Convergence to equilibrium for second order integrodifferential equations with polynomially decaying kernels

Tomáš Bárta*

October 22, 2018

Abstract

In this paper we study convergence to equilibrium and rate of convergence for a class of abstract second order evolution equations with convolutionary damping term. We focus on polynomially decaying convolution kernels and show how the rate of convergence depends on the decay of the kernel, decay of the right-hand side and the Łojasiewicz exponent of the leading non-linear operator. Similar results were recently shown for exponentially decaying kernels.

keywords: Evolutionary integrodifferential equations, convergence to equilibrium, Łojasiewicz–Simon inequality

msc: 45K05, 35L90, 47J35, 45N05, 45M05

1 Introduction

Integrodifferential equations of the type

$$u_{tt} - \Delta u + f(x,u) + \int_0^t k(s)\Delta u(t-s)\,\mathrm{d}s = g \quad \text{in } \mathbb{R}_+ \times \Omega \tag{1}$$

arise naturally in the theory of viscoelasticity, and therefore were studied by many authors (see e.g. [6], [7], [8]).

In [9], H. Yassine proved that solutions of (1) converge to an equilibrium and estimated the rate of convergence for a class of problems with exponentially decaying kernels k. In particular, he has shown polynomial rate of convergence for polynomially decaying g. In the present paper, we show similar results for polynomially decaying kernels k. In particular, if $k(t) \le C(1+t)^{-p}$ with $p > p_0$ (for an explicitly computed p_0 depending on decay of g and

*Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 18675 Praha 8, Czech Republic

the Łojasiewicz exponent of $-\Delta + f$), then we have the same polynomial decay estimates as for exponentially decaying kernels. For $p \in (2, p_0]$ the rate of convergence depends on p. For $p \leq 2$ convergence to equilibrium remains open.

Moreover, in the present paper we work in more abstract settings covering (1) as a special case. In particular, we study the equation

$$\ddot{u}(t) + E'(u(t)) - \int_0^t k(t-s)Au(s)ds = g(t) \tag{IDE}$$

in a Hilbert space H, where $-A$ is a dissipative self-adjoint operator and the operator $E_A(u) = E(u) + \frac{1}{2}\int_0^{+\infty} k(s)\,\mathrm{d}s\|A^{1/2}u\|^2$ satisfies the Łojasiewicz gradient inequality

$$|E_\mu(u) - E_\mu(\phi)|^{1-\theta} \leq C\|E'_\mu(u)\|_{H^{-1}(\Omega)}.$$

The present results also apply to the finite-dimensional case $H = \mathbb{R}^n$.

For other results based on the Łojasiewicz inequality giving rate of convergence in a finite-dimensional case see [10] (for completely positive kernels singular at zero). For the infinite-dimensional case, see [4] for abstract equations containing an additional damping term $B\dot{u}$ (which helps to stabilize the solution), and see [1] for abstract semilinear equations with polynomially decaying kernels.

The paper is organized as follows. Section 2 contains basic definitions and settings and formulation of the main result. Section 3 contains some preliminary results on convolutions. Energy estimates are derived in Section 4, while in Section 5 the proof of the main result is given in a series of Lemmas.

2 Definitions and the main result

Let $V \hookrightarrow H \hookrightarrow V^*$ be Hilbert spaces with the scalar products $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_*$ and norms $\|\cdot\|_1$, $\|\cdot\|$, $\|\cdot\|_*$ respectively. Let the embeddings be dense and let V^* be the dual of V with the duality satisfying $(u,v)_{V^*,V} = \langle u,v\rangle$ if $u \in H$, $v \in V$. Further, let $J : V' \to V$ be the duality mapping defined by $\langle h,v\rangle_* = (h, Jv)_{V^*,V}$ for all h, $v \in V^*$.

Throughout the paper c, C are general positive constants independent of t, their values vary from expression to expression.

