Absorption in Universal Algebra and CSP

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Abstract

The algebraic approach to Constraint Satisfaction Problem led to many developments in both CSP and universal algebra. The notion of absorption was successfully applied on both sides of the connection. This article introduces the concept of absorption, illustrates its use in a number of basic proofs and provides an overview of the most important results obtained by using it.

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1 Introduction

Absorption is a simple concept, which has found several interesting applications in universal algebra and constraint satisfaction. The aim of this survey is to show *what* results have been achieved using absorption and, more importantly, to explain *how* absorption is applied to prove these results.

1.1 Results

In constraint satisfaction, absorption is mostly applied in the study of the computational and descriptive complexity of the Constraint Satisfaction Problem (CSP) over a fixed finite relational structure (also known as a *template* or a *constraint language*). In this paper, a *relational structure* $\mathbb{A} = (A; R_1, \dots, R_k)$ consists of a **finite** set A, called a *domain* or a *universe*, and a finite sequence of (finitary) relations $R_1, \dots, R_n$. A *primitive positive formula*, or *pp-formula*, over $\mathbb{A}$ is a first order formula over $\mathbb{A}$ that uses only existential quantification, conjunction, and equality. The *constraint satisfaction problem over* $\mathbb{A}$, written CSP($\mathbb{A}$), is the problem of deciding whether an input pp-sentence is true. Thus, for a relational structure $\mathbb{A}$ with a ternary relation $R \subseteq A^3$ and a binary relation $S \subseteq A^2$, an instance of CSP($\mathbb{A}$) is e.g.

$$(\exists x_1)(\exists x_2)(\exists x_3)(\exists x_4)\, R(x_1, x_3, x_2) \wedge S(x_1, x_1) \wedge S(x_1, x_4) \wedge (x_2 = x_1).$$

The clauses in the instance are often called *constraints* as they are constraining the possible values of the tuples of variables.

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Known results suggests that, for any relational structure $\mathbb{A}$, the problem CSP($\mathbb{A}$) is tractable (i.e. solvable in polynomial time), or NP-complete. The conjecture postulating this separation is known as the *CSP dichotomy conjecture* [31]. The concept of absorption allowed to confirm this conjecture for the CSPs over digraphs with no sources or sinks [12] and to greatly simplify the proof of the dichotomy theorem [4] for conservative CSPs [25] (i.e. CSPs over structures that contain all unary relations).

A closely related line of research studies the power of consistency methods in CSP. The applicability of consistency algorithms to CSPs with fixed template was determined [11] using absorption (independently in [21] using different tools). Moreover, it was shown that basic algorithms, such as $(2, 3)$-minimality [7] or Singleton Arc Consistency [42], solve all the CSPs solvable by local consistency. At the same time, the templates solvable by local consistency were proved to be exactly those with CSPs having robust approximation algorithms [10] — all these proofs are based on absorption.

A significant step towards understanding the power of "linear consistency" and characterizing the CSPs in NL has been made in [14], and a related result studying robust approximation with a polynomial loss appeared in [28] — both of these proofs rely on absorption as well.

The contributions of absorption to universal algebra mostly concern equational conditions for finite algebras. In this paper, an *algebra* $\mathbf{A} = (A; f_1, f_2, \dots)$ consists of a **finite** universe A and a set of (finitary) operations on $\mathbf{A}$, called the *basic operations* (this set sometimes needs to be indexed so that, e.g, one can define direct products). A *term operation* of $\mathbf{A}$ is any operation on A obtained by composing the basic operations. An *equational condition* stipulates the existence of term operations satisfying certain *identities*, that is, universally quantified equations (the term "equational condition" is nonstandard and used instead of a closely related, but different concept of a Mal'tsev condition). Equational conditions often characterize properties of invariant relations, for instance, the existence of a term operations m satisfying the identities $m(x, x, y) = y = m(y, x, x)$ characterizes permutability of compatible equivalences in a sense which is made precise in Theorem 8. Nontrivial information about the shape of invariant relations under some equational condition is also the core of some CSP results, such as the aforementioned dichotomy theorem for digraphs with no sources and sinks, see the discussion after Theorem 13.

Equational conditions are intimately related to the fixed template CSPs in that the complexity of CSP($\mathbb{A}$) is determined by the equational conditions satisfied by the associated algebra of polymorphisms, see Subsection 1.3 for a brief explanation and references. A chief product of absorption in this context is a characterization of the conjectured bordeline [23] between tractable and NP-complete CSPs by means of cyclic operations [9]. Another contributions of absorption are new equational and relational conditions for properties that are important in CSP and/or universal algebra, including congruence distributivity (see Section 5.3), modularity [1], and meet semi-distributivity [13]. For polymorphism algebras of relational structures, a surprising collapse of equational conditions has emerged [5, 3], which also impacted some other computational problems parametrized by relational structures [20, 27]. Not so closely related to computational complexity is the connection of solvability and absorption discovered in [13] (see Theorem 28), which allowed to greatly simplify the proofs of some classical universal algebraic results.

The results on the CSP and universal algebra coming from absorption have been used in several other works, including the reduction of valued CSP to CSP in [40] (which uses cyclic operations), or further characterizations of the conjectured borderline between tractable/NP-complete CSPs in [49, 8, 9].

1.2 Why is absorption useful

The success of absorption is a product of three factors.

Absorption transfers connectivity. The connectivity in the slogan is meant in a wide sense, it might by strong connectivity or connectivity in a directed graph as well as any other property resembling connectivity. The most basic example of "transferring connectivity" appears already in Proposition 2. Further in Section 4.2 we display the notion of a Prague instance which can be viewed as a connectivity condition. Finally, in Section 7.2 we exhibit other properties which are transferred by absorption; these properties do not resemble connectivity — it is usually hidden in the proofs.

Connectivity is common. The reason why absorbing connectivity, or structural conditions in general, is useful is that equational conditions are often reflected in structural/connectivity properties of compatible relations (see Section 5). In the CSP, connectivity can be provided by local consistency checking algorithms running in polynomial time, and transferring it to smaller instances sometimes allows to construct a solution. Section 4 gives some examples of this phenomenon.

Absorption is common. The two factors would not be so useful if absorption was rare. Fortunately, quite mild assumption enforces either a significant restriction on the shape of compatible relations, or an interesting absorption. This is shown in Section 6 together with some applications.

1.3 CSP and universal algebra

The link between the fixed template CSP and universal algebra hinges on two Galois connections: the Pol–Inv Galois connection between relational structures and algebras [33, 19] and the Mod-Id Galois connection between classes of algebras and sets of identities [17]. The first connection implies that the complexity CSP($\mathbb{A}$) depends only on a certain algebra associated to $\mathbb{A}$ [19, 37, 36], and the second one that only the equational conditions satisfied by the algebra matter [23].

W proceed to introduce definitions and results that are behind [36] and that are essential for understanding the next section. Other concepts and results are introduced when the need for them arises; the index at the end of the paper is constructed to help with such scattered definitions. For further details we refer the reader to the recent survey on CSP basics [6] or its revision [15].

A *homomorphism* between two relational structures $\mathbb{A} = (A; R_1, \ldots, R_n)$ and $\mathbb{A}' = (A'; R'_1, \ldots, R'_n)$ is a map from A to A' which, when computed coordinatewise, maps R_i to R'_i. The *n-th power* of relational structure $\mathbb{A} = (A; R_1, \ldots, R_n)$ is $\mathbb{A}^n$ with the universe A^n and relations R'_i defined coordinatewise (i.e. a tuple of elements of $\mathbb{A}^n$ is in R'_i if they are in R_i on every coordinate). Polymorphisms of a structure generalizes endomorphisms. An operation $f : A^n \to A$ is a *polymorphism* of $\mathbb{A}$ if it is a homomorphism from $\mathbb{A}^n$ to $\mathbb{A}$. When f is a polymorphism of $\mathbb{A} = (A; R)$, we also say that f is *compatible* with R, or that R is invariant under f.

To every relational structure $\mathbb{A}$, we associate an algebra $\mathbf{A}$, denoted $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$, on the same domain whose operations are all the polymorphisms of $\mathbb{A}$. By [19, 37, 36], a relation is pp-definable (that is, definable by a pp-formula) from $\mathbb{A}$ if and only if it is compatible with every polymorphism of $\mathbb{A}$. Since CSP($\mathbb{B}$) can be easily reduced to CSP($\mathbb{A}$) whenever $\mathbb{A}$ pp-defines $\mathbb{B}$ (ie. every relation of $\mathbb{B}$ is pp-definable from $\mathbb{A}$), it follows that $\mathbf{A}$ determines the complexity of CSP($\mathbb{A}$).

Given an algebra $\mathbf{A}$, a subset A' of A closed with respect to basic operations of $\mathbf{A}$ is called

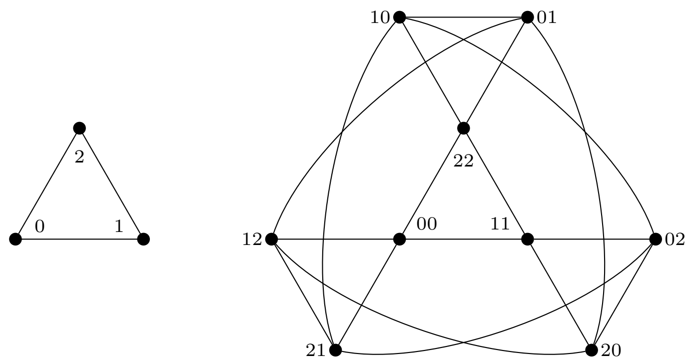


Figure 1 The underlying graph of $\mathbb{K}_3^c$ and its second power.

a *subuniverse* of $\mathbf{A}$, and defines a subalgebra of $\mathbf{A}$ (denoted by $\mathbf{A}' \leq \mathbf{A}$). A subuniverse is the same as a unary relation invariant under all the basic operations in $\mathbf{A}$.

2 Example

In this section, we work out an example, which illustrates several basic concepts and motivates the concept of absorption (this concept actually emerged in a quite similar context). We will show that the undirected complete graph with constants, that is, the relational structure

$$\mathbb{K}_3^c = (\{0,1,2\}; R, C_0, C_1, C_2), \quad R = \{(x,y) \in \{0,1\}^2 : x \neq y\}, C_i = \{i\}$$

has no other polymorphisms than the projections. By remarks in Subsection 1.3, this is equivalent to proving that each relation on $\{0,1,2\}$ is pp-definable from $\mathbb{K}_3^c$. In particular, any computational problem parametrized by a relational structure, whose complexity depends on the pp-definability strength of the structure, is bound to be hard over $\mathbb{K}_3^c$.

In our example, a polymorphisms of arity n is a homomorphism from the n-th power of $\mathbb{K}_3^c$ to $\mathbb{K}_3^c$. Figure 1 shows the second power of the relation R; the relations corresponding to C_i in this power are $\{(ii)\}$.

2.1 One and two element sets are preserved by polymorphisms.

The first observation is that every vertex in the power graph is mapped to an element which appears on some coordinate; in other words, each subset is a subuniverse of the polymorphism algebra $\mathrm{Pol}(\mathbb{K}_3^c)$.

In the power graph, the element $(2,\ldots,2)$ is in the relation corresponding to C_2 and therefore $f : (2,\ldots,2) \mapsto 2$. This shows that $\{2\}$ is a subuniverse of the polymorphism algebra; similarly, the other two singletons are subuniverses as well. The neighbors of $(2,\ldots,2)$ must be mapped to neighbors of 2, but each vertex indexed by 0's and 1's is a neighbor of $(2,\ldots,2)$ and thus it has to be mapped into $\{0,1\}$. That $\{0,2\}$ and $\{1,2\}$ are subuniverses is shown similarly.

A more compact way to show that $U = \{0,1\}$ is invariant under every polymorphism is by observing that this unary relation is pp-definable from $\mathbb{K}_3^c$ by the formula

$$U(x) \text{ iff } (\exists y)\, C_2(y) \wedge R(y,x) \ .$$

In words, U is the set of R-neighbors of the invariant relation C_2.

2.2 Unary and binary polymorphisms are trivial.

The relational structure $\mathbb{K}_3^c$ has no endomorphisms other than the identity. Indeed, we have already observed that each unary f maps i to i.

For binary polymorphisms, consider the picture in Figure 1 and let f be any homomorphism from the second power of $\mathbb{K}_3^c$ into $\mathbb{K}_3^c$. The function f maps (i, i) to i for every i, and thus the values on the inner triangle are fixed. Choose an arbitrary vertex, say $(0, 1)$. By previous section, it can be mapped to 0 or to 1. Without loss of generality, assume it maps to 0. Then $(1, 0)$, as a neighbor of $(0, 1)$ and $(2, 2)$ which are mapped to 0 and 2 respectively, has to map 1. Further $(2, 0)$, as a neighbor of $(0, 1)$ and $(1, 1)$, needs to be mapped to 2. Continuing in this way we establish that f is the first projection, and if $(0, 1)$ were mapped to 1 we would obtain a second projection.

