone is Taylor

Fact: Each Taylor clone contains a Taylor minimal clone

Fun facts I (no guarantee!!!)

Theorem?

For a Taylor minimal clone $\mathbf{A}$ *and a unary* B *in* $\mathbb{A}$ *TFAE*

- B *is binary absorbing*
- *for each operation* f *in* $\mathbf{A}$ *and each essential variable* i, $f(A,\dots,A,B,A,\dots,A) \subseteq B$
- B *is a cube term blocker*
- B *is semilattice & majority & affine upward-closed*

Fun facts II (no guarantee!!!)

Theorem?

For a Taylor minimal clone $\mathbf{A}$ *and* $B \subseteq A$, *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* ...

(i) B *is ternary absorbing*

(ii) B *is a center*

(iii) B *is absorbing*

(iv) B *is semilattice & affine upward-closed*

(v) B *is "singleton ternary absorbing"*

Fun facts III (no guarantee!!!)

- [Brady] Complete classification of 3-element affine-free Taylor minimal clones
- Complete classification on small domains possible $\to$ source of examples
- Taylor minimality closed under H,S,P
- 5 omitting colors theorem, e.g.
  - semilattice & affine free $\Leftrightarrow$ majority $\Leftrightarrow$ near-unanimity
  - majority & affine free $\Leftrightarrow$ 2-semilattice $\Leftrightarrow$ binary cyclic
- 1 missing
- many questions...

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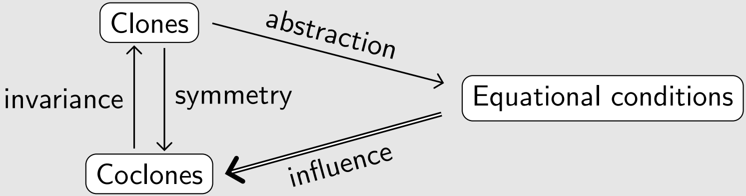

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What next?

- Organize, unify, simplify, ...
- $\rightarrow$ (slightly) infinite
- $\rightarrow$ weighted
- $\rightarrow$ clonoids