Let us assume

(E) $E : V \to \mathbb{R}$ is of class C^2 such that $JE''(u) : V \to V$ extends to a bounded linear mapping from H to H with $\|JE''(u)\|$ being uniformly bounded for u from a compact subset of V.

(A) $-A : V \to V^*$ is a linear dissipative self-adjoint operator with $A^{1/2} : V \to H$ being bounded.

(k) $k : \mathbb{R}_+ \to (0, +\infty)$ is continuous and differentiable and there exist $c_k > 0$ and $p > 2$ such that

$$k'(t) \leq -c_k k^{1+\frac{1}{p}}(t) \quad \text{for all } t \geq 0. \tag{2}$$

(g) $g:\mathbb{R}_+\to\mathbb{R}_+$ is square integrable and there exists C_g, $\delta>0$ such that

$$\int_t^{+\infty}\|g(s)\|^2\,\mathrm{d}s\le C_g(1+t)^{-1-\delta}\quad\text{for all } t\ge 0.$$

Let us observe that (k) implies $k(t)\le C(1+t)^{-p}$ for some $C>0$ and all $t\ge 0$. In fact, since k is positive we can divide (2) by $k^{1+\frac{1}{p}}(t)$, integrate from 0 to t and we obtain

$$\int_0^t k^{-1-\frac{1}{p}}(s)k'(s)\,\mathrm{d}s=-p(k^{-\frac{1}{p}}(t)-k^{-\frac{1}{p}}(0))\le -c_k t,$$

which leads to

$$k^{\frac{1}{p}}\le\frac{1}{\frac{c_k}{p}t+k^{-\frac{1}{p}}(0)}\le\frac{1}{c(t+1)}$$

for and appropriate $c>0$ and we have $k(t)\le C(1+t)^{-p}$. So, $k\in L^1(\mathbb{R}_+)$ and we can denote $K_\infty=\int_0^{+\infty}k(s)\,\mathrm{d}s$.

Let us define $E_A:V\to\mathbb{R}$ by

$$E_A(u)=E(u)+\frac{1}{2}K_\infty\|A^{1/2}u\|^2.$$

Then $E_A\in C^2(V)$ and for $u\in V$ we have $E_A'(u)=E'(u)+K_\infty Au\in V^*$ and $E_A''(u)=E''(u)+K_\infty A$ being a bounded linear operator from V to V^*. We assume

(LI) E_A satisfies the Łojasiewicz gradient inequality, i.e. for every $\phi\in V$ there exists $\theta_\phi\in(0,\frac{1}{2}]$, ρ, $C>0$ such that

$$|E_A(u)-E_A(\phi)|^{1-\theta}\le C\|E_A'(u)\|_*\quad\text{for all } u \text{ with } \|u-\phi\|_1\le\rho.$$

Condition (E) appears in other works dealing with abstract second order equations (see e.g. [5] and [3]) and it is satisfied for

$$E(u)=\frac{1}{2}\int_\Omega\|\nabla u(x)\|^2\,\mathrm{d}x+\int_\Omega\int_0^u f(x,s)\,\mathrm{d}s\,\mathrm{d}x$$

which is the energy corresponding to (1). Condition (E) allows to define $\langle E''(u)v,v\rangle_*:=\langle JE''(u)v,v\rangle$ for $v\in H$ and estimate this expression (we use it in the proof of (15)). Let us mention that also the remaining assumptions generalize those of [9] and the operator E_A corresponds to E_μ from [9].