2.3 Polymorphisms of higher arities.

Consider now an arbitrary polymorphism f of arity $n \geq 3$. We define binary operations f_i, $i \in [n]$, by

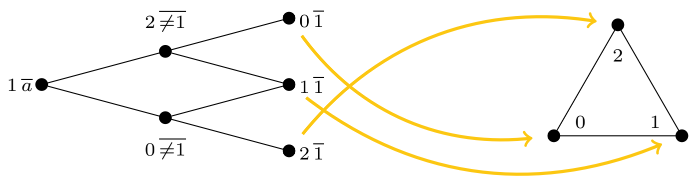
$f_i(x, y) = f(x, \ldots, x, y, x, \ldots, x)$ with y at the i-th place

The set of polymorphisms of any relational structure is a *clone*, that is, it contains all the projections and is closed under composition. In particular, the binary operations f_i are also polymorphisms of $\mathbb{K}_3^c$. Since the only binary polymorphisms of our structure are projections, for every $i \in [n]$, either $f_i(x, y) = x$ for all $x, y \in \{0, 1, 2\}$, or $f_i(x, y) = y$ for all $x, y \in \{0, 1, 2\}$. We distinguish two cases.

(a) There exist i such that $f_i(x, y) = y$.
(b) For all i, $f_i(x, y) = x$.

2.4 Polymorphisms with $f_i(x, y) = y$ for some i.

For simplicity assume $i = 1$. The reasoning is illustrated by the following figure.



Take an arbitrary tuple and, without loss of generality, assume that it has 1 on the first coordinate (tuple $1\overline{a}$ in the figure). Find a neighbour of this tuple with 2 on the first coordinate and elements different from 1 on the remaining coordinates. This element is denoted by $2\,\overline{\neq 1}$ and an analogous element with 0 on the first coordinate is denoted by $0\,\overline{\neq 1}$. Both of theses elements are adjacent to $1\,\overline{1}$ (the vertex with 1's only); the first is also adjacent to $0\,\overline{1}$ and the second to $2\,\overline{1}$.

The three elements $1\,\overline{1}, 0\,\overline{1}$, and $2\,\overline{1}$ are mapped to $1, 0$, and 2, respectively, which forces $2\,\overline{\neq 1} \mapsto 2$ and $0\,\overline{\neq 1} \mapsto 0$. This in turn forces $1\,\overline{a} \mapsto 1$, i.e. the polymorphism is the first projection.

From what we have just shown, it follows that $f_i(x, y) = y$ cannot simultaneously hold for two different i's. There is a deeper reason to it. If, say, $f_1(x, y) = f_2(x, y) = y$, then the ternary polymorphism

$$m(x, y, z) = f(x, z, y, y, \dots, y)$$

is a Mal'tsev operation, that is, it satisfies $m(x, x, y) = y = m(y, x, x)$ for all $x, y \in \{0, 1, 2\}$. A Mal'tsev polymorphism drastically restricts the shape of relations, for instance, a binary relation invariant under a Mal'tsev operation is rectangular (see Section 5.2), which is not the case for R.

2.5 Polymorphisms with $f_i(x, y) = x$ for all i.

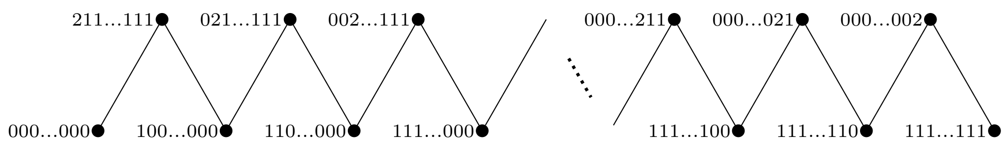
This is the most interesting part of the analysis. In this case, the polymorphism f satisfies

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y) = x \ .$$

Operations satisfying these identities are called *near unanimity* (or *NU*) operations.

It is possible to derive a contradiction by an ad hoc argument (as in the previous section) by considering where various vertices of the power graph need to be mapped. We will show a nicer argument, which can be used in more general situations.

Consider the following part of the powergraph:



Every vertex in the bottom row is adjacent to $(2, \dots, 2)$ which is mapped to 2, so bottom elements are mapped to $\{0, 1\}$ (as already observed). But, also, every vertex in the top row is adjacent to a vertex of the form $(2, \dots, 2, 0, 2, \dots, 2)$, which is mapped to 2 by the assumption on f. Thus all the vertices in the path are mapped into $\{0, 1\}$ and we get a path from 0 to 1 of even length – this is clearly impossible.

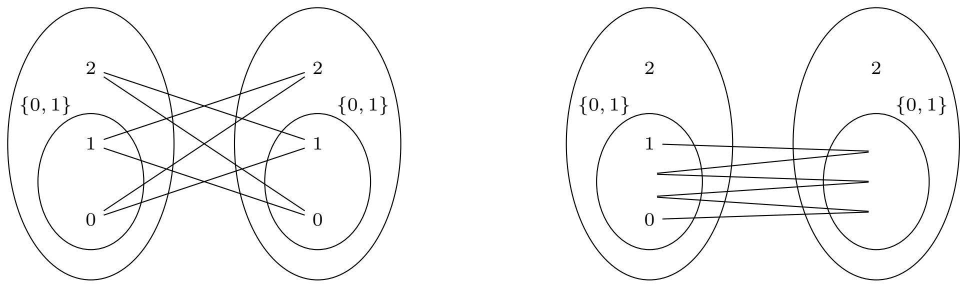
It is often useful to look at a binary relation $R \subseteq A^2$ as a bipartite graph. The partite sets are disjoint copies of A (one copy is on the left, the other one on the right) and edges correspond to pairs in R. The relation R in our example is shown in Figure 2 on the left. Note that the elements 0 and 1 (on the left) are disconnected in the subgraph induced by both copies of $\{0, 1\}$. However, the above path provides a connection from 0 to 1 (see the right part of Figure 2), a contradiction again.

Crucial for the given argument was a pleasant property of the set $\{0, 1\}$ and operation f: not only are elements consisting of 0's and 1's mapped to $\{0, 1\}$, f tolerates one exception.

3 Absorption and more absorption

The notion of absorption (defined below) generalizes the property of singletons and the set $\{0, 1\}$ that made the reasoning in Subsection 2.5 possible. However, before diving into the definitions we make a couple of remarks.

First, we restrict to idempotent algebras. An algebra $\mathbf{A}$ is *idempotent* if, for every operation f and all $a \in A$, $f(a, \dots, a) = a$. Equivalently, one can require that every one-element subset of A is a subuniverse of $\mathbf{A}$. This is not a severe restriction: In the CSP, one can often restrict to the relational structures that contain the singleton unary relations; their



(a) R as a bipartite graph **(b)** A connection from 0 to 1

Figure 2 Bipartite viewpoint

polymorphism algebras are idempotent. In universal algebra, many properties of algebras depend only on certain idempotent algebras associated to them, their full idempotent reducts. We wish to stress that in all definitions and theorems we will implicitly assume that **algebras are finite and idempotent**.

Second, the operation defining absorption in an algebra is not always one of the basic operations of the algebra – it can be any term operation. Polymorphisms of any relational structure are closed under composition, so there is no difference in such a situation.

3.1 Absorption

There are in fact several useful notions of absorption: (directed) Jónsson absorption (see Section 5.3) or (directed) Gumm absorption [3, 1]. The one that appears the most useful resembles near unanimity operations.

▶ **Definition 1** (Absorption)**.** A subalgebra $\mathbf{B}$ of $\mathbf{A}$ is *absorbing* with respect to an n-ary term operation f of $\mathbf{A}$ if $f(a_1, \dots, a_n) \in B$ whenever the set of indices $\{i : a_i \notin B\}$ has at most one element. The fact is denoted $\mathbf{B} \trianglelefteq_f \mathbf{A}$, or $\mathbf{B} \trianglelefteq \mathbf{A}$ if f is not important. We also say that $\mathbf{B}$ absorbs $\mathbf{A}$, that f absorbs $\mathbf{A}$ into $\mathbf{B}$, and so on.

In Subsection 2.5, we have observed that $\{0, 1\}$ absorbs $\mathbf{A} = \mathrm{Pol}(\mathbb{K}_3^c)$ with respect to f. More formally, we should say that the subalgebra of $\mathrm{Pol}(\mathbb{K}_3^c)$ with universe $\{0, 1\}$ absorbs $\mathbf{A}$, but subalgebras are determined by their universes, so we can safely disregard this formal distinction when $\mathbf{A}$ is clear from the context.

Algebras with absorbing subuniverses are common. For example, most two-element algebras have proper absorbing subalgebras: It is known that if a two-element algebra contains an operation which is not affine over the two-element field, then it contains the binary minimum operation, or the binary maximum operation, or the majority (the only ternary NU operation on a two–element universe). In the first case, $\{0\}$ is absorbing; in the second case, $\{1\}$ is absorbing; and in the third case, both singletons are absorbing. These absorptions are behind the polynomial algorithm for Horn-SAT, and can be used to construct polynomial algorithms for 2-SAT as well.

In any algebra with a near–unanimity operation, every one-element subalgebra is absorbing with respect to this operation. The converse is also true, if every one element subuniverse absorbs $\mathbf{A}$, then $\mathbf{A}$ has a near unanimity term. It is not immediate, since the absorptions can be witnessed by different operations, but this problem can be fixed by composing terms in a way introduced in the next paragraph.

If $\mathbf{B} \trianglelefteq_f \mathbf{A}$ and $\mathbf{C} \trianglelefteq_g \mathbf{A}$, where f is n-ary and g is m-ary, then both absorptions are also witnessed by the *star composition* of f and g (denoted $f \star g$) which is an nm-ary operation defined by

$$f \star g(x_1, \ldots, x_{nm}) = f(g(x_1, \ldots, x_m), g(x_{m+1}, \ldots, x_{2m}), \ldots, g(x_{nm-m+1}, \ldots, x_{nm})).$$

Several other simple properties of absorption can be shown using the star composition, e.g. one can prove that the relation “is an absorbing subuniverse of” is transitive, or that an intersection of absorbing subuniverses is again absorbing.

3.2 Absorption from absorption: propagation

This section explains how to use compatible relations, or subpowers (defined in the next paragraph), in order to propagate the property of “being an absorbing subuniverse" from one subuniverse to another. An example of such a situation appeared already in Subsection 2.5: the fact that $\{2\}$ was absorbing implied that the set $\{0, 1\}$, the set of all R-neighbors of $\{2\}$, was absorbing as well.

Before stating the general version of the property, we recall several basic definitions. The *n-th power* of $\mathbf{A}$ with universe A is the algebra with universe A^n and the operations computed coordinatewise. A *subpower* of $\mathbf{A}$ is any subuniverse (or a subalgebra) of a power of $\mathbf{A}$. In other words, a subpower of $\mathbf{A}$ is an n-ary relation invariant under coordinate-wise action of any operation of $\mathbf{A}$. When $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$, then subuniverses of $\mathbf{A}$ are exactly the relations pp-definable from $\mathbb{A}$.

It is easy to prove that the set of all the subpowers of an algebra is closed under pp-definitions. The following proposition gives an analogue for absorption. The proof of the proposition is left as an exercise.

▶ **Proposition 1** (Propagation of absorption). *Let $\mathbf{A}$ be an algebra and let $\mathbf{R} \leq \mathbf{A}^n$ be a subpower defined from subpowers $\mathbf{S}_1, \ldots, \mathbf{S}_k$ by a pp-formula ϕ. Moreover, let $\mathbf{S}'_1 \trianglelefteq \mathbf{S}_1$, $\ldots$, $\mathbf{S}'_k \trianglelefteq \mathbf{S}_k$.*

Then the subpower defined by the pp-formula obtained from ϕ by replacing each $S_i(\ldots)$ by $S'_i(\ldots)$ absorbs $\mathbf{R}$.

Note that $\mathbf{A} \trianglelefteq \mathbf{A}$ for any algebra $\mathbf{A}$, so, in the proposition, $\mathbf{S}'_i$ can be equal to $\mathbf{S}_i$.

The proposition above is often used to “walk” with absorption, a first example of the “walking” was already in Subsection 2.5. To put the construction into slightly more general terms, consider subalgebras $\mathbf{R} \leq \mathbf{A}^2$ and $\mathbf{B} \trianglelefteq \mathbf{A}$. The set C of out-neighbors of B in the directed graph with edge-set R is pp-defined by the formula

$$C(y) \text{ iff } (\exists x)\, B(x) \wedge R(x, y).$$

It absorbs the subalgebra $\mathbf{D}$ of $\mathbf{A}$ defined by

$$D(y) \text{ iff } (\exists x)\, A(x) \wedge R(x, y) \quad \text{equivalently} \quad D(y) \text{ iff } (\exists x)\, R(x, y).$$

In particular, if every $a \in A$ has an in-neighbor, then $D = A$ and we get that $\mathbf{C}$ absorbs $\mathbf{A}$. This construction will be generalized in the next section.

4 Connectivity

In this section, we will show that if a smaller subpower absorbs a bigger one, then some structural properties (like connectivity) of the bigger subpower transfer to the smaller one.

A similar situation appeared in Section 2.5: the relation R defined a connected bipartite graph, but its restriction to $\{0,1\}$ (on both sides) was absorbing and disconnected – this contradiction concluded the proof in Section 2.

