Universal Algebra Today
Part II

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Original plan

Yesterday:

- **Today**: Universal Algebra yesterday
- **Tommorow**: Universal Algebra today
- **Thursday**: Universal Algebra tommorow

Today:

- **Today**: Universal Algebra today
- **Thursday**: Universal Algebra tommorow

Reality

- **Yesterday**: Universal algebra yesterday
- **Today**: Universal Algebra yeasterday and today
- **Thursday**: Universal Algebra today and tommorow

Summary of yesterday

- (function) clones are generalizations of (permutation) groups
- important object: $\{n\text{-ary operations in } \mathbf{A}\}$ – it is an invariant relation
- theorems
  - clones $\leftrightarrow$ coclones
  - homomorphisms $\leftrightarrow$ EHSP
- theorems talk about 2 levels of abstraction

Subdirect representation

properties of congruences $\leftrightarrow$ properties of relations

My old problem

For $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$, consider

- $\alpha \circ \beta = \beta \circ \alpha$
- $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

My questions

- Why should I care about such properties of congruences?
- Why is it important for a *variety* to have these properties, not so much for a single algebra?

Subdirect representation

$$\text{Clone} + n\text{-congruences meeting to } 0$$
$$\leftrightarrow$$
$$\text{subdirect product} + \text{projection kernels}$$

- ($\leftarrow$) Assume $\mathbf{R} \leq_{sd} \mathbf{A} \times \mathbf{B}$.
  - $\eta_1 \in \mathrm{Con}(\mathbf{R})$: $(a, b), (a', b') \in R$ are in η_1 iff $a = a'$
  - η_2 similarly
  - $\mathbf{R}/\eta_1 \cong \mathbf{A}$, $\mathbf{R}/\eta_2 \cong \mathbf{B}$, $\eta_1 \wedge \eta_2 = 0$
- ($\rightarrow$) Consider $\mathbf{C}$, $\alpha, \beta \in \mathrm{Con}(\mathbf{C})$, $\alpha \wedge \beta = 0$
  - then

$$\mathbf{C} \cong \mathbf{R} \leq_{sd} \mathbf{C}/\alpha \times \mathbf{C}/\beta$$
$$c \mapsto ([c]_\alpha, [c]_\beta)$$

  - projection kernels correspond to α, β

Remember: congruences are $\pm$ projection kernels

Example (without operations)

- $R \subseteq_{sd} A \times B$, $A = \{0,1\}, B = \{2,3\}$
  $R = \{c,d,e\} = \{(0,2),(0,3),(1,3)\}$
  - $\eta_1 = cd|e$, $\eta_2 = c|de$,
  - blocks cd, e of η_1 correspond to $0,1$
  - blocks c, de of η_2 correspond to $2,3$
- $C = \{c,d,e\}$, $\alpha = cd|e$, $\beta = c|de$
  - denote α-blocks cd,e by $0,1$; β-blocks c, de by $2,3$
  - $A := C/\alpha = \{0,1\}$, $B := C/\beta = \{2,3\}$
  - $c \mapsto (0,2), d \mapsto (0,3), e \mapsto (1,3)$ is the isomorphism with $R \leq_{sd} A \times B$

Projection kernels permute

Consider $R \subseteq_{sd} A \times B$, $\alpha = \eta_1, \beta = \eta_2$
Draw it as a bipartite graph.

What does $\alpha \circ \beta = \beta \circ \alpha$ mean?

- Consider $(a,b),(a',b') \in R$ in the equivalence on the left
- ie. there is $(a'',b'') \in R$ such that $(a,b) \sim_\beta (a'',b'') \sim_\alpha (a',b')$
- ie. $b'' = b$, $a'' = a'$
- ie. $(a,b') \in R$
- similarly on the right $\dots (a',b) \in R$
- **It means "no Z" ... rectangularity**

Definition

$R \subseteq A_1 \times A_2 \times \cdots \times A_n$ is rectangular if $\mathbf{ab}, \mathbf{a'b'}, \mathbf{ab'} \in R \Rightarrow \mathbf{a'b} \in R$
(should hold also for permuted coordinates of R)

Join of projection kernels

Consider $R \subseteq_{sd} A \times B$, $\alpha = \eta_1, \beta = \eta_2$
Draw it as a bipartite graph.

What is $\alpha \vee \beta$?