We say that $u\in C(\mathbb{R}_+,V)\cap C^1(\mathbb{R}_+,H)$ is *a weak solution* to (IDE) if (IDE) holds in V^* for every $t>0$. If $u\in C^1(\mathbb{R}_+,V)\cap C^2(\mathbb{R}_+,H)$ and (IDE) holds in H for every $t>0$, then we call u *a strong solution.*

Theorem 2.1. *Let (E), (A), (k), (g), (LI) hold and let u be a strong solution to* (IDE) *such that $\{(u(t), \dot{u}(t)), t \geq 0\}$ is bounded in $V \times H$ and $\{u(t), t \geq 0\}$ is relatively compact in V. Then there exists $\phi \in V$ such that*

$$\lim_{t \to +\infty} \|u_t(t)\| + \|u(t) - \phi\|_1 = 0.$$

Moreover, for every $\varepsilon > 0$ there exists $C > 0$ such that

$$\|u(t) - \phi\| \leq C(1+t)^{-\mu}, \qquad \forall\ t \geq 0, \tag{3}$$

where $\mu = \min\{\frac{\theta}{1-2\theta}, \frac{p-2-\varepsilon}{2}, \frac{\delta}{2}\}$, $\theta = \theta_\phi$.

Proposition 2.2. *Let the assumptions of Theorem 2.1 hold. If*

$$\|A^{1/2}(u(t) - \phi)\| \leq C(t+1)^{-\omega} \tag{4}$$

for some C, $\omega > 0$ and all $t > 0$, then (3) *holds with $\mu = \min\{\frac{\theta}{1-2\theta}, \frac{p-1}{2}, \frac{\delta}{2}\}$.*

Condition (4) *holds, e.g., if $\{\|Au(t)\|, t > 0\}$ is bounded.*

Remark 2.3. *If every weak solution can be approximated by strong solutions, then Theorem 2.1 and Proposition 2.2 hold also for weak solutions. In fact, we need a strong solution to derive* (9) *only. From Lemma 4.2 on, everything holds for weak solutions.*

3 Preliminaries

Let us denote

$$K(t) = \int_0^t k(s)\,\mathrm{d}s \qquad \text{and} \qquad \mathcal{K}(t) = \int_0^t k^{1-\frac{1}{p}}(s)\,\mathrm{d}s.$$

Further, for a function $l \in L^2_{loc}(\mathbb{R}_+)$ we define

$$(k \circ l)(t) = \int_0^t k(t-s)\|l(s) - l(t)\|^2\,\mathrm{d}s.$$

Lemma 3.1. *Let (k) hold. Then*

$$K_\infty - K(t) = \int_t^{+\infty} k(s)\,\mathrm{d}s \leq Ck(t)^{1-\frac{1}{p}}$$

for some $C > 0$ and all $t \geq 0$.

Proof. Since both sides of the inequality tend to zero as $t \to +\infty$, it is sufficient to compare derivatives, i.e. to show

$$-k(t) \geq Ck(t)^{-\frac{1}{p}}k'(t)$$

or, equivalently, $k'(t) \leq -\frac{1}{C}k(t)^{1+\frac{1}{p}}$. This is true by (k). □

Lemma 3.2. *If $k \in L^1_{loc}([0,+\infty))$ is non-negative and $f \in L^1_{loc}([0,+\infty), H)$, then*

$$\left\| \int_0^t k(t-s)(f(s)-f(t))\,\mathrm{d}s \right\|^2 \leq \mathcal{K}(t)(k^{1+\frac{1}{p}} \circ f)(t)$$

holds for all $t \geq 0$. If, moreover, (2) holds with $p > 2$, then there exists $C > 0$ such that

$$\left\| \int_0^t k(t-s)(f(s)-f(t))\,\mathrm{d}s \right\|^2 \leq C(k^{1+\frac{1}{p}} \circ f)(t) \qquad \forall\, t \geq 0.$$

If $k' \in L^1_{loc}([0,+\infty))$ is non-positive and $f \in L^1_{loc}([0,+\infty), H)$, then

$$\left\| \int_0^t k'(t-s)(f(s)-f(t))\,\mathrm{d}s \right\|^2 \leq (k(0)-k(t))(k' \circ f)(t)$$