We will illustrate the slogan "absorption transfers connectivity" using two examples: First we study a single binary relation and obtain a result which will be used later to prove, e.g., the Loop Lemma (which is Theorem 13). Later we focus on a more complex example: a microstructure graph arising from an instance of a CSP.

In order to simplify the applications of Proposition 1, we will be working with subdirect subpowers. A subset R of $A_1 \times \cdots \times A_n$ is called *subdirect* if, for each i, the projection of R onto the i-th coordinate is equal to A_i.

4.1 Absorbing linkedness

The first notion that is preserved by absorption is "linkedness". A subset $R \subseteq A^2$ is called *linked* if R is connected when regarded as a bipartite graph (exactly like in Section 2.5). Similarly, we can talk about $a, b \in A$ being linked, but, as every element of A has two copies in the bipartite graph, we need to specify whether we mean left or right a and left or right b.

The same relation $R \subseteq A^2$ can also be regarded as a directed graph and we talk about in/out-neighbors, sinks, sources, directed walks, etc. The digraph R is *smooth* if R is subdirect in A^2, in other words, R has no sources and no sinks. The *smooth part* of R is the maximal subset B of A such that $R \cap B^2$ is smooth.

Note that the linkedness (i.e. the connectivity of the bipartite graph) is equivalent to neither strong nor weak connectivity of the directed graph (but it implies the weak one). The following proposition states that the linkedness transfers to absorbing subuniverses.

▶ **Proposition 2.** *Let $\mathbf{R} \leq \mathbf{A}^2$ be subdirect, linked, and let $\mathbf{A}$ have a proper absorbing subalgebra. Then there exists $\mathbf{B}$, a proper subalgebra of $\mathbf{A}$, such that $\mathbf{R} \cap \mathbf{B}^2$ is a linked and subdirect in $\mathbf{B}^2$.*

Proof. First we look for a proper absorbing subalgebra $\mathbf{D} \trianglelefteq \mathbf{A}$ such that $D \cap A^2$ has a nonempty smooth part. This is achieved by "walking": think of $C \subseteq A$ as being on the left side of the bipartite graph, define C' as the set of all neighbors (i.e. on the right) of vertices from C, and put $(C, \text{left}) \sqsubseteq (C', \text{right})$ — this is a step from left to right. Similarly, for a step from C' on the right to C'' on the left (C'' contains all the neighbors of C'), put $(C', \text{right}) \sqsubseteq (C'', \text{left})$. Finally, close $\sqsubseteq$ under composition with itself.

Let $\mathbf{B}'$ be proper absorbing subalgebra of $\mathbf{A}$. Whenever $(B', \text{left}) \sqsubseteq (C, \text{left})$ or $(B', \text{left}) \sqsubseteq (C, \text{right})$, then C, by Proposition 1, is an absorbing subuniverse of $\mathbf{A}$. Since R is linked, $(B', \text{left}) \sqsubseteq (A, \text{left})$ and therefore there is D, say on the left, such that $(B', \text{left}) \sqsubseteq (D, \text{left})$ and by stepping to the right from D we obtain A, i.e. every vertex in the right A has an neighbor in the left D.

Looking at R as a directed graph, this property of D means that every vertex in A has an incoming edge from a vertex in D. It follows that there exists an arbitrarily long directed walk entirely in D, which immediately provides a directed cycle in the directed graph induced by R on D. Therefore, the smooth part of $D \cap A^2$, denoted by B, is nonempty.

The subuniverse D absorbs $\mathbf{A}$ by Proposition 1. Moreover, by the same proposition, the smooth part of $D \cap R^2$ (i.e. B) absorbs the smooth part of R (which is the whole A). This last fact holds since the smooth part of a directed graph can be pp-defined as the set of vertices with a directed walk of length $|A|$ from them and to them.

Finally, a generalization of the argument from Section 2.5 shows that $B \cap A^2$ is linked: Take any $a, b \in B$ (on the left) and a link $a = a_0, a_1, a_2, \ldots, a_{2k} = b$ from a to b (even members are on the left, the odd ones on the right). Consider a term operation f witnessing $\mathbf{B} \trianglelefteq \mathbf{A}$. Then, for any i, the sequence

$$f(\underbrace{a,\dots,a}_{(i-1)\times}, a_0, b, \dots, b),\ f(a,\dots,a,a_1,b,\dots,b),\ \dots\ , f(a,\dots,a,a_{2k},b,\dots,b)$$

provides a link from

$$f(\underbrace{a,\dots,a}_{i\times}, b, \dots, b) \text{ to } f(\underbrace{a,\dots,a}_{(i-1)\times}, b, \dots, b)$$

which lies fully in B. By concatenating these links we get a link in B from $a = f(a, \dots, a)$ to $b = f(b, \dots, b)$. ◀

The proof above exhibits a structure common to almost all the proofs using absorption. It splits into two stages:

- **Walking stage** finds a substructure which is "subdirect" and "absorbing" (here finds $\mathbf{B}$ absorbing $\mathbf{A}$ such that the restriction of $\mathbf{R}$ to $\mathbf{B}^2$ is subdirect in $\mathbf{B}^2$).
- **Reducing stage** uses absorption to transfer an additional property (here linkedness of $\mathbf{R}$ is transferred to the restriction of $\mathbf{R}$ by $\mathbf{B}^2$).

In the remaining part of the paper, we will see more proofs following this pattern. Here we present a corollary which is an easy consequence of the proposition above.

▶ **Corollary 2.** *Let* $\mathbf{R}$ *be a subdirect linked subalgebra of* $\mathbf{A}^2$ *and assume that* every non–singleton subalgebra of $\mathbf{A}$ *(including* $\mathbf{A}$ *itself) has a proper absorbing subalgebra. Then* $\mathbf{R}$ *contains a constant pair.*

Proof. The corollary is proved by a repeated application of Proposition 2. Indeed, after a first application of Proposition 2 to $\mathbf{R}$ and $\mathbf{A}$, we obtain $\mathbf{B}$ and if $|B| = 1$, we have a constant tuple in $\mathbf{R}$. Otherwise, $\mathbf{R} \cap \mathbf{B}^2$ is linked and subdirect in $\mathbf{B}$ and, as $\mathbf{B}$ has a proper absorbing subuniverse, we can apply Proposition 2 again. In a finite number of steps, we arrive at a one–element B, which finishes the proof. ◀

The structure of this proof is also typical, most of the proofs using absorption perform a sequence of reductions decreasing the sizes of underlying algebras until each one has only one element.

Note that algebras with a near-unanimity term, or algebras with a semilattice term, or products of such algebras all satisfy the assumptions of the corollary.

4.2 Connectivity in CSP

The results presented in this section are parts of a proof of the bounded width conjecture of Feder and Vardi [31]. This conjecture, its motivation and resolution, are discussed in more detail in Section 7. In here we focus on a single obstacle, which had to be overcome in order for the proof to work. The obstacle can be phrased as follows: is there a single consistency algorithm solving the CSP over all the binary templates with near-unanimity polymorphisms? We ought to note that already Feder and Vardi [31] showed that near-unanimity templates can be solved in polynomial time. However, their algorithm depends on the arity of the near-unanimity term and therefore does not reach our goal.

We move on to a list of consistency notions in search of a consistency notion that will be transferred by absorption the way the linkedness was transferred in Proposition 2. We start, however, with a consistency notion which plays the role played by subdirectness in Proposition 2 – it is the most basic and important consistency notion, the *arc consistency.*

4.2.1 Arc consistency of a CSP instance

In practical applications, arc consistency is often used to quickly disqualify some of the instances with no solutions. Unfortunately, a rather strong structure of the template is needed [31, 29] for arc consistency to solve the associated CSP.

We say that an instance with a variable set V is arc consistent with $\{P_x\}_{x\in V}$ if for every constraint $R(x_1,\dots,x_n)$, the relation R is subdirect in $P_{x_1}\times\dots\times P_{x_n}$. The following algorithm turns an arbitrary instance into an arc consistent instance with the same set of solutions.

for every variable x **do** $P_x := A$
repeat
  for every constraint $R(x_1,\dots,x_n)$ **do**
    let $R' := R\cap\prod_i P_{x_i}$
    for $i=1$ **to** n **do** $P_{x_i} := P_{x_i}\cap \operatorname{proj}_i R'$
    substitute constraint $R(x_1,\dots,x_n)$ with $R'(x_1,\dots,x_n)$
  end for
until none of the P_x's changed

It is clear that the output instance is arc consistent and has the same set of solutions as the input instance. The AC algorithm *derives a contradiction* if at least one of the sets P_x is empty; this means that the algorithm correctly detected an unsolvable instance.

Note that all the sets P_x as well as the new relations in the output instance are pp-definable from the relations in the original instance, therefore all the polymorphisms of the template are compatible with the new instance. In other words, if $\mathbb{A}$ is a template and $\mathbf{A}=\operatorname{Pol}(\mathbb{A})$ the associated algebra, then every P_x is a subuniverse of $\mathbf{A}$ and determines a subalgebra $\mathbf{P}_x$ of $\mathbf{A}$.

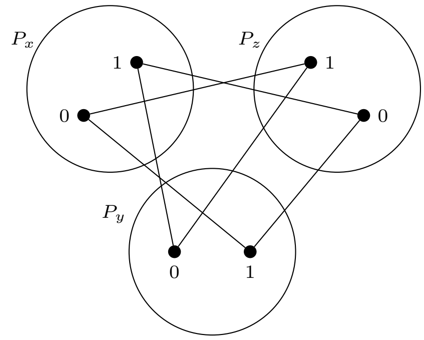
Arc consistency solves a CSP over $\mathbb{A}$ if it derives a contradiction on every unsolvable instance over $\mathbb{A}$. Such CSPs are said to have *width 1* and include the CSP over the template $\mathbb{A}=(\{0,1\};C_0,C_1,\leq)$, which is essentially the problem of finding a directed path in a directed graph, or over $\mathbb{A}=(\{0,1\};\{0,1\}^3\setminus\{(1,1,0)\},\{0,1\}^3\setminus\{(1,1,1\}))$, which is Horn-3-SAT.

A simple example of an instance where the arc consistency algorithm fails to detect a problem uses the template $\mathbb{A}=(\{0,1\};\neq)$, whose CSP is essentially 2-colorability, and the instance

$$(\exists x)(\exists y)(\exists z)\, x\neq y\wedge y\neq z\wedge z\neq x,$$

which corresponds to the triangle graph. The arc consistency algorithm on this instance does not update any constraints and outputs the original instance with the sets $P_x=P_y=P_z=\{0,1\}$.

The following picture shows the problematic instance as a multipartite graph called the *microstructure graph* of the instance: the graph has a copy of P_x for every variable x and the edges between P_x and P_y are given by a constraint $R(x,y)$.



Such a graph is well defined if all the constraints are binary, and every pair of variables appears in at most one constraint. This is not a very restrictive condition and we discuss such instances in the next section.

Note that the template $\mathbb{A} = (\{0,1\}; \neq)$ has a majority polymorphism, and therefore arc consistency fails to work for binary templates with near unanimity polymorphisms. In order to answer the questions posted in the section above we need to work with a different consistency notion.

4.2.2 (2,3)-consistent, simplified instances

In order to simplify presentation, we impose the following restrictions on CSP instances:

- all the constraints are binary and
- for every two distinct variables x, y, there is a unique constraint $P_{xy}(x, y)$ and $P_{xy} = P_{yx}^{-1}$.

We call an instance *simplified* if these conditions are satisfied. We note that, by an appropriate preprocessing, every instance can be turned into a simplified instance on a, possibly different, template with the same complexity of the associated CSP.

The arc consistency algorithm, over a simplified instance, finds algebras $\mathbf{P}_x$ and restricts the constraints so that $\mathbf{P}_{xy}$ is subdirect in $\mathbf{P}_x \times \mathbf{P}_y$. If arc consistency fails to solve a particular CSP($\mathbb{A}$) (as was the case in the example in the previous section) we may try to solve the problem by enforcing a stronger form of consistency. The next, after arc consistency, standard consistency notion is the $(2,3)$-consistency, also known as path consistency.

▶ **Definition 3.** A simplified instance is $(2,3)$-*consistent* if it is arc consistent and for every pairwise different variables x, y, z and any $(a, b) \in P_{xy}$ there exists $c \in P_z$ such that $(a, c) \in P_{xz}$ and $(b, c) \in P_{yz}$.

Both arc consistency and $(2,3)$-consistency have simple interpretations in the microstructure graphs of simplified instances: arc-consistency means that every vertex is adjacent to some vertex in every other partite set, $(2,3)$-consistency further ensures that, for any pairwise different x, y, z, every edge in $P_x \cup P_y$ (or, more precisely, in the union of the disjoint copies of P_x and P_y in the microstructure graph) extends to a triangle in $P_x \cup P_y \cup P_z$.

Already Feder and Vardi [31] noted that, over a template with majority polymorphism, every simplified $(2,3)$-consistent instance has a solution. The reasoning, however, did not extend to near unanimity operations of higher arities.