- it is $\alpha \cup (\alpha \circ \beta) \cup (\alpha \circ \beta \circ \alpha) \cup \dots$
- ie. (a, b) and (a', b') are equivalent if a, a' are connected in the bipartite graph (equivalently, b, b' are connected) ... they are linked

Definition

$R \subseteq_{sd} A \times B$

$a, a' \in A$ are linked if they are connected in the bipartite graph picture.

R is linked if each pair $a, a' \in A$ is linked

Note: R is linked iff $\alpha \vee \beta = 1$

Projection kernels distribute

Consider $R \subseteq_{sd} A \times B \times C$, $\alpha = \eta_1, \beta = \eta_2, \gamma = \eta_3$
Draw it as a colored bipartite graph: draw (a, b, c) as (b, c) colored by a.
What does $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ mean?

- Consider $(a, b, c), (a', b', c') \in R$ in the equivalence on the left
- ie. there $a = a'$ (the edges (b, c) and (b', c') have the same color)
- and b, b' are linked by edges of arbitrary colors
- Consider $(a, b, c), (a', b', c') \in R$ in the equivalence on the right
- ie. $a = a'$ and b, b' (equivalently c, c') are linked by edges colored by a
- $\supseteq$ is always true
- **It means "If b, b' are both incident to an a-colored edge and they are linked, then they are linked by a-colored edges**
  … "CD property"

Mal'tsev conditions

Mal'tsev's Mal'tsev condition

Theorem

$\mathbf{A}$ ... *clone, A finite. TFAE*

- *All relations in* $\operatorname{Inv}\mathbf{A}$ *are rectangular*
- $\mathbf{A}$ *contains a Mal'tsev operation* $f(y,x,x) = f(x,x,y) = y$

Proof of $\Rightarrow$.

- Consider subuniverse R of $\mathbf{Free_A}(2) \times \mathbf{Free_A}(2)$ generated by $(\pi_2,\pi_1),(\pi_1,\pi_1),(\pi_1,\pi_2)$
- It is a subset of $A^{A^2} \times A^{A^2}$ – an invariant relation of arity $2A^2$
- $R = \{(f(\pi_2,\pi_1,\pi_1), f(\pi_1,\pi_1,\pi_2)) : f \in \mathbf{A} \text{ ternary}\}$
  $= \{((x,y) \mapsto f(y,x,x), (x,y) \mapsto f(x,x,y)) : f \in \mathbf{A} \text{ ternary}\}$
- R is rectangular $\Rightarrow (\pi_2,\pi_2) = ((x,y) \mapsto y, (x,y) \mapsto y) \in R$

□

Mal'tsev's Mal'tsev condition, more traditionally

Theorem

$\mathbf{A}$ ... *clone, A finite. TFAE*

- *All relations in* $\mathrm{Inv}\,\mathbf{A}$ *are rectangular*
- *If* $\mathbf{A} \to \mathbf{B}$ *and* $\alpha, \beta \in \mathrm{Con}(\mathbf{B})$, *then* $\alpha \circ \beta = \beta \circ \alpha$
- $\mathbf{A}$ *contains a Mal'tsev operation* $f(y,x,x) = f(x,x,y) = y$

- The rest follows easily from subdirect representation
- Equivalence of 2nd and 3rd: no finiteness needed

Jónsson's Mal'tsev condition

Theorem (Jónsson)

A ... *clone, A finite. TFAE*

- *All relations in* Inv **A** *have "Property CD"*
- *If* **A** $\to$ **B** *and* $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{B})$, *then* $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
- **A** *contains Jónsson operations (too complicated to put it here)*

Proof.

The same □

Equational conditions

Definition (For this talk)

Equational condition ... condition (for a clone) of the form
"there exist $f_1, f_2, \ldots$ satisfying identities ...

- Neither the number of operations nor equations needs to be finite
- A bit different than strong Mal'tsev condition or Mal'tsev condition
- Does it have a name?
- If $\mathbf{A} \to \mathbf{B}$ and $\mathbf{A}$ satisfies an equational condition, then so does $\mathbf{B}$
- Actually, for fixed $\mathbf{A}$, "$\mathbf{A} \to \mathbf{B}$" is an equational condition

Three levels of abstraction

Definition ([Neumann])

Homomorphism order of clones ... define $\mathbf{A} \leq \mathbf{B}$ if $\mathbf{A} \to \mathbf{B}$, glue $\mathbf{A}, \mathbf{B}$ if $\mathbf{A} \leq \mathbf{B} \leq \mathbf{A}$

- It is the lattice of interpretability types of varieties
- It is the lattice of strength of equational conditions
- Clones are now organized

Three levels of abstraction

- Function clone
- (Abstract) clone ... remember only identities
- Equational condition ... remember only position in this lattice
  Trivial for permutation groups!

