The Complexity of Homomorphism Factorization

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The Homomorphism Factorization Problem

We assume throughout that all algebras are finite.
Fix an algebraic language $\mathcal{L}$.

Problem (The Homomorphism Factorization Problem)

Given a homomorphism $f\colon X \to Z$ between $\mathcal{L}$-algebras X and Z and an intermediate $\mathcal{L}$-algebra Y, decide whether there are homomorphisms $g\colon X \to Y$ and $h\colon Y \to Z$ such that $f = hg$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dotted, "\exists g?"'] & & Z \\ & Y \arrow[ur, dotted, "\exists h?"'] & \end{tikzcd}$$

Figure: The general form of the commutative diagram for Homomorphism Factorization Problems.

Variants of the Homomorphism Factorization Problem

Problem (I. The Homomorphism Problem)

When $|Z| = 1$, the homomorphisms f and h from the HFP must be constant, reduces to the problem of deciding whether, given $\mathcal{L}$-algebras X and Y, there is a homomorphism $g\colon X \to Y$.

$$\begin{tikzcd}
X \arrow[rr, "f(x)=\bullet"] \arrow[dr, dotted, "\exists g?"'] & & \bullet \\
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Figure: The commutative diagram for the Homomorphism Problem.

The Homomorphism Factorization Problem also generalizes the Retraction Problem and the Isomorphism Problem.

Variants of the Homomorphism Factorization Problem

Problem (II. The Exists Right-Factor Problem)

Given $\mathcal{L}$-algebras X, Y, and Z, and homomorphisms $f\colon X \to Z$ and $h\colon Y \to Z$, decide whether there is a homomorphism $g\colon X \to Y$ such that $f = hg$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dotted, "\exists g?"'] & & Z \\ & Y \arrow[ur, "h"'] & \end{tikzcd}$$

Figure: The commutative diagram for the Exists Right-Factor Problem.

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Unless stated otherwise, we assume all graphs are loop-free and connected.

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Rich Languages

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Theorem (B.)

The Homomorphism Factorization Problem for rich languages is NP-complete.

Two Unary Operations

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Definition ($G^\dagger$)

For every v in V_G, there are two corresponding elements v_1 and v_2 in $G^\dagger$, and for each edge (u, v) in E_G, there are two elements, $a_{(u,v)}$ and $b_{(u,v)}$, in $G^\dagger$. We assign to $G^\dagger$ two unary operations: $f(\cdot)$ and $g(\cdot)$.

Table: The operations of $G^\dagger$.

	$f(\cdot)$	$g(\cdot)$
u_1	u_1	u_2
u_2	u_1	u_2
$a_{(u,v)}$	u_1	$b_{(u,v)}$
$b_{(u,v)}$	v_2	$a_{(u,v)}$

HFPs for Algebras with Two Unary Operations

Theorem (B.)

Let G and H be directed graphs with at least two vertices. There exists a homomorphism $\phi\colon G \to H$ if and only if there exists a homomorphism $\psi\colon G^{\dagger} \to H^{\dagger}$.

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Corollary

The Homomorphism Problem for finite algebras in a language with two unary operations is NP-complete. Consequently, the Homomorphism Factorization Problem for finite algebras in a language with two unary operations is also NP-complete.

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We instead focus on the Exists Right-Factor Problem for semigroups.

Graph Encoding into Semigroups

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Definition (Semigroup X_G)

The universe of X_G consists of an element, v, for each v in V_G; an element, $\chi_{u,v}$, for each u, v in V_G such that (u, v) is not an element of E_G (let $\chi_{u,v} = \chi_{v,u}$); and auxiliary elements b, b^2, c, and 0. We assign to X_G a binary operation, $\cdot$, that encodes whether two vertices are adjacent or not adjacent.

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We also define a target semigroup Z by encoding the graph consisting of a single loop on a vertex a.

Homomorphism Factorization for Semigroups

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Show that the Exists Right-Factor Problem is NP-complete using either the two unary operations construction or the semigroup construction, depending on the number and arity of the operations. □

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Proof Idea.

Show that the Exists Right-Factor Problem is NP-complete using either the two unary operations construction or the semigroup construction, depending on the number and arity of the operations. □

In addition, using these and other encodings it was possible to classify the computational complexity of other variants of the Homomorphism Factorization Problem for rich languages – for example, the Homomorphism Problem for algebras with a single, non-associative binary operation is NP-complete, and the Retraction Problem for semigroups is also NP-complete.