At present, we know that the $(2,3)$-consistency is transferred by absorption in the same way the linkedness is transferred in Proposition 2:

▶ **Proposition 3.** *Take a $(2,3)$-consistent simplified instance such that at least one $\mathbf{P}_x$ has a proper absorbing subuniverse. Then there exist $\mathbf{P}'_x \trianglelefteq \mathbf{P}_x$ (at least one proper) and $\mathbf{P}'_{xy} \trianglelefteq \mathbf{P}_{xy}$ which form a $(2,3)$-consistent instance.*

This would finish our search for a consistency notion transferred by absorption, except for the fact that more involved tools are required in order to prove this proposition. These tools are presented and discussed in Section 7, while here we continue the search for a consistency notion for which an analogue of Proposition 2 can be proved directly.

4.2.3 Prague instances

In this section, we study notions weaker than $(2,3)$-consistency. A concept underlying all the definitions in this section is the notion of a pattern. In a simplified instance, *a pattern* p is a sequence of variables $(x_1, x_2, \dots, x_{k-1}, x_k)$. If the first and the last variable of a pattern coincide, we call it a *circle*. An element a is connected to b by $p = (x_1, x_2, \dots, x_{k-1}, x_k)$ if there exists $(a = a_1, a_2, \dots, a_{k-1}, a_k = b)$ such that $(a_i, a_{i+1}) \in P_{x_i x_{i+1}}$ if $x_i \neq x_{i+1}$, and $a_i = a_{i+1}$ otherwise. We write $p + q$ for the concatenation of patterns, and kp for the concatenation of k copies of p.

The following notion of consistency is a first approximation to the notion of a Prague instance:

▶ **Definition 4.** A simplified instance is a *circle instance* if it is arc-consistent and for every pattern $p = (x = x_1, x_2, \dots, x_k = x)$, every $a \in P_x$ is connected to itself by p.

Unfortunately, the notion suffers from the same problem as $(2,3)$-consistency. One can state a result transferring the consistency:

▶ **Proposition 4.** *Take a simplified circle instance such that at least one $\mathbf{P}_x$ has a proper absorbing subuniverse. Then there exist $\mathbf{P}'_x \trianglelefteq \mathbf{P}_x$ (at least one proper) which together with $P'_{xy} = P_{xy} \cap (P'_x \times P'_y)$ form a simplified circle instance.*

The proof, yet again, requires tools from Section 7.

In order to introduce the final consistency notion of this section, we need one more definition. In a simplified instance, $a, b \in P_x$ are connected *in the set of variables* I (x needs to belong to I) if they are connected by a pattern with all the variables in I. Equivalently, we can restrict the microstructure graph to P_y's with $y \in I$ and ask for the usual undirected connectivity of vertices. We say that a, b are connected *in the variables of* p if they are connected in I, where the set I consists of the variables that appear in p.

Finally, we can define the notion of a Prague instance, a consistency notion for which an analogue of Propositions 2, 3 or 4 can be shown directly.

▶ **Definition 5.** A simplified instance is a *Prague instance* if it is arc consistent and for every circle pattern at x and every $a, b \in P_x$, the vertices a, b are connected by kp, for some natural number k, whenever they are connected in the variables of p.

It is left as an exercise for the reader to prove that every $(2,3)$-consistent instance is a circle and a Prague instance. Moreover, the number k, in the definition of Prague instance, can be chosen to depend only on p and not on a, b.

Note that the example in Section 4.2.1, although arc consistent, is neither a circle nor a Prague instance. Indeed, $0, 1 \in P_x$ are connected in $\{x, y, z\}$ but not connected by any power of (x, y, x, z, x) (nor any power of (x, y, z, x, y, z, x)) and therefore does not contradict the following proposition (which is an analogue of Proposition 2).

▶ **Proposition 5.** *Take a simplified Prague instance such that at least one $\mathbf{P}_x$ has a proper absorbing subuniverse. Then there exist $\mathbf{P}'_x \trianglelefteq \mathbf{P}_x$ (at least one proper) which together with $P'_{xy} = P_{xy} \cap P'_x \times P'_y$ form a simplified Prague instance.*

Proof. As usual, the proof splits into two stages. In the walking stage, we find $\mathbf{P}'_x$'s such that

- $\mathbf{P}'_x \trianglelefteq \mathbf{P}_x$ for every x,
- for some x, the algebra $\mathbf{P}'_x$ is a proper subalgebra of $\mathbf{P}_x$, and
- putting $P'_{xy} = P_{xy} \cap P_x \times P_y$ produces an arc consistent instance.

In the reduction stage, we will show that this instance is a Prague instance.

The walking stage is similar to the one in the proof of Proposition 2: we put $(B, x) \sqsubseteq (C, y)$ whenever the set C consists of elements of P_y which have a neighbor in B in the bipartite graph P_{xy} (B is on the left, while C on the right). Yet again, we close $\sqsubseteq$ under composition with itself and disregard the pairs with the full sets P_x, i.e. the pairs of the form (x, P_x).

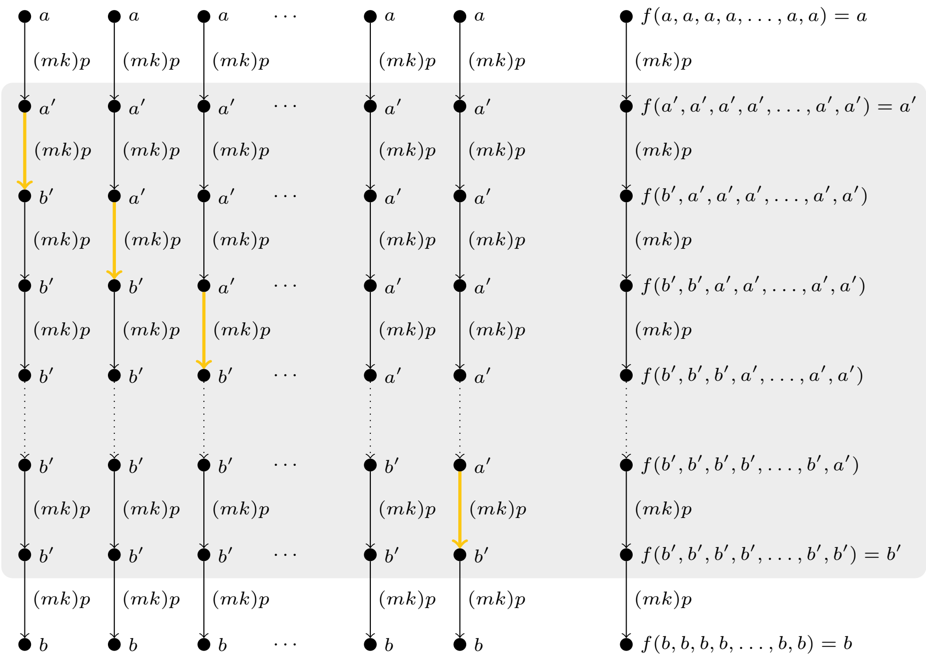
Let $\mathbf{B}_0$ be a proper absorbing subuniverse of $\mathbf{P}_{x_0}$. We walk, as far as we can, from (B_0, x_0) to end up in a maximal strong component of $\sqsubseteq$ (which may contain a single pair). We denote the set of all the pairs in this component by $\mathcal{P}$. Note that if $(B, x) \in \mathcal{P}$ and C consists of neighbors of B in P_{xy} (from left to right), then C is either P_y, or the pair (C, y) belongs to $\mathcal{P}$.

If (B, x) and (B', x) are in $\mathcal{P}$, then $(B, x) \sqsubseteq (B', x) \sqsubseteq (B, x)$. Let p denote the pattern describing the walk witnessing $(B, x) \sqsubseteq (B', x) \sqsubseteq (B, x)$. Pick $a \in B' \setminus B$ and $b \in B$ such that a is reachable from b by the appropriate initial part of p. But then b is connected to a in the vertices of p but, as $a \notin B$, not by any kp, a contradiction.

Thus, for a given x, there is at most one pair $(B, x) \in \mathcal{P}$ and we let $P'_x = B$ in such a case. If there is no such a pair, we put $P'_x = P_x$. Each set P'_x absorbs $\mathbf{P}_x$ (exactly like in the proof of Proposition 2), at least one of them is proper (actually all that arise from $\mathcal{P}$ are proper), and they define an arc consistent instance. That last property follows from the choice of $\mathcal{P}$ as the maximal strong component. We are done with the walking stage and proceed to the reducing stage.

Let a be in $\mathbf{P}'_x$ and p be a pattern such that $b \in \mathbf{P}'_x$ is connected to a in the variables of p. Find m and a', b' such that a' is reachable from itself and from a by mp in the new instance; and, similarly, from b' one can reach b' and b by mp also in the new instance. Since a' and b' are connected in the variables of p in the original instance, we get k such that b' is reachable from a' in by $(mk)p$ in the original instance.

Now we take a term operation f witnessing the absorptions $P'_x \trianglelefteq \mathbf{P}_x$ and apply it as shown on the following picture. The black arrows are realizations of $(mk)p$ in the new instance, and the yellow arrows are realizations of the same pattern in the original instance. On the right-hand side of the picture is the result of pointwise application of f to the realizations of $(mk)p$ and the grey part indicates where absorption is used to ensure that the resulting elements are in the new instance.



Thus b can be reached from a by a sufficiently large multiple of p in the new instance. This finishes the proof of the reduction. ◀

The following corollary, which is an analogue of Corollary 2, immediately follows from the previous proposition.

▶ **Corollary 6.** *Take a simplified Prague instance. If every non–singleton subalgebra of every* $\mathbf{P}_x$ *(including* $\mathbf{P}_x$ *itself) has a proper absorbing subuniverse, then the instance has a solution.*

The corollary gives a polynomial time algorithm for CSPs over templates with a near-unanimity polymorphism, semilattice polymorphism, etc. using a single consistency notion: the notion of a Prague instance. This settles the question posed at the beginning of Section 4.2.

5 Equational descriptions

In this section, we present several results showing how equational conditions impact properties of invariant relations. In fact, equational conditions influence invariant relations of all algebras in a *variety* rather than an individual algebra. We start by defining this concept.

An equivalence on universe of an algebra $\mathbf{A}$ is a *congruence* if it is invariant, as a binary relation, under the operations in $\mathbf{A}$; in other words, it is a subpower of $\mathbf{A}$. An algebra $\mathbf{A}$ can be factored modulo its congruence α to obtain a *quotient* $\mathbf{A}/\alpha$: the compatibility of $\mathbf{A}$ with α ensures that operations can be defined using arbitrarily chosen representatives. A *variety generated by an algebra* $\mathbf{A}$, denoted $\mathcal{V}(\mathbf{A})$, is the smallest class of algebras containing $\mathbf{A}$ and closed under taking powers, subalgebras, and quotients (and isomorphic copies).

Varieties provide the second step in the algebraic approach to the CSP: from the algebra $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$ to the variety generated by $\mathbf{A}$. This step is meaningful as every relational structure $\mathbb{B}$ compatible with an (as always finite) algebra $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ defines a CSP not harder than $\mathrm{CSP}(\mathbf{A})$ [23]. The "not harder" statement can be understood as an existence of a LOGSPACE reduction from $\mathrm{CSP}(\mathbb{B})$ to $\mathrm{CSP}(\mathbb{A})$, but the connection between $\mathbb{A}$ and $\mathbb{B}$ is much closer: many structural properties are transferred from $\mathbb{A}$ to $\mathbb{B}$.

We can talk about $\mathbb{B}$'s, compatible with algebras in the variety generated by $\mathbf{A}$, totally bypassing the algebraic nomenclature and using pp-interpretations instead [18]. A relational structure $\mathbb{B} = (B, S_1, \ldots, S_m)$ is *pp-interpretable* in $\mathbb{A}$ if there are

- relation $R' \subseteq A^n$ and equivalence α on R', both pp-definable[1] in $\mathbb{A}$;
- relations $S'_1, \ldots, S'_m$ on R'/α also pp-definable[1] in $\mathbb{A}$

such that $\mathbb{B}$ and $(R'/\alpha, S'_1, \ldots, S'_m)$ are isomorphic. It is easy to see that a relational structure is pp-interpretable in $\mathbb{A}$ if and only if it compatible with an algebra in $\mathcal{V}(\mathrm{Pol}(\mathbb{A}))$.

A part of the Mod–Id Galois connection gives a link between identities and varieties [17]: An algebra $\mathbf{B}$ (of the same signature as $\mathbf{A}$) is in $\mathcal{V}(\mathbf{A})$ if and only if $\mathbf{B}$ satisfies all the identities satisfied by $\mathbf{A}$, equivalently, by all members of $\mathcal{V}(\mathbf{A})$.