Summary: Universal algebra yesterday

Summary

- (function) clones are generalizations of (permutation) groups
- important object: $\{n$-ary operations in $\mathbf{A}\}$ – it is an invariant relation
- properties of congruences ($\pm =$ projection kernels)
  $\rightarrow$ connectivity properties of relations
- theorems
  - clones $\leftrightarrow$ coclones
  - homomorphisms $\leftrightarrow$ EHSP
  - properties of all invariant relations $\leftrightarrow$ equational conditions
- theorems talk about 3 levels of abstraction

Universal algebra today

Tools for general algebras

Classic:

- Commutator theory [Smith], ...
- Tame congruence theory [Hobby, McKenzie], ...

More recent:

- Absorption theory [Barto, Kozik], ...
- Bulatov's theory [Bulatov]
- Zhuk's theory [Zhuk]

Comments and outline

- Concepts from commutator theory important in all others
- Absorption, Bulatov, Zhuk concern mostly (but not exclusively) finite, idempotent, equationally nontrivial algebras
- and they are related (how closely?)
- Tame congruence theory, Bulatov – mostly on the algebraic side
  Absorption, Zhuk – mostly on the relational side

Outline

- Abelianness (the simplest concept from commutator theory)
- Idempotent equationally nontrivial clones

Abelian clones

The Term Condition

Definition

A clone $\mathbf{A}$ is Abelian if $\forall f \in \mathbf{A}$ n-ary $\forall x, y \in A, \mathbf{u}, \mathbf{v} \in A^{n-1}$

$$f(x, \mathbf{u}) = f(x, \mathbf{v}) \;\Rightarrow\; f(y, \mathbf{u}) = f(y, \mathbf{v})$$

Example:

- If $\mathbf{R}$ is a ring, $\mathbf{M}$ is an $\mathbf{R}$-module, then $\mathrm{Clo}(\mathbf{M})$ is Abelian
- $\mathrm{Clo}(\mathbf{M}) = \{(x_1, \dots, x_n) \mapsto r_1 x_1 + \dots + r_n x_n : \dots\}$
- We will also need:
  $\mathrm{Clo}(\mathbf{M} + \text{constants}) = \{(x_1, \dots, x_n) \mapsto r_1 x_1 + \dots + r_n x_n + m : \dots\}$

Facts:

- Clone of a group $\mathbf{G}$ is Abelian $\Leftrightarrow$ $\mathbf{G}$ is commutative
- Clone of a ring $\mathbf{R}$ is Abelian $\Leftrightarrow$ $\mathbf{R}$ has zero multiplication

Relational definition of Abelian clones

Proposition

Clone **A** *is Abelian iff* $\Delta = \{(a,a) : a \in A\}$ *is a congruence block of* $\mathbf{A}^2$.

Proof.

Hint: the congruence generated by some set can be described by unary polynomials, this translates to the Term Condition □

Example:

- Take **A** with $A = \{0,1,2\}$ and assume that the following relation is in $\mathrm{Inv}(\mathbf{A})$:
$$R = \{(a,b,c) \in A^3 : a - b + c = 2 \bmod 3\}$$
- Is **A** necessarily Abelian?

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$$R = \{(a,b,c) \in A^3 : a - b + c = 2 \bmod 3\}$$
- Is $\mathbf{A}$ necessarily Abelian?
- Hint: consider
$$\alpha := \{((a,b),(a',b')) \in A^2 \times A^2 : (\exists c)\ R(a,b,c) \text{ and } R(a',b',c)\}$$

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- Is **A** necessarily Abelian?
- Hint: consider

$$\alpha := \{((a, b), (a', b')) \in A^2 \times A^2 : (\exists c)\ R(a, b, c) \text{ and } R(a', b', c)\}$$

- It is a congruence, the block corresponding to $c = 2$ is Δ

Fundamental Theorem on Abelian Clones, 1st version

Theorem (Smith)

If $\mathbf{A}$ *is Abelian and contains a Mal'tsev operation* m*,*
then $\mathbf{A} \subseteq \mathrm{Clo}(\mathbf{M} + \textit{consts})$ *for some module* $\mathbf{M}$ $(M = A)$ *over a ring* $\mathbf{R}$

Proof.