Proof. The first part follows easily by Hölder inequality since

$$\left(\int_0^t k^{\frac{1}{2}(1-\frac{1}{p})}(t-s) \cdot k^{\frac{1}{2}(1+\frac{1}{p})}(t-s)\|f(s)-f(t)\|\,\mathrm{d}s \right)^2 \leq \int_0^t k^{1-\frac{1}{p}}(t-s)\,\mathrm{d}s\ (k^{1+\frac{1}{p}} \circ f)(t). \tag{5}$$

Moreover, (2) implies that $k(t) \leq c(1+t)^{-p}$, so $k^{1-\frac{1}{p}} \leq c(1+t)^{-p+1}$ is integrable. Therefore, $\mathcal{K}$ is bounded and the second part follows.

Similarly, the third part follows by writing $k' = \sqrt{-k'}\sqrt{-k'}$, applying Hölder inequality as in (5) and using $\int_0^t -k'(t-s)\,\mathrm{d}s = k(0) - k(t)$. □

The following lemma is taken from [2, Lemma 2.4].

Lemma 3.3. *Let $p > 1$ and $\sigma \geq 0$. Assume that $k(t) \leq C_1(1+t)^{-p}$ and $\|u(t)\|^2 \leq C_2(1+t)^{-\sigma}$ for some C_1, $C_2 > 0$ and all $t \geq 0$. If $0 \leq \sigma \leq 1$, then for every $1 > r > \frac{1-\sigma}{p}$ there exists $K > 0$ such that*

$$k \circ u \leq K \left(k^{1+\frac{1}{p}} \circ u \right)^{\frac{(1-r)p}{1+(1-r)p}} \qquad \textit{for all } t \geq 0.$$

If $\sigma > 1$, then there exists $K > 0$ such that

$$k \circ u \leq K \left(k^{1+\frac{1}{p}} \circ u \right)^{\frac{p}{1+p}} \qquad \textit{for all } t \geq 0.$$

4 Energy estimates

For a fixed strong solution u to (IDE) and $v = \dot{u}(t)$ let us define

$$\mathcal{E}_1(t) = \frac{1}{2}\|v(t)\|^2 + E(u(t)) + \int_t^{+\infty} \langle g(s), v(s) \rangle\ \mathrm{d}s.$$

Using (IDE) we have

$$\frac{d}{dt}\mathcal{E}_1(t) = \int_0^t k(t-s)\left\langle A^{1/2}u(s), A^{1/2}v(t)\right\rangle \,\mathrm{d}s. \tag{6}$$

Further, we define

$$\begin{aligned}\mathcal{E}(t) =& \mathcal{E}_1(t) - \frac{1}{2}K(t)\|A^{1/2}u(t)\|^2 + \frac{1}{2}k \circ A^{1/2}u(t)\\ =&\frac{1}{2}\|v(t)\|^2 + E_A(u(t)) + \frac{1}{2}(K_\infty - K(t))\|A^{1/2}u(t)\|^2\\ &+ \frac{1}{2}(k \circ A^{1/2}u)(t) + \int_t^{+\infty} \langle g(s), v(s)\rangle \,\mathrm{d}s.\end{aligned} \tag{7}$$

Lemma 4.1. *The inequalities*

$$\begin{aligned}\mathcal{E}(t) \leq& \frac{1}{2}\|v(t)\|^2 + E_A(u(t)) + \frac{1}{2}(K_\infty - K(t))\|A^{1/2}u(t)\|^2\\ &+ \frac{1}{2}(k \circ A^{1/2}u)(t) + \int_t^{+\infty} \langle g(s), v(s)\rangle \,\mathrm{d}s\end{aligned} \tag{8}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = \frac{1}{2}(k' \circ A^{1/2}u)(t) - \frac{1}{2}k(t)\|A^{1/2}u(t)\|^2 \leq 0 \tag{9}$$

hold for every $t > 0$. Moreover, $\lim_{t\to+\infty}\mathcal{E}(t)$ exists and we denote it $\mathcal{E}_\infty$.