The identities true in a variety are closely connected to particular members of the variety: the free algebras. The *free algebra* in $\mathcal{V}(\mathbf{A})$ over n generators can be described as the subalgebra of $\mathbf{A}^{A^n}$ whose universe is the set of n-ary term operations of $\mathbf{A}$. For a k-ary basic or term operation f of $\mathbf{A}$, the corresponding operation f in the free algebra acts as composition: for any $g_1, \ldots, g_k$ in the free algebra, we have

$$f(g_1, \ldots, g_k) : (x_1, \ldots, x_n) \mapsto f(g_1(x_1, \ldots, x_k), g_2(x_1, \ldots, x_k), \ldots, g_k(x_1, \ldots, x_k))$$

The free algebra is the smallest (with respect to inclusion) subalgebra of $\mathbf{A}^{A^n}$ containing the projections $\pi_1, \pi_2, \ldots, \pi_n$ (where $\pi_i : (x_1, \ldots, x_n) \mapsto x_i$). We also say that the free algebra is *generated by* the projections. Very often, important properties of algebras are determined by the structure of subpowers of $\mathbf{F}$, in particular, smooth digraphs on the free algebra on two elements are used in Sections 5.2 and 6.3. Free algebras are also behind both the Pol–Inv and Mod–Id Galois connections.

Looking from the relational side, note that if $\mathbf{A}$ is the algebra of polymorphisms of a relational structure $\mathbb{A}$, i.e. $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$, then the universe of the n-generated free algebra in $\mathcal{V}(\mathbf{A})$ is the set of n-ary polymorphisms of $\mathbb{A}$.

5.1 Decomposable relations and near unanimity

The near unanimity operations were among the most prominent operations in the previous sections. One of the relational descriptions is by means of decomposable relations, which appear naturally in the study of CSP (comp. [31]).

A k-decomposition of an n-ary relation R over A is another n-ary relation R_k over A defined by:

$$(a_1, \ldots, a_n) \in R_k \text{ if } \big(\text{for all } 1 \leq j_1 < \cdots < j_k \leq n \text{ tuple } (a_{j_1}, \ldots, a_{j_k}) \in \mathrm{proj}_{\{j_1,\ldots,j_k\}}(R)\big)$$

where $\mathrm{proj}_{\{j_1,\ldots,j_k\}}(R)$ is the k-ary relation obtained by taking elements of R and selecting only the coordinates $j_1, \ldots, j_k$. Clearly, $R \subseteq R_k$ and the relation R is called *k-decomposable* if $R_k = R$. Note that, for example, 2-decomposability of all the relations in the template of a CSP allows a trivial transformation of every instance to an equivalent simplified instance (it suffices to take the binary projections of constraints and intersect them if necessary).

The Baker–Pixley theorem [2] below provides identities equivalent to decomposability of all relations in a variety.

▶ **Theorem 7.** *For an algebra* $\mathbf{A}$ *the following are equivalent:*

[1] The relation α is viewed here as a binary relation on R' and a $(2n)$-ary relation over A. The relations S'_i are k_i-ary over R' or $(k_i n)$-ary over A and independent on the choice of representatives for an α-class on any coordinate.

1. *the algebra* $\mathbf{A}$ *has a near-unanimity term operation of arity* $k+1$*;*
2. *for every* $\mathbf{B} \in \mathcal{V}(\mathbf{A})$*, every subpower of* $\mathbf{B}$ *is* k*-decomposable.*

Proof. To prove item 1 from 2 we consider the free algebra $\mathbf{F}$ for $\mathbf{A}$ over 2 generators and let $\mathbf{R} \leq \mathbf{F}^{k+1}$ be generated by the tuples which are π_1 on all coordinates except for one where they are π_2.

As $\mathbf{R}$ is the smallest subalgebra of $\mathbf{F}^{k+1}$ containing the generators, its elements are obtained by applying the term operations of $\mathbf{F}$ to the generators. More formally,

$$
\begin{aligned}
R &= \{f\big((\pi_1,\dots,\pi_1,\pi_2),(\pi_1,\dots,\pi_1,\pi_2,\pi_1),\dots,(\pi_2,\pi_1,\dots,\pi_1)\big) : f \text{ is a } k+1\text{-ary term op.}\}\\
&= \{\big(f(\pi_1,\dots,\pi_1,\pi_2),\dots,f(\pi_1,\pi_2,\pi_1,\dots,\pi_1),f(\pi_2,\pi_1,\dots,\pi_1)\big) : f \text{ as above}\}\\
&= \{\big((x,y)\mapsto f(x,\dots,x,y),\dots,(x,y)\mapsto f(x,y,x,\dots,x),\dots,(x,y)\mapsto f(y,x,\dots,x)\big) : f \text{ as above}\}
\end{aligned}
$$

To simplify the notation, we will write $f(x,\dots,x,y)$ instead of $(x,y)\mapsto f(x,\dots,x,y)$, so that the generators are $(x,\dots,x,y),(x,\dots,x,y,x),\dots,(y,x,\dots,x)$ and

$$
R = \{\big(f(x,\dots,x,y),\dots,f(x,y,x\dots,x),f(y,x,\dots,x)\big) : f \text{ is } (k+1)\text{-ary term op. of } \mathbf{A}\}.
$$

Since R is decomposable, the tuple $(x,\dots,x)$ is in R. The description of the elements of $\mathbf{R}$ implies that there is a $(k+1)$-ary term f such that

$$
\big(f(x,\dots,x,y),\dots,f(x,y,x\dots,x),f(y,x,\dots,x)\big) = (x,\dots,x).
$$

This f is clearly a near unanimity operation.

It remains to show that if an algebra has a $(k+1)$-ary near unanimity term operation, denote it by f, then all the subpowers of any $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ are k-decomposable.

Let $\mathbf{R} \leq \mathbf{B}^{(k+1)}$ and let R_k be the k-decomposition of R. To show that $R_k \subseteq R$, we take an arbitrary tuple $(a_1,\dots,a_{k+1})$ from R_k and will show that it belongs to R. The structure of R_k implies that for every i there is a tuple in R which differs from $(a_1,\dots,a_{k+1})$ only on the i-th coordinate. Applying the near-unanimity operation f to such tuples we get

$$
f\big((?,a_2,\dots,a_{k+1}),(a_1,?,a_2,\dots,a_{k+1}),\dots,(a_1,\dots,a_k,?)\big) = (a_1,\dots,a_{k+1}).
$$

This shows that $R_k = R$, i.e. that R is k-decomposable. For relations of higher arity, the reasoning is similar and we conclude that all the subpowers of $\mathbf{B}$ are k-decomposable. ◀

Note that the algebras in the statement of the theorem can be equivalently characterized as algebras with every one-element subuniverse absorbing (comp. Section 3).

In case that $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$, item 1 says that $\mathbb{A}$ has a near unanimity polymorphism and item 2 is equivalent to the following statement: for every $\mathbb{B} = (B;R)$ pp-interpretable in $\mathbb{A}$, the relation R is k-decomposable.

We proceed to studying other classes of algebras. Every class will be defined by identities and posses a structural counterpart (playing a role similar to the one played by decomposability for near unanimity algebras).

5.2 Rectangular relations and Mal'tsev term

Rectangularity is another natural property of relations. A subset R of $B \times C$ is called *rectangular* if it is a disjoint union of products of the form $B' \times C'$, where $B' \subseteq B$ and $C' \subseteq C$. Equivalently R is rectangular if, when regarded as a bipartite graph, it is a disjoint union of bicliques, which means that every two linked elements $b \in B$ and $c \in C$ are adjacent. The following theorem [45] is a counterpart of Theorem 7 for rectangular relations.

▶ **Theorem 8.** *For an algebra* $\mathbf{A}$ *the following are equivalent:*
1. *the algebra* $\mathbf{A}$ *has a Maltsev term operation i.e.* f *such that*

$$f(x,x,y) = f(y,x,x) = y$$

2. *for any* $\mathbf{B} \in \mathcal{V}(\mathbf{A})$*, any* $R \leq \mathbf{B}^2$ *is rectangular.*

Proof. In order to prove the implication from 2 to 1, we proceed exactly like in the proof o Theorem 7: We choose $\mathbf{R}$ to be the subalgebra of $\mathbf{F}^2$ ($\mathbf{F}$ is still the free algebra on two generators) generated by $(\pi_1, \pi_2), (\pi_1, \pi_1)$ and (π_2, π_1) also denoted as $(x,y), (x,x), (y,x)$. The relation $\mathbf{R}$ is rectangular and thus includes the pair (y,y). This implies that we have a ternary term operation which generates this pair in $\mathbf{R}$, i.e.

$$f\big((x,y),(x,x),(y,x)\big) = (y,y).$$

This operation is clearly the required Mal'tsev operation.

For the other implication, take any $\mathbf{R} \leq \mathbf{B} \times \mathbf{C}$ and let $(a,b), (a',b), (a',b') \in R$. Then $f((a,b),(a',b),(a',b')) = (a,b') \in R$ which proves rectangularity. ◀

A more familiar form of the second item is the following.

3. for any $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ and α, β congruences on $\mathbf{B}$ the α and β permute, i.e. $\alpha \circ \beta = \beta \circ \alpha$,

We leave it as an exercise to the reader to show that this additional property is equivalent to item 2. A hint: to prove that 3 implies 2, use the kernels of the projections of R onto the two coordinates.

For relational structures associated to algebras with near unanimity operation, the tractability of CSP was provided by [31] or by Corollary 6. The relational structures with associated Mal'tsev algebras define tractable CSPs as well [22], but the algorithm is beyond the scope of this article.

5.3 Congruence distributivity

In this section, we describe a class of algebras that significantly benefited from absorption. It also motivated a study of weaker forms of absorption (comp. Section 5.3.2) which, in many cases, allow for stronger version of theorems. For instance, in Proposition 5, the standard absorption can be substituted with any of the weaker forms from Section 5.3.2.

In order to define the class, we need to introduce a new notion: For equivalences α, β on a set A,
- the smallest equivalence containing α and β is denoted by $\alpha \vee \beta$ and
- by $\alpha \wedge \beta$ we denote the intersection of α and β (which, accidentally, is the larges equivalence contained in both).

We say that an algebra $\mathbf{A}$ generates a *congruence distributive variety* (CD variety), if for every $\mathbf{B} \in \mathcal{V}(\mathbf{A})$, and every α, β, γ congruence on $\mathbf{B}$, we have

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Note that equivalence on the right side is always contained in the one on the left, so only one of the inclusions bears consequences.

To illustrate the connection between distributivity of congruences and the structure of relations, we consider $R \subseteq B \times C \times D$ and assume that the projection kernels η_B, η_C, and η_D, satisfy the distributive law, i.e.

$$\eta_B \wedge (\eta_C \vee \eta_D) \subseteq (\eta_B \wedge \eta_C) \vee (\eta_B \wedge \eta_D).$$

Two triples (b, c, d) and (b', c', d') are equivalent modulo the left equivalence, if they are equivalent modulo η_B and modulo $\eta_C \vee \eta_D$. The former simply means that $b = b'$, while the latter takes place exactly when there is a sequence $(b, c, d) = (b_1, c_1, d_1)$, (b_2, c_1, d_2), (b_3, c_2, d_2), (b_4, c_2, d_3), $\ldots$, $(b_{2k-1}, c_k, d_k) = (b', c', d')$; put otherwise, c and c' are linked in the projection of R to the second and third coordinates.

The two triples are equivalent modulo the right side if and only if such a sequence exists with $b_1 = b_2 = \cdots = b_{2k-1} = b$. To give this description a more lucid form, we regard R as a B-labeled bipartite graph with partitions C and D – a triple (b, c, d) corresponds to an edge (c, d) labeled by b (so the edges can have multiple labels). Now the inclusion can be interpreted in the following way: If $c, c' \in C$ are incident to a b-labeled edge and they are linked, then they are linked by b-labeled edges.

The identities characterizing congruence distributivity are derived from the connectivity property of a subpower of a free algebra, by a proof similar to the proofs in Sections 5.1 and 5.2: Let $\mathbf{F}$ denote the 2–generated free algebra for $\mathbf{A}$ and let $\mathbf{R}$ be the subalgebra of $\mathbf{F}^3$ generated by

$$(x, x, x), (y, x, y), (x, y, y)$$

which can be described as

$$R = \left\{ \left((p(x, y, x), p(x, x, y), p(x, y, y) \right) : p \text{ is a ternary term operation of } \mathbf{A} \right\}$$

Since $\mathbf{A}$ generates a CD variety, the relation R satisfies the connectivity property discussed above.

The vertices x and y are incident to x-labeled edges (namely (x, x) and (y, y) coming from the generators of the algebra) and they are linked. Therefore, they must be linked by x-labeled edges. This produces a sequence $(x, x = b_1, c_1), (x, b_2, c_1), (x, b_2, c_2), (x, b_3, c_2), \ldots, (x, b_n = y, c_{n-1})$, and the ternary term operations $p_1, \ldots, p_{2n-1}$ generating this sequence satisfy

$$\begin{aligned}
x &= p_1(x, x, y) && \\
p_i(x, y, y) &= p_{i+1}(x, y, y) && \text{for odd } 1 \le i \le 2n-3 \\
p_i(x, x, y) &= p_{i+1}(x, x, y) && \text{for even } 1 \le i \le 2n-2 \\
p_i(x, y, x) &= x && \text{for all } 1 \le i \le 2n-1 \qquad (\ddagger) \\
p_{2n-1}(x, x, y) &= y. &&
\end{aligned}$$

Term operations satisfying such identities are called *Jónsson terms*, and the following theorem [38] states an equivalence in the spirit of Theorems 7 and 8.