Need to define group operations on A, ring $\mathbf{R}$ and the ring action. How?

Preparation:

- Assume the conclusion is true. What is m?
- $m(x_1, x_2, x_3) = r_1x_1 + r_2x_2 + r_3x_3 + a$ for some $r_1, r_2, r_3 \in R$, $a \in M$
- $m(y, x, x) = r_1y + r_2x + r_3x + a = y$,
  $m(x, x, y) = r_1x + r_2x + r_3y + a = y$ for each $x, y \in M$
- Plug in $x = y = 0 \Rightarrow a = 0$. Plug $x = 0 \Rightarrow r_1y = y$. Similarly $r_3y = y$. Finally $r_2x = -x$.
- **The unique Maltsev operation is** $m(x_1, x_2, x_3) = x_1 - x_2 + x_3$.

$\square$

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Proof.

- Select $0 \in A$ arbitrarily
- Define $x + y := m(x, 0, y)$, $-x := m(0, x, 0)$ (it must work!)
- Similar considerations lead to
  $R := \{$unary polynomials f with $f(0) = 0\}$, and ring operations
- Term Condition $\Rightarrow$ $\mathbf{R}$ is a ring, $\mathbf{M}$ an $\mathbf{R}$-module
- Similarly, $\mathbf{A} \subseteq \mathrm{Clo}(\mathbf{M} + \text{consts})$

$\square$

Taylor clones

Idempotent clones

Definition

Clone **A** is idempotent

if $f(x, x, \dots, x) = x$ for each f in **A**

$\Leftrightarrow$ unary part of **A** is trivial

$\Leftrightarrow$ all singleton unary relations are in $\mathrm{Inv}(\mathbf{A})$

Why this assumption?

- Complementary to group/semigroup theory
- Many useful equational conditions are idempotent
- Gives some information about general clones

Taylor clones

Definition

Proj ... the clone of projections on (say) 2-element set.

A is equationally nontrivial if $\mathbf{A} \not\to \mathbf{Proj}$

Equivalently:

- Not at the bottom of the homomorphism order
- Satisfies some nontrivial equational (Mal'tsev) condition
- $\mathbf{Proj} \notin \mathrm{HSP}(\mathbf{A})$

Definition

A is Taylor if it is equationally nontrivial and idempotent.

Getting rid of powers

Proposition (Bulatov)

If $\mathbf{A}$ *is finite and not Taylor (ie.* $\mathbf{Proj} \in \mathrm{HSP}(\mathbf{A})$*),*
then $\mathbf{Proj} \in \mathrm{HS}(\mathbf{A})$*.*

Proof.

- By the proof of Birkhoff, $\mathbf{Proj} \in \mathrm{HSP}^{\mathrm{fin}}(\mathbf{A})$
- For simplicity, assume $\mathbf{Proj} \in \mathrm{HS}(\mathbf{A}^2)$
- ie. $\mathbf{R}$ is binary in $\mathrm{Inv}(\mathbf{A})$, $\alpha \in \mathrm{Con}(\mathbf{R})$, $\mathbf{R}/\alpha \cong \mathbf{Proj}$
- Draw R as a bipartite graph, colored by blocks of α (2 colors)
- **Case 1:** $(\exists a, b, c \in A)\ (a, b) \not\sim_\alpha (a, c)$
- Then neighbors of a (ie. $X = \{x : (a, x) \in R\}$) form a subuniverse, since it is pp-definable (using the singleton $\{a\}$ and R)
- The "image of α" is a congruence β of $\mathbf{X}$ (again, pp-definable), $\mathbf{X}/\beta \cong \mathbf{Proj}$

Getting rid of powers

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- ie. $\mathbf{R}$ is binary in $\mathrm{Inv}(\mathbf{A})$, $\alpha \in \mathrm{Con}(\mathbf{R})$, $\mathbf{R}/\alpha \cong \mathbf{Proj}$
- Draw R as a bipartite graph, colored by blocks of α (2 colors)
- **Case 2:** $(\forall a, b, c \in A)\ (a, b) \sim_\alpha (a, c)$
- The projection of R to the 1st coordinate form a subuniverse X,
- The "projection of α" is a congruence β of $\mathbf{X}$, $\mathbf{X}/\beta \cong \mathbf{Proj}$

$\square$