Proof. Estimate (8) is obvious. The equality in (9) follows from (6),

$$\begin{aligned}\left(-\frac{1}{2}K(t)\|A^{1/2}u(t)\|^2\right)' &= -\frac{1}{2}k(t)\|A^{1/2}u(t)\|^2 - K(t)\left\langle A^{1/2}u(t), A^{1/2}v(t)\right\rangle,\\ \left(\frac{1}{2}(k \circ A^{1/2}u)(t)\right)' &= \frac{1}{2}(k' \circ A^{1/2}u)(t) - \int_0^t k(t-s)\left\langle A^{1/2}u(s) - A^{1/2}u(t), A^{1/2}v(t)\right\rangle \mathrm{d}s,\end{aligned}$$

and $\int_0^t k(t-s)\left\langle A^{1/2}u(t), A^{1/2}v(t)\right\rangle \mathrm{d}s = K(t)\left\langle A^{1/2}u(t), A^{1/2}v(t)\right\rangle$. Since $k' \leq 0$, we have $k' \circ A^{1/2}u \leq 0$ which yields the inequality in (9). Hence, $\mathcal{E}$ is non-increasing. Since $(u(t), v(t))$ is bounded in $V \times H$ and range of u is relatively compact in V, $\mathcal{E}(t)$ is bounded. So, the limit exists and the Lemma is proved. □

Let us define

$$I(t) = -\left\langle v(t), \int_0^t k(t-s)(u(t) - u(s))\,\mathrm{d}s\right\rangle + \frac{1}{2}\int_t^{+\infty} \|g(s)\|^2 \,\mathrm{d}s,$$

Lemma 4.2. *There exist constants ε_0, C_I, $c_I > 0$, $T > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the inequalities*

$$I(t) \leq \frac{1}{2}\|v(t)\|^2 + K(t)\ (k \circ u)(t) + \frac{1}{2}\int_t^{+\infty} \|g(s)\|^2 \,\mathrm{d}s. \tag{10}$$

and

$$\frac{d}{dt}I(t) \leq -\frac{1}{2}K_\infty\|v(t)\|^2 + \frac{\varepsilon}{8}\|E_A'(u(t))\|_* - \frac{c_I}{\varepsilon}(k' \circ A^{1/2}u)(t) \\ + C_I(K_\infty - K(t))^2\|A^{1/2}u(t)\|^2 + \frac{C_I}{\varepsilon}\mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t) \tag{11}$$

hold on $(T, +\infty)$.

Proof. The inequality (10) follows immediately from Lemma 3.2. Let us prove (11). We have

$$\frac{d}{dt}I(t) = -\int_0^t k(s)\,\mathrm{d}s\|v(t)\|^2 - \left\langle v(t), \int_0^t k'(t-s)(u(t)-u(s))\,\mathrm{d}s\right\rangle - \frac{1}{2}\|g(t)\|^2 \\ + \left\langle E'(u(t)) + \int_0^t k(t-s)Au(s)\,\mathrm{d}s - g(t), \int_0^t k(t-s)(u(t)-u(s))\,\mathrm{d}s\right\rangle \tag{12}$$

(here the second line corresponds to the derivative of v replaced by the other terms from (IDE) and the first line contains the remaining terms). By Lemma 3.2 and $\|u\|^2 \leq C_0\|A^{1/2}u\|^2$ we can estimate the second term in (12) by

$$\frac{\varepsilon}{2}\|v(t)\|^2 + \frac{C_0}{2\varepsilon}(k(t)-k(0))(k' \circ A^{1/2}u)(t).$$