▶ **Theorem 9.** *The following are equivalent for an algebra* $\mathbf{A}$*.*

1. $\mathbf{A}$ *has Jónsson terms.*
2. *For each subalgebra* $\mathbf{R}$ *of* $\mathbf{B} \times \mathbf{C} \times \mathbf{D}$ *regarded as a* B*-labeled bipartite graph as above and for each* $b \in B$*,* $c, c' \in C$*,*
   - *if* $c, c' \in C$ *are incident to a* b*-labeled edge and*
   - *they are linked,*

   then they are linked by b*-labeled edges.*
3. *Any three congruences* α*,* β*, and* γ *of any algebra in* $\mathcal{V}(\mathbf{A})$ *satisfy*

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

5.3.1 Near unanimity and directed Jónsson terms

Near unanimity terms are stronger than Jónsson terms. This can be seen in the structure of subpowers i.e. one can prove item 2 of Theorem 9 using near unanimity operations; or by a direct syntactic argument as follows.

Let $\mathbf{A}$ be an algebra with an n-ary near unanimity term operation f. Define term operations $q_1(x, y, z), \ldots, q_n(x, y, z)$ by putting

$$q_i(x, y, z) = f(x, \ldots, x, \underset{(n-i+1)}{\overset{}{\underset{\uparrow}{y}}}, z, \ldots, z).$$

These terms satisfy

$$\begin{aligned} x &= q_1(x, x, y) && \\ q_i(x, y, y) &= q_{i+1}(x, x, y) && \text{for all } 1 \leq i < n \\ q_i(x, y, x) &= x && \text{for all } 1 \leq i \leq n \\ q_n(x, y, y) &= y. && \end{aligned} \tag{‡}$$

Term operations satisfying such identities are called *directed Jónsson terms*. It is easy to see that putting $p_{2i}(x, y, z) = q_i(x, y, z), p_{2i+1}(x, y, z) = q_i(x, z, z)$ we obtain Jónsson terms and therefore directed Jónsson terms imply a Jónsson terms.

It can be shown that the reverse implication also holds, even for infinite algebras [1]. Moreover, directed Jónsson terms have their own relational condition, i.e. we can extend Theorem 9 by two additional, equivalent conditions:

4. $\mathbf{A}$ has directed Jónsson terms.
5. For each subalgebra $\mathbf{R}$ of $\mathbf{B} \times \mathbf{C} \times \mathbf{C}$ regarded as a B-labeled digraph $b \in B$, $c, c' \in C$,
   - if both c and c' have b-labeled loops
   - and there is a directed walk from c to c'

   then there is a directed b-labeled walk from c to c'.

5.3.2 Jónsson absorption

The definition of absorption is similar to the conditions imposed on near unanimity terms. In a similar way, we can talk about Jónsson absorption or directed Jónsson absorption: we say that $\mathbf{B}$ *Jónsson absorbs (directed Jónsson absorbs)* $\mathbf{A}$ when there is sequence of terms like in the definition of Jónsson terms (directed Jónsson terms) but with the condition (‡) replaced by $p_i(B, A, B) \subseteq B$.

Quite a few results (e.g. Proposition 5) can be strengthen by relaxing the assumptions and allowing Jónsson absorption or directed Jónsson absorption instead of the absorption from Definition 1. However, as an analysis of such relaxations is beyond the scope of this article, we move to yet another application of absorption.

6 Absorption in Taylor algebras and its consequences

Proper absorption is common, even if algebras do not satisfy restrictive conditions. In fact, relatively mild assumption on the algebra forces either a strong restriction on the shape of compatible relations, or proper absorption.

From the algebraic perspective, the aforementioned mild assumption is, roughly, that the algebra is not “equationally trivial”. By this, we mean that it has a set of term operations which satisfy some identities that cannot be satisfied by projections. By the Mod–Id Galois correspondence, this is equivalent to requiring that no algebra in $\mathcal{V}(A)$ is a G-set, where a *G-set* (in our idempotent world) is an algebra whose every operation is a projection and which has at least 2 elements. The theorem of Taylor [50] provides an equational characterization of this class:

▶ **Theorem 10.** *The following are equivalent for an algebra* $\mathbf{A}$*.*
1. $\mathcal{V}(\mathbf{A})$ *does not contain a* G*-set.*
2. $\mathbf{A}$ *has a* Taylor *term operation, that is, a term operation* t *that for each* i *satisfies an identity of the form*

$$t(\dots, \underset{\substack{\uparrow\\ i}}{x}, \dots) = t(\dots, \underset{\substack{\uparrow\\ i}}{y}, \dots),$$

where $\dots$ *stand for some sequences of* x*'s and* y*'s.*

An algebra satisfying the equivalent conditions in Theorem 10 is called a *Taylor* algebra. Taylor algebras are central to the algebraic approach to CSP: whenever $\mathrm{Pol}(\mathbb{A})$ does not contain a Taylor operation then $\mathrm{CSP}(\mathbb{A})$ is NP-complete and the tractability conjecture (also known as the algebraic dichotomy conjecture) states that, otherwise it is solvable in polynomial time. The reason behind the hardness part is that if $\mathrm{Pol}(\mathbb{A})$ is not Taylor, then $\mathbb{A}$ pp-interprets every relational structure compatible with the G-set in the variety, but every relational structure is compatible with a G-set.

6.1 Absorption theorem

The following theorem is used to produce a proper absorption [9].

▶ **Theorem 11** (Absorption theorem)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be Taylor algebras (of the same signature) and* $\mathbf{R}$ *a subdirect linked subalgebra of* $\mathbf{A} \times \mathbf{B}$*. Then either* $\mathbf{A}$ *or* $\mathbf{B}$ *has a proper absorbing subalgebra, or* $R = A \times B$*.*

Proof strategy. Assume that neither of the algebras have a proper absorbing subalgebra. The strategy of the proof is the following.

- We produce a transitive term operation in $\mathbf{A}$ and $\mathbf{B}$. An operation t on A is *transitive* if, for each $a, b \in A$ and each coordinate i, there exists a tuple $(a_1, \dots, a_n)$ with $a_i = a$ such that $t(a_1, \dots, a_n) = b$. Such terms are produced by star composing Taylor term (see Proposition 2.7 in [9]) using the fact that $\mathbf{A}$ has no proper absorbing subalgebra with respect to the binary operations which appear in the Taylor equations.
- We show that a maximal set $X \subseteq A$ (or $Y \subseteq B$) such that each $a \in A$ ($b \in B$) is adjacent to a common neighbor of all elements of X (Y), absorbs $\mathbf{A}$ ($\mathbf{B}$). Having X or Y nonempty can be obtained by replacing R by a suitable relational composition of the form $R \circ R^{-1} \circ R \circ \dots$.
- The last item implies that necessarily $X = A$ or $Y = B$. In the first case, R has a nonempty *right center* – the set of elements $b \in B$ adjacent to every element of A. We show that the right center absorbs $\mathbf{B}$. Therefore it is equal to B and then $R = A \times B$.

◀

The theorem implies that non-rectangularity of a binary relation $\mathbf{R} \leq \mathbf{A} \times \mathbf{B}$ enforces proper absorption in a subalgebra of $\mathbf{A}$ or $\mathbf{B}$. Indeed, let R be a non-rectangular relation. Viewing R as a bipartite graph we get a connected component which is not a biclique. The elements of this component which are on the left form a set A' which is a subuniverse of $\mathbf{A}$, and similarly elements on the right form, denoted by B', form a subuniverse of $\mathbf{B}$. The relation $R \cap (A' \times B')$ is subdirect and linked in $\mathbf{A}' \times \mathbf{B}'$ and is not the full product. The absorption theorem guarantees a proper absorption in $\mathbf{A}'$ or $\mathbf{B}'$.

▶ **Example 12.** Going back to the example in Section 2, we will use the Absorption theorem to show that the polymorphism algebra $\mathbf{A}$ of $\mathbb{K}_3^c$ is not Taylor. This is equivalent to proving that $\mathbb{K}_3^x$ pp-interprets every finite structure. In Section 2, we proved a stronger claim, but, by the discussion above, the weaker claim still implies that $\mathrm{CSP}(\mathbb{K}_3^c)$ is NP-complete.

Assume for a contradiction that $\mathbf{A}$ is Taylor. The inequality relation R is a subdirect subalgebra of $\mathbf{A}^2$ and it is linked. By the absorption theorem, $\mathbf{A}$ has a proper absorbing subalgebra. This however directly contradicts Proposition 2, as there is no $\mathbf{B}$ with $R \cap B^2$ subdirect and linked.

6.2 Loop lemma

The absorption theorem makes it possible to relax the assumptions of Corollary 2 in two ways:

- instead of assuming that every non-singleton subalgebra of $\mathbf{A}$ has a proper absorbing subuniverse, we assume that it is Taylor,
- instead of assuming that the relation is linked, we assume that R has *algebraic length* 1, i.e. there is a closed oriented walk in R with one more forward than backward edges.

The generalized theorem [12, 9] states:

▶ **Theorem 13** (Loop lemma). *Let $\mathbf{A}$ be a Taylor algebra. If $\mathbf{R} \leq \mathbf{A}^2$ is subdirect and has algebraic length 1, then it has a loop.*

Proof. We present only a sketch of the proof. The proof supposes that $\mathbf{A}$ has more than one element (as otherwise the claim holds) and restricts $\mathbf{R}$ to a proper subalgebra of $\mathbf{A}$ while preserving subdirectness and algebraic length 1. The reasoning splits in two parts depending on the existence of a proper absorbing subuniverse in $\mathbf{A}$.

If $\mathbf{A}$ has an absorbing subuniverse, then the standard two-stage reasoning, as in the proof of Proposition 2, can be used. In the walking stage, we find $\mathbf{B}$ a proper absorbing subuniverse of $\mathbf{A}$ such that $\mathbf{R}$ restricted to $\mathbf{B}$ is subdirect. Then, in the reduction stage, we show that R restricted to $\mathbf{B}$ has algebraic length 1.

If $\mathbf{A}$ has no absorbing subuniverse, we take the smallest n such that R composed n-times with itself is linked. Such an n exists since the algebraic length of R is one. Since $\mathbf{A}$ has no absorbing subuniverses, the n-fold composition of R with itself is full (note that if $n = 1$ we found the needed loop). We set $\mathbf{B}'$ to consist of all the vertices on the right-hand side of a linked component of R composed with itself $(n-1)$-times. It is easy to see that R restricted to $\mathbf{B}'$ contains at least one cycle and we define $\mathbf{B}$ as the set of all the elements which are in some cycle in $\mathbf{B}'$. Direct graph-theoretical considerations, using the fact that the n-fold composition of R is full, show that $\mathbf{R}$ restricted to $\mathbf{B}$ has algebraic length one. ◀

A relatively simple consequence of the loop lemma is the CSP dichotomy over relational structures consisting of a single subdirect binary relation [12, 9] (these are exactly the smooth digraphs in the terminology of that paper). Theorem 13 is used as the main tool in showing that a core of such a graph is a disjoint union of directed cycles (which puts the CSP of such a graph in P), or it has no Taylor polymorphism and the CSP is NP-complete.

6.3 Siggers term

The Loop lemma, and earlier a similar result for undirected graphs [34, 24], can be used to prove [49, 39] that every Taylor algebra has Taylor operations of a very particular form.

▶ **Corollary 14** (Taylor implies Siggers). *Every Taylor algebra has a* 4*-ary term operation* s *and a* 6*-ary term* s' *such that*

$$s(x, y, z, x) = s(y, x, y, z) \text{ and } s'(x, y, x, z, y, z) = s'(y, x, z, x, z, y).$$

Proof. The argument is very similar to the reasoning in the proof of Theorem 8. Consider $\mathbf{F}$ – the free algebra on three generators and let $\mathbf{R}$ be the subalgebra of $\mathbf{F}^2$ generated by (x, y), (y, x), (z, y), and (x, z) for s (or $(x, y), (y, x), (x, z), (z, x), (y, z), (z, y)$ for s'). In both cases, $\mathbf{R}$ is subdirect in $\mathbf{F}^2$ and has algebraic length one. (For s', the relation R is additionally symmetric.) The loop lemma provides a loop in $\mathbf{R}$ which implies the appropriate term. ◀

The operations from the corollary are called Sigger's terms (a 4-ary one and a 6-ary one). The identities of 4-ary Siggers can be rewritten as $s(a, r, e, a) = s(r, a, r, e)$ which serves as an easy mnemonic. As both of the Sigger's operations are automatically Taylor, Corollary 14 provides an alternative characterization of Taylor algebras. It is quite surprising that (for finite idempotent algebras!) nontrivial identities always imply one specific nontrivial identity, a relatively nice and simple one at that. Even more surprising is a very recent result of Olšák [48] providing a specific nontrivial identities satisfied by any idempotent Taylor algebra, not necessarily finite. His proof also uses absorption in a substantial way.

6.4 Cyclic terms

Another equational characterization of Taylor algebras uses cyclic operations [9]. An operation t of arity $n \geq 2$ on a set A is *cyclic* if $t(x_1, \ldots, x_n) = t(x_2, \ldots, x_n, x_1)$.