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Our results suggested two approaches for determining when Homomorphism Factorization Problems are in P:

1. Classify the complexity of Homomorphism Factorization Problems for algebraic languages that are not rich.
2. Determine which specific varieties in rich languages have Homomorphism Factorization Problems in P.

Results in Case 1 have been found in, or can be derived from, prior research into monounary algebras. For this talk, we will focus on Case 2.

Case 2: A Strategy for Additional Polynomial Time Cases

Recall the general commutative diagram for Homomorphism Factorization Problems:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dotted, "\exists g?"'] & & Z \\ & Y \arrow[ur, dotted, "\exists h?"'] & \end{tikzcd}$$

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$$\begin{tikzcd}
X \arrow[loop left, "r"] \arrow[rr, "f"] \arrow[dr, "g"'] & & Z \\
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Let $X' = r(X)$. We have:

$$\begin{tikzcd}
X \arrow[r, bend right, "r"] \arrow[rrr, bend left=40, "f"] \arrow[drr, bend right, "g"'] & X' \arrow[l, bend right, "\supseteq"] \arrow[rr, "f|_{X'}"] \arrow[dr, "g'"'] & & Z \\
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Proposition

f factors through Y if and only if $f|_{X'}$ factors through Y.

Case 2: A Strategy for Additional Polynomial Time Cases

Definition (f-Core)

A is an **f-core of X** if A is minimal with respect to the existence of an onto, f-respecting retraction, $r\colon X \to A$. If X is its own f-core, X is an **f-core**.

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Definition (Bounded f-Core)

Variety $\mathcal{V}$ has *bounded f-cores* if, for any finite algebra Z, there exists a function s such that for any finite algebra X in $\mathcal{V}$ and any surjective homomorphism $f\colon X \to Z$ for which X is an f-core, $|X| \leq s(|Z|)$.

Bounded f-Cores and the Exists Right-Factor Problem

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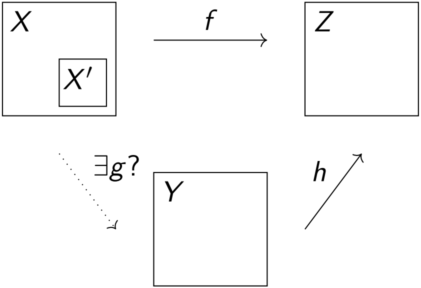

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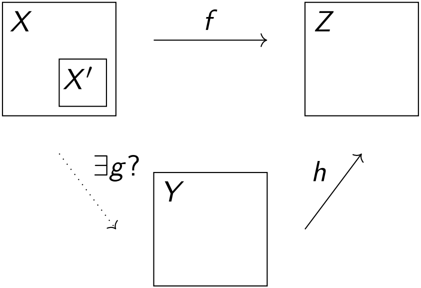

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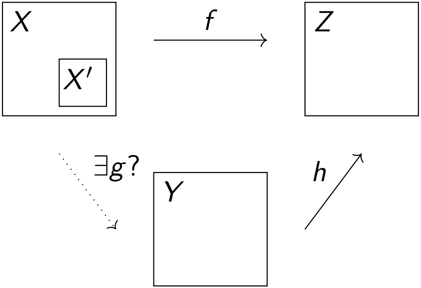

There are $|Y|^{|X'|} \leq |Y|^{s(|Z|)}$ many choices for g; the number of possible solutions to the Exists Right-Factor Problem with fixed Z is bounded above by a polynomial in $|Y|$.

Bounded f-Cores and P

Theorem (B.)

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1. $\mathcal{V}$ *has bounded* f*-cores.*
2. *The* f*-cores of finite algebras in* $\mathcal{V}$ *can be found in polynomial time.*
3. *Given a finite algebra* X *in* $\mathcal{V}$*, a retraction from* X *to its* f*-core can be found in polynomial time.*

Varieties with Bounded f-Cores

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Theorem (Vector Spaces)

Let F be a field. The variety of vector spaces over F has bounded f-cores, and the Exists Right-Factor Problem with fixed Z is in P.

Can We Do Better?

The arguments for both Boolean algebras and vector spaces can be extended to show that the Homomorphism Factorization Problem for these languages is also in P. Could the presence of bounded f-cores be a sufficient condition for this to hold in general?

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Thank you for your time.