Further, we have

$$\int_0^t k(t-s)Au(s)\,\mathrm{d}s = \int_0^t k(t-s)(Au(s)-Au(t))\,\mathrm{d}s + Au(t)\int_0^t k(s)\,\mathrm{d}s \\ = \int_0^t k(t-s)(Au(s)-Au(t))\,\mathrm{d}s + Au(t)K_\infty + Au(t)(K(t)-K_\infty), \tag{13}$$

and therefore

$$E'(u(t)) + \int_0^t k(t-s)Au(s)\,\mathrm{d}s - g(t) \\ = E_A'(u(t)) + \int_0^t k(t-s)(Au(s)-Au(t))\,\mathrm{d}s + (K(t)-K_\infty)Au(t) - g(t),$$

By Cauchy–Schwarz and Lemma 3.2 we have

$$\left\langle E_A'(u(t)), \int_0^t k(t-s)(u(t)-u(s))\,\mathrm{d}s\right\rangle \leq \frac{\varepsilon}{8}\|E_A'(u(t))\|_*^2 + \frac{2}{\varepsilon}\mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t),$$

$$\left\langle -Au(t)(K_\infty - K(t)), \int_0^t k(t-s)(u(t)-u(s))\,\mathrm{d}s\right\rangle \\ \leq \frac{1}{2}(K_\infty - K(t))^2\|A^{1/2}u(t)\|^2 + \frac{1}{2}\mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t),$$

$$\left\langle g(t), \int_0^t k(t-s)(u(t)-u(s))\,\mathrm{d}s \right\rangle \le \frac{1}{2}\|g(t)\|^2 + \frac{C_0}{2}\mathcal{K}(t)\ (k^{1+\frac{1}{p}} \circ A^{1/2}u)(t),$$

and

$$\begin{aligned}&\left\langle \int_0^t k(t-s)(Au(s)-Au(t))\,\mathrm{d}s, \int_0^t k(t-s)(u(t)-u(s))\,\mathrm{d}s \right\rangle \\ &= \left\| \int_0^t k(t-s)(A^{1/2}u(t) - A^{1/2}u(s)\,\mathrm{d}s \right\|^2 \le \mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t).\end{aligned}$$

Together we have

$$\begin{aligned}\frac{d}{dt}I(t) \le &- \left(K(t) - \frac{\varepsilon}{2}\right)\|v(t)\|^2 + \frac{\varepsilon}{8}\|E'_A(u(t))\|_*^2 + \frac{C_0}{2\varepsilon}(k(t)-k(0))(k' \circ A^{1/2}u)(t) \\ &+ \frac{1}{2}\left(K_\infty - K(t)\right)^2\|A^{1/2}u(t)\|^2 + \left(\frac{3+C_0}{2} + \frac{2}{\varepsilon}\right)\mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t),\end{aligned}$$

which for large t and small ε yields (11). □

Let us define

$$J(t) = \langle E'_A(u(t)), v(t)\rangle_* + C'_0 \int_t^{+\infty} \|g(s)\|^2\,\mathrm{d}s.$$

Lemma 4.3. *Inequalities*

$$J(t) \le \frac{1}{2}\|E'_A(u(t))\|_*^2 + \frac{1}{2}\|v(t)\|_*^2 + C'_0 \int_t^{+\infty} \|g(s)\|^2\,\mathrm{d}s. \tag{14}$$

and

$$\begin{aligned}\frac{d}{dt}J(t) \le &- \frac{1}{4}\|E'_A(u(t))\|_*^2 + C\|v(t)\|_*^2 + \frac{1}{2}(K_\infty - K(t))^2\|A^{1/2}u(t)\|^2 \\ &+ \mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t)\end{aligned} \tag{15}$$

hold.