▶ **Theorem 15** (Taylor implies cyclic). *If* $\mathbf{A}$ *is Taylor and* p *is a prime number greater than* $|A|$*, then* $\mathbf{A}$ *has a cyclic term operation of arity* p*.*

Proof. The proof splits into two, uneven, parts. The first important step is to reduce the problem to a question about compatible relations. Namely, it is enough to prove that each nonempty p-ary subpower $\mathbf{R}$ of $\mathbf{A}$ which is invariant under cyclic shifts (called *cyclic relation*) contains a constant tuple. This fact provides term operations which are cyclic with respect to one tuple. These "local cyclic operations" can be composed to a global one which finishes the proof.

In order to prove the constant tuples in cyclic relations we employ the Loop lemma. Consider a cyclic subpower $\mathbf{R} \leq \mathbf{A}^p$ and let $\mathbf{S}$ be the projection of $\mathbf{R}$ onto all but the last coordinate. By cyclic invariance, $\mathbf{S}$ is equal to the projection to all but the first coordinate, therefore the binary relation

$$T = \{((a_1, \ldots, a_{p-1}), (a_2, \ldots, a_p)) : (a_1, \ldots, a_p) \in R\}$$

is a subdirect subalgebra of $\mathbf{S}^2$. If we knew that T has algebraic length one, the Loop lemma would give us a loop in T, which clearly implies a constant tuple in T. Showing that T has algebraic length 1 is the technical core of the proof. ◀

The cyclic terms strengthen the characterization of Taylor algebras by means of *weak NU operations* [47] – these are such that their value on the tuples $(x, \ldots, x, y, x, \ldots, x)$ does not depend on the position of y (but is not necessarily equal to x like for the NU operations). Corollary 14 also follows from Theorem 15 as both Siggers operations can be obtained by identification of variables in a cyclic operation of suitable chosen prime arity, as observed in [43].

The cyclic terms, or more precisely their characterization by cyclic relations, allow to formulate the algebraic CSP dichotomy conjecture by means of properties of pp-definable relations as follows:

▶ **The Algebraic CSP Dichotomy Conjecture.** *Let $\mathbb{A}$ be a relational structure. If, for some (equivalently all) prime number $p > |A|$, every p-ary cyclic relation pp-definable from $\mathbb{A}$ has a constant tuple, then* $\mathrm{CSP}(\mathbb{A})$ *is tractable. Otherwise, it is NP-complete.*

6.5 Conservative CSPs

One of the biggest classes of CSPs known to exhibit the dichotomy was obtained by Bulatov [25]. His result confirms the algebraic dichotomy conjecture for all *conservative* CSPs, that is, CSPs over relational structures that contain all the unary relations.

▶ **Theorem 16** (Bulatov). *Assume that $\mathbb{A}$ contains all the unary relations and let $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$. If $\mathbf{A}$ is a Taylor algebra, then* $\mathrm{CSP}(\mathbb{A})$ *is tractable, else it is NP-complete.*

Note that the conservativity of $\mathbb{A}$ is equivalent to the fact that each subset of A is a subuniverse of $\mathbf{A}$.

Bulatov's proof of Theorem 16 uses his technique of local analysis of finite algebras and is rather long and technical. Absorption allowed to provide a significantly shorter proof [4].

One ingredient of this alternative proof is the following fact stated in Theorem 29: If $\mathbf{P}$ is a Taylor algebra such that no subalgebra of $\mathbf{P}$ has a proper absorbing subniverse (such algebras are called *hereditarily absorption free*), then $\mathbf{P}$ has a Maltsev term operation (comp. Theorem 29). For conservative algebras, this fact can be easily proved from Theorem 15: Consider a cyclic term t and observe that $t(a, a, \ldots, a, b)$ is necessarily equal to b for any $a, b \in P$, since the result must be either a or b (from conservativity) and it cannot be a as otherwise $\{a\}$ would absorb $\{a, b\}$. But then $m(x, y, z) = t(x, y, y, \ldots, y, z)$ is a Maltsev term.

The next ingredient is a certain "Rectangularity theorem" for conservative algebras (Theorem III.7 in [4]). Its simplified version is as follows.

▶ **Proposition 6.** *Let $\mathbf{P}, \mathbf{P}'$ be conservative Taylor algebra and R a subdirect subalgebra of $\mathbf{P} \times \mathbf{P}'$. Let, moreover, $\mathbf{Q}$ and $\mathbf{Q}'$ be minimal absorbing subalgebras of $\mathbf{P}$ and $\mathbf{P}'$ such that $R \cap (Q \times Q') \neq \emptyset$ and $R \cap (Q \times (P' \setminus Q')) \neq \emptyset$. Then $Q \times Q' \subseteq R$.*

Proof. The conservativity and the last assumption on R can be used to show that $R \cap (Q \times Q')$ is a subdirect, linked subuniverse of $\mathbf{Q} \times \mathbf{Q}'$ (we omit the proof here). Then the claim follows from the minimality of Q, Q' and the Absorption theorem. ◀

The Rectangularity theorem together with the reduction techniques shown in the proof of Proposition 5 are used to transform a CSP instance into an arc consistent instance with each $\mathbf{P}_x$ hereditarily absorption free and which has a solution whenever the original instance had. The idea is, imprecisely, that Proposition 5 allows to find a subinstance of a Prague instance where $\mathbf{P}_x$'s in the subinstance are minimal absorbing subuniverses of $\mathbf{P}_x$'s in the original instance. The Rectangularity theorem now guarantees the propagation of a solution to a suitably chosen subinstance. We continue in this way until every $\mathbf{P}_x$ is hereditarily absorption free. After the transformation is performed, all $\mathbf{P}_x$'s have Maltsev operations and we can apply the Bulatov–Dalmau algorithm for Maltsev constraints mentioned in Section 5.2.

Using Maróti's technique from [46], Bulatov has revisited and significantly simplified his original proof [26]. Notably, some parts of the revised proof (including e.g. the Rectangularity theorem) turned out to be very similar to the sketched proof by means of absorption – the interaction between absorption and Bulatov's local analysis deserves further attention.

7 Applications of absorption to local consistency checking

One of the main achievements of absorption is the characterization of the CSPs solvable by "local consistency methods". In general, a local consistency checking algorithm (LCC algorithm), operates on a family of local solutions to a CSP by removing the local solutions which are "inconsistent". The arc-consistency checking algorithms was an example of such an algorithm.

Systems of linear equations over the p-element field can be solved in polynomial time, but not by any LCC algorithm [31]. The restriction of the problem to equations involving at most 3 variables have the same properties and is equivalent to $\mathrm{CSP}(\mathbb{Z}_p)$, where the domain of $\mathbb{Z}_p$ is the p-element field $GF(p)$ and the relations are affine subspaces of $GF(p)^3$.

The solvability by local consistency checking is preserved by pp-interpretations and homomorphic equivalence [44], therefore a necessary condition for $\mathrm{CSP}(\mathbb{A})$ to be solvable by LCC is that $\mathbb{A}$ does not pp-interpret a structure homomorphically equivalent (that is with homomorphisms to and from) with $\mathbb{Z}_p$. We call structures satisfying this necessary condition $\mathbb{Z}_p$–avoiding.

The bounded width theorem states that, as conjectured in [44], this necessary condition is also sufficient. The theorem was proved using absorption [11] and independently by Bulatov [21] using his local analysis technique.

▶ **Theorem 17.** *If $\mathbb{A}$ is $\mathbb{Z}_p$-avoiding, then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time by local consistency checking.*

The plan for this section is first to prove the theorem for simplified instances using Prague instances from Section 4.2.3. Then we will move on to discuss other consistency notions for simplified instances, including consistency notions easier to compute. The section is concluded by an overview of results concerning consistency notions and algorithms that do not assume the simplicity of instances.

7.1 Prague instances and local consistency checking

Recall the definition of the simplified Prague instance from Section 4.2.3, and note that Proposition 5 reduces the Prague instance if at least one of the $\mathbf{P}_x$'s has a proper absorbing subuniverse. Unfortunately, this is not always the case for $\mathbb{Z}_p$-avoiding structures. Luckily, we can use the following lemma (Lemma 7.6 in [11], see also [13]) instead.

▶ **Lemma 18.** *Let $\mathbb{A}$ be $\mathbb{Z}_p$-avoiding. If $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$ is simple (has only trivial congruences) and has no proper absorbing subalgebra, then for every $b \in A$ there is an n-ary operation f of $\mathbf{A}$ and elements $a_1, \dots, a_n$ such that*

$$f(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n) = b \text{ for all } i \text{ and all } a \in A.$$

In the situation of the lemma, we say that f *points to* b. This operation is the missing tool necessary to tackle the following theorem:

▶ **Theorem 19.** *Let $\mathbb{A}$ be $\mathbb{Z}_p$-avoiding. Then every non-trivial Prague instance over $\mathbb{A}$ has a solution.*

Proof. The basic idea, and the structure of the proof is the same as in the proof of Theorem 13: we will show that every nontrivial Prague strategy over $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$ has a solution by shrinking the sets P_x until they are singletons. In case some $\mathbf{P}_x$ has a proper absorbing subalgebra, we use Proposition 5. It remains to deal with the case of no absorption.

In this case we choose x so that $|P_x| > 1$ and take a maximal proper congruence α_x of $\mathbf{P}_x$. For any $y \neq x$ consider the quotient of P_{xy} modulo α_x:

$$R_y = \{(a/\alpha_x, b) : (a, b) \in P_{xy}\} \leq \mathbf{P}_x/\alpha_x \times \mathbf{P}_y.$$

The partition of P_x/α_x into the components of linkedness of R_y defines a congruence of $\mathbf{P}_x/\alpha_x$, which, as α_x is maximal, is either the equality relation, or the full relation $P_x/\alpha_x \times P_x/\alpha_x$. We will say that the variable y is of type (1) when the relation is equality and of type (2) if it is full.

In both cases, we get a non-trivial information about P_{xy}. In the first case, R_y^{-1} is a graph of a surjection $P_y \to P_x/\alpha_x$. Its kernel, denoted α_y, is a congruence of $\mathbf{P}_y$ and P_{xy} modulo $\alpha_x \times \alpha_y$ is an isomorphism between P_x/α_x and P_y/α_y. In the second case, R_y is linked. Neither $\mathbf{P}_y$ nor $\mathbf{P}_x/\alpha_x$ have any proper absorbing subuniverses (because absorption can be lifted from quotients) and the Absorption theorem implies that R_y is the full product. Translating this to the original relation P_{xy} we get that each vertex in P_y is adjacent to an element in each α_x-block.

To shrink the instance, we choose one equivalence block of α_x and, for all y's of type (1), we choose a block of α_y mapped to the chosen block of α_x by R_y. For all the other y we take the whole P_y. Such a new, strictly smaller simplified instance is arc consistent because of the information we obtained in the previous paragraph.

So far the argument only required that $\mathbf{A}$ is Taylor, but now we need to use Lemma 18 to choose an operation pointing in $\mathbf{P}_x/\alpha_x$ to the chosen block of α_x. Using this operation, together with the structure of the R_y's, we are able to imitate the reasoning from the proof of Proposition 5 and prove that the new instance is Prague. This finishes the reduction. ◀

The theorem, modulo the reduction to simplified instances, proves Theorem 17. This reduction is rather direct but results in very inefficient consistency notions for the original CSP. In Section 7.3 we discuss consistency notions which can be applied without reducing to simplified instances.

7.2 $(2, 3)$-consistency, circle instances and semidefinite programming

To complete section 4.2.2, we will provide a sketch of a proof of Proposition 3. The proposition states that if, in a simplied $(2, 3)$-consistent instance, at least one of the $\mathbf{P}_x$'s has an absorbing subuniverse then the instance has a proper $(2, 3)$-consistent subinstance.

In order to prove this proposition, we require the following theorem which, despite surprising assumptions, is extremely useful [42].

▶ **Theorem 20.** *Let an instance of the CSP be arc consistent and such that for every variable x and $a \in P_x$, the map $x \mapsto a$ extends to a solution. If, for every x, we have $\mathbf{P}'_x \trianglelefteq \mathbf{P}_x$, and the restriction of the instance to P'_x's is arc consistent, then the restriction to P'_x's has a solution.*

We will not provide a sketch of the proof, only mention that it hinges on the following proposition from [14] which vaguely resembles the loop lemma:

▶ **Proposition 7.** *Let $\mathbf{R}' \trianglelefteq \mathbf{R}$ be subdirect subpowers of $\mathbf{A}^n$ and for every $a \in A$ the tuple $(a, \ldots, a)$ belongs to $\mathbf{R}$. Then $\mathbf{R}'$ contains a constant tuple.*

Proof. We present a sketch of the proof which splits into the usual stages. The walking stage finds a proper $\mathbf{B}$ such that $\mathbf{R}' \cap \mathbf{B}^n$ is subdirect in $\mathbf{B}^n$. The reduction stage is easy, indeed, having such $\mathbf{B}$ we can restrict both $\mathbf{R}'$ and $\mathbf{R}$ to $\mathbf{B}^n$. The assumptions of the theorem are satisfied for such restrictions and we can repeat the argument until $\mathbf{A}$ is a one-element algebra and the theorem holds.

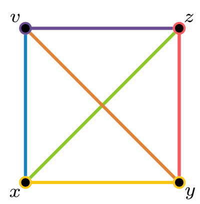
In order to complete the walking stage, we use pp-formulas which are trees and define a pre-order on subalgebras of $\mathbf{A}$: $\mathbf{B} \sqsubseteq \mathbf{B}'$ if $\mathbf{B}'$ can be pp-defined from $\mathbf{R}'$ and $\mathbf{B}$ by a tree pp-formula. We can show, *using absorption*, that this preorder contains elements which are not below the empty set, and, by the (omitted) definition of tree pp-formulas, we can choose an appropriate $\mathbf{B}$ there. ◀

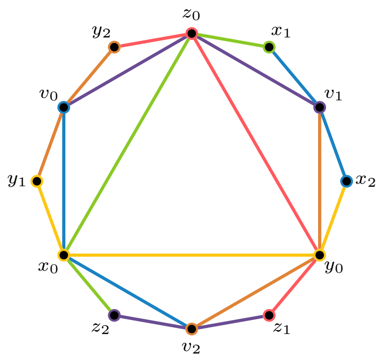
Before launching into the proof of Proposition 3, we need to establish one more fact. Given a simplified CSP instance, we can construct another simplified instance (usually infinite) which has a solution if and only if the original instance contains a $(2, 3)$-consistent subinstance. The idea is to, following Definition 3, construct the instance in steps:

1. start with a copy $P_{x_0 y_0}$ of any constraint P_{xy} from the original instance
2. for each new constraint $P_{x_i y_j}$, consider every variable w (different from x and y) and introduce into the constructed instance new constraints P_{x_i, w_k} and P_{y_j, w_k} (where k is such that w_k is a new variable) and repeat.

The following example illustrates first few steps of this procedure:

▶ **Example 21.** The following picture presents an instance on four variables $\{x, y, z, v\}$ with the constraints $\mathbf{P_{xy}}, \mathbf{P_{yz}}, \mathbf{P_{zv}}, \mathbf{P_{vx}}, \mathbf{P_{zx}}, \mathbf{P_{yv}}$ together with a part of the instance which is responsible for its $(2, 3)$-consistency.





The constructed instance is infinite, but by compactness argument (using the fact that $\mathbf{A}$ is finite), a large enough, finite part of the infinite instance plays the same role. This final remark allows us to finish the proof of Proposition 3.

Proof of Proposition 3. The proof starts with a $(2, 3)$-consistent instance over $\mathbf{P}_x$'s. The walking stage which, for every x, finds $\mathbf{P}'_x \trianglelefteq \mathbf{P}_x$ defining a proper arc consistent, absorbing subinstance of the original instance goes in exactly the same way as it was done in the proof of Proposition 5. The proof is actually simpler as we work with strictly stronger assumptions here.

The reduction stage follows from Theorem 20. Indeed, take a finite part of the simplified instance responsible for $(2, 3)$-consistency of the original instance. After setting P_{x_i} to P_x in this new instance, we get that for every $a \in P_{x_i}$ the map $x_i \mapsto a$ extends to a solution. The restriction of this instance to $P'_{x_i} = P'_x$ is an arc consistent, absorbing subinstance which, by Theorem 20, has a solution in P'_{x_i}'s.

Thus every finite part of the instance responsible for consistency of the original instance can be solved in the P'_x's and therefore, but compactness reasoning, we can find a $(2, 3)$-consistent subinstance of the original instance inside P'_x's, which proves the proposition. ◀

The analogue for circle instance, Proposition 4, can be proved in an almost identical way. Using Proposition 4 and a refinement of the non-absorbing part of the proof of Theorem 19 we can establish the following theorem.

▶ **Theorem 22.** *Let $\mathbb{A}$ be $\mathbb{Z}_p$-avoiding. Then every non-trivial circle instance over $\mathbb{A}$ has a solution.*

Now we are very close to defining a consistency notion which corresponds directly to semidefinite programming (SDP) relaxations of CSP instances. Each simplified CSP instance can be relaxed to a problem solvable by the semidefinite programming. This relaxation has very useful properties: among other things, it allows us to "almost solve almost solvable instances" of CSP of bounded width [10]. More precisely, if the instance is "almost solvable", i.e. solvable after forgetting small number of constraints, we can use the solution to the SDP relaxation to produce a instance which does not forget many constraints and is *pq*-consistent [41].

▶ **Definition 23.** An arc consistent simplified instance is *pq*-consistent if for every $a \in P_x$ and every p, q circle patterns at x, a is reachable from itself via $j(p + q) + p$ for some j.

The *pq*-consistency implies solvability for instances which are $\mathbb{Z}_p$-avoiding by a proof almost identical to the proof for circle instances:

▶ **Theorem 24.** *Let $\mathbb{A}$ be $\mathbb{Z}_p$-avoiding. Then every pq-consistent instance over $\mathbb{A}$ has a solution.*

7.3 Consistency notions for all instances

A characterization of the set of templates whose CSP is solvable by local consistency checking was conjectured by Feder and Vardi in [31]. Even after the conjecture was confirmed [11], it was not clear whether a single consistency notion suffices to deal with all these problems. Note that up till now, all the consistency notions worked for simplified instances, and it is not hard to generalize them to all instances over binary constraints. But incorporating constraints of higher arities destroys the uniformity of the reasoning.

The first result identifying a consistency notion that works for all the $\mathbb{Z}_p$-avoiding templates is [7, 21]. The result uses the concept of $(2, 3)$-minimality which will not be defined in this paper. The proof of this result in [7] is a small refinement of the proof of Theorem 17.

▶ **Theorem 25.** *Let $\mathbb{A}$ be $\mathbb{Z}_p$-avoding. Then every $(2, 3)$-minimal instance over $\mathbb{A}$ has a solution.*

Further results established other consistency notions which work for all the $\mathbb{Z}_p$-avoiding templates. Here we define Singleton Arc Consistency (SAC) [30], a well established notion of consistency. We present an algorithm for SAC using a pseudocode similar to that used for arc consistency

for every variable x **do** add constraint $P_x := A$ to the instance

repeat
  for every variable x and every $a \in P_x$ **do**
    run arc consistency with additional, temporary constraint $x = a$
    if the last AC derived a contradiction **do** substitute P_x with $P_x \setminus \{a\}$
  end for
until none of the P_x's changed

Similarly as in the case of arc consistency, we say that an instance is a SAC instance if it can be returned by the algorithm above. The following theorem states that SAC works, uniformly, for all the CSPs solvable by local consistency checking.

▶ **Theorem 26.** *Let $\mathbb{A}$ be $\mathbb{Z}_p$-avoiding. Then every SAC instance over $\mathbb{A}$ has a solution.*

This theorem follows from generalizations of Theorems 22 and 24 to arbitrary instances.

8 Abelianess versus absorption

One of the chief achievements of universal algebra is finding suitable generalizations of several concepts in group theory, like abelianess, solvability, and commutator [32, 35]. Here we only introduce the most basic concept of an abelian algebra.

▶ **Definition 27.** An algebra $\mathbf{A}$ is *abelian* if one of the equivalent conditions is satisfied:

- For every term function t elements a, b and tuples $\bar{c}, \bar{d}$:

  $$t(a, \bar{c}) = t(a, \bar{d}) \text{ implies that } t(b, \bar{c}) = t(b, \bar{d});$$

- the set $\{(a, a) : a \in \mathbf{A}\}$ is a block of some congruence on $\mathbf{A}^2$.

Examples of abelian algebras include the polymorphism algebras of $\mathbb{Z}_p$ from Section 7. This indicates that abelian algebras are natural obstacles for proving the dichotomy conjecture by means of refining the local consistency algorithms. In fact, $\mathbb{A}$ is not $\mathbb{Z}_p$-avoiding if and only if some subalgebra of $\mathrm{Pol}(\mathbb{A})$ has a nontrivial abelian quotient [51].

8.1 Abelianess prevents absorption

The notion of absorption is, in a sense, complementary to abelianess: the following theorem says that an abelian algebra has no non-trivial absorbing subuniverses. In fact, even a weaker property, solvability, prevents absorption [13].

▶ **Theorem 28.** *If $\mathbf{A}$ is abelian, then no subalgebra of $\mathbf{A}$ has a proper absorbing subalgebra.*

Proof. We only show a special case, that an abelian algebra cannot have a 1-element absorbing subuniverse.

Assume that $\{a\}$ absorbs an algebra $\mathbf{A}$ and α is a congruence of $\mathbf{A}^2$ from the definition of abelianess (the second item). Then $\{(a, a)\}$ absorbs $\mathbf{A}^2$ and thus $\{(a, a)/\alpha\}$ absorbs $\mathbf{A}^2/\alpha$. This in turns implies that $(a, a)/\alpha$ absorbs $\mathbf{A}^2$, but, from the abelianess of $\mathbf{A}$, $(a, a)/\alpha$ is not linked while $\mathbf{A}^2$ is. But the linkedness is absorbed by a version of the argument in Proposition 2, a contradiction. ◀

The algebras satisfying the conclusion are called *hereditarily absorption free*, or *HAF* for short (note that they have already appeared in Section 6.5). This theorem is interesting in combination with the following simple consequence of the Absorption theorem [13].

▶ **Theorem 29.** *If $\mathbf{A}$ is HAF and Taylor, then $\mathbf{A}$ has a Maltsev term operation.*

Proof. In context of Sections 5.2 and 6.1 the proof is natural: to prove Maltsev it suffices to show that the free algebra $\mathbf{F}$ has rectangular subpowers. On the other hand, if an algebra $\mathbf{B}$ is Taylor and HAF, then all its subalgebras have rectangular subpowers because of the Absorption theorem. It is, therefore, enough to show that $\mathbf{F}$ is HAF. But $\mathbf{F}$ is a subpower of $\mathbf{A}$ and every subpower of a HAF algebra is HAF. We leave it for the reader as an exercise to prove the latter fact. ◀

By combining the last two theorems, we get that each abelian (or just solvable) Taylor algebra has a Maltsev term operation. This fact was known before absorption [35], but its proof was quite long and used heavy machinery.

8.2 Absorption theorem for higher arity relations

Abelianess is also an obstacle for generalizing the Absorption theorem to higher arities. The following example shows that a naive generalization does not work in general – even if all the binary projections of a ternary relation are full and the relation is not full, no absorbing subalgebra needs to exist.

▶ **Example 30.** Consider an algebra $\mathbf{A}$ over $\{0,1\}$ with a single ternary plus, i.e. $x, y, z \mapsto x + y + z \bmod 2$. The relation $\{(a,b,c) : a+b+c = 0 \mod 2\}$ is a non-full subuniverse of $\mathbf{A}^3$ and has full binary projections. However it is easy to see, that the algebra has no non-trivial absorbing subuniverses.

Actually, every abelian algebra can participate in a problematic ternary relation. Indeed, take an abelian algebra $\mathbf{A}$ and let α be a congruence on $\mathbf{A}^2$ from the definition of abelianess. It is easy to see that $\{(a,b,(a,b)/\alpha) : a,b \in \mathbf{A}\}$ is a non-trivial, subdirect subuniverse of $\mathbf{A}^2 \times (\mathbf{A}^2/\alpha)$ and has all the binary projections linked.

We finish with a theorem witnessing that abelianess is the only obstacle. Its proof is left as a harder exercise for the reader.

▶ **Theorem 31.** *Let $\mathbf{A}_1, \dots \mathbf{A}_n$ be Taylor algebras (in the same signature) and let $\mathbf{R}$, subdirect in $\prod_{i=1}^n \mathbf{A}_i$, be such that all the binary projections of $\mathbf{R}$ are linked. Then*
1. *some $\mathbf{A}_i$ has a proper absorbing subuniverse, or*
2. *some $\mathbf{A}_i$ has a proper congruence α such that $\mathbf{A}_i/\alpha$ is abelian, or*
3. *$R = \prod_{i=1}^n A_i$.*

9 Conclusions

The simple concept of absorption proved surprisingly useful in universal algebra and CSP. The main contribution of absorption to CSP is a proof of the characterization of CSPs of bounded width, and the main contribution to algebra is the existence of cyclic terms. However, the concept is not yet well understood even for finite Taylor algebras.

The main obstacle to applying absorption outside of CSPs of bounded with is the incompatibility with abelian algebras. In particular, to prove the CSP dichotomy conjecture one needs to be able to operate on instances which in some parts have absorption, but in other parts are e.g. abelian. Currently, apart from very few basic results, we lack the knowledge to work with such instances.

Another challenge in the field is to bridge the gap between the absorption theory and the local approach used by Bulatov. The structure of an algebra imposed by Bulatov's colored graphs is similar to the one imposed by absorption (or lack of absorption), but the concepts are seemingly different.

An active direction of research is to extend the results obtained by absorption to infinite algebras. In this direction, we already know that the characterization by directed Jónsson terms extends [1], an analogue of a Sigger's term exists [48], etc. However, many questions remain. In particular, we do not know the correct extent and statement of the loop lemma for infinite algebras, although some facts are known [16].

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