Proof. The inequality (14) is obvious. To prove (15) let us compute

$$\begin{aligned}\frac{d}{dt}J(t) =& \langle E'_A(u(t)), -E'(u(t)) - \int_0^t k(t-s)Au(s)\,\mathrm{d}s + g(t)\rangle_* \\ &+ \langle E''(u(t))v(t) + K_\infty Av(t), v(t)\rangle_* - C'_0\|g(t)\|^2.\end{aligned}$$

Using (13) we have

$$\begin{aligned}-E'(u(t)) - \int_0^t k(t-s)Au(s)\,\mathrm{d}s =& - (E'_A(u(t))) + (K_\infty - K(t))Au(t) \\ &+ \int_0^t k(t-s)(Au(t) - Au(s))\,\mathrm{d}s.\end{aligned}$$

So,

$$
\begin{aligned}
\frac{d}{dt}J(t) =& -\|E_A'(u(t))\|_*^2 + \langle E''(u(t))v(t) + K_\infty A v(t), v(t)\rangle_* - C_0'\|g(t)\|^2 \\
&+ \langle E_A'(u(t)), (K_\infty - K(t))Au(t) + \int_0^t k(t-s)(Au(t) - Au(s))\,\mathrm{d}s + g(t)\rangle_* \\
\le& -\frac{1}{4}\|E_A'(u(t))\|_*^2 + C\|v(t)\|_*^2 - C_0'\|g(t)\|^2 \\
&+ \frac{1}{2}(K_\infty - K(t))^2\|A^{1/2}u(t)\|^2 + \mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t) + \frac{1}{2}\|g(t)\|_*^2 \\
\le& -\frac{1}{4}\|E_A'(u(t))\|_*^2 + C\|v(t)\|_*^2 + \frac{1}{2}(K_\infty - K(t))^2\|A^{1/2}u(t)\|^2 \\
&+ \mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t)
\end{aligned}
$$

and the lemma is proved. □

Lemma 4.4. *The function*

$$
H(t) = \mathcal{E}(t) + \varepsilon^2 I(t) + \varepsilon^3 J(t)
$$

satisfies

$$
\begin{aligned}
H(t) \le& C\|v(t)\|^2 + E_A(u(t)) + C\|E_A'(u(t))\|_*^2 + C(k \circ A^{1/2}u)(t) \\
&+ C(K_\infty - K(t))\|A^{1/2}u(t)\|^2 + \int_t^{+\infty} \langle g(s), v(s)\rangle\,\mathrm{d}s + C\int_t^{+\infty} \|g(s)\|^2\,\mathrm{d}s
\end{aligned} \tag{16}
$$

and

$$
\frac{d}{dt}H(t) \le -c\left(\|v(t)\|^2 + \|E_A'(u(t))\|_*^2 + k(t)\|A^{1/2}u(t)\|^2 + (k^{1+\frac{1}{p}} \circ A^{1/2}u)(t)\right). \tag{17}
$$

Proof. The first inequality follows immediately by the upper bounds for $\mathcal{E}$, I, J derived in Lemmas 4.1, 4.2, 4.3, and by boundedness of $\mathcal{K}$, $(k \circ u) \le (k \circ A^{1/2}u)$ and $\|v\|_* \le C\|v\|$.

To prove (**??**) let us estimate

$$\begin{aligned}\frac{d}{dt}H(t) \leq & \frac{1}{2}(k' \circ A^{1/2}u)(t) - \frac{1}{2}k(t)\|A^{1/2}u(t)\|^2 - \varepsilon^2 K_\infty \|v(t)\|^2 \\ & + \varepsilon^2 \frac{\varepsilon}{8}\|E_A'(u(t))\|_*^2 - \varepsilon^2 \frac{c_I}{\varepsilon}(k' \circ A^{1/2}u)(t) \\ & + \varepsilon^2 C_I (K_\infty - K(t))^2 \|A^{1/2}u(t)\|^2 + \varepsilon^2 \frac{C_I}{\varepsilon}\mathcal{K}(t)\ (k^{1+\frac{1}{p}} \circ A^{1/2}u)(t) \\ & - \varepsilon^3 \frac{1}{4}\|E_A'(u(t))\|_*^2 + \varepsilon^3 C \|v(t)\|_*^2 \\ & + \varepsilon^3 \frac{1}{2}\left(K_\infty - K(t)\right)^2 \|A^{1/2}u(t)\| + \varepsilon^3 \mathcal{K}(t)(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t) \\ \leq & \left(\frac{1}{2} - c_I\varepsilon\right)(k' \circ A^{1/2}u)(t) - \frac{1}{2}\left(k(t) - \varepsilon^2(2C_I - \varepsilon)(K_\infty - K(t))^2\right)\|A^{1/2}u(t)\|^2 \\ & - \varepsilon^3 (K_\infty - \varepsilon C)\|v(t)\|^2 - \varepsilon^3 \frac{1}{8}\|E_A'(u(t))\|_*^2 \\ & + \varepsilon\left(C_I + \varepsilon^2\right)\mathcal{K}(t)\ (k^{1+\frac{1}{p}} \circ A^{1/2}u)(t).\end{aligned}$$

Moreover, by (2) we have for $\varepsilon > 0$ small enough

$$(k' \circ A^{1/2}u)(t) + C\varepsilon(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t) \leq \frac{1}{2}(k' \circ A^{1/2}u)(t) \leq -c(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t),$$

and by Lemma 3.1 $(K_\infty - K(t))^2 \leq Ck(t)$. Then, for $\varepsilon > 0$ small we obtain (17). □

5 Proof of the main result

Using the energy estimates derived in the previous section we first prove

Proposition 5.1. *The following holds.*

(i) $v \in L^2(\mathbb{R}_+, H)$, $E_A'(u(\cdot)) \in L^2(\mathbb{R}_+, V^*)$ *and* $k^{1+\frac{1}{p}} \circ A^{1/2}u \in L^1(\mathbb{R}_+)$.

(ii) For all $\phi \in \omega(u)$ *we have* $E_A(\phi) = \mathcal{E}_\infty$ *and* $\lim_{t\to+\infty} E_A(u(t)) = \mathcal{E}_\infty$.

(iii) $\lim_{t\to+\infty}\|v(t)\| = \lim_{t\to+\infty}(k \circ A^{1/2}u)(t) = 0$.

(iv) $\lim_{t\to+\infty} I(t) = \lim_{t\to+\infty} J(t) = 0$, $\lim_{t\to+\infty} H(t) = \mathcal{E}_\infty$.

Proof. The function H defined in Lemma 4.4 is decreasing (by conclusions of the Lemma) and bounded below (due to boundedness of $(u(\cdot), v(\cdot))$). So, H has a limit H_∞. Then integrating the inequality (17) over $(t, +\infty)$ yields (i).

Let $\phi \in \omega(u)$ and $t_n \nearrow +\infty$ such that $u(t_n) \to \phi$ in V as $n \to \infty$. Then for any fixed $s > 0$ we have $u(t_n + s) = u(t_n) + \int_{t_n}^{t_n+s} v(r)\,\mathrm{d}r \to \phi + 0$ in H due to (i). By relative compactness of the trajectory we have $u(t_n+s) \to \phi$ in V and by continuity of $E_A : V \to \mathbb{R}$

and the Lebesque dominated convergence theorem we have $\int_0^1 E_A(u(t_n+s))\,\mathrm{d}s \to E_A(\phi)$. By integrating (7) we obtain

$$\mathcal{E}_\infty = \lim_{n\to\infty}\int_0^1 \mathcal{E}(t_n+s)\,\mathrm{d}s = \lim_{n\to\infty}\int_0^1 E_A(u(t_n+s))\,\mathrm{d}s = E_A(\phi),$$

where the second equality follows since all other terms on the right-hand side of (7) (after integration) tend to zero by (i) and $\lim_{t\to+\infty} K(t) = K_\infty$. We have proved that $E_A(\phi) = \mathcal{E}_\infty$. Precompact range of u then implies by standard arguements $E_A(u(t)) \to \mathcal{E}_\infty$ and (ii) is proved.

Then taking limit in (7) we have $\mathcal{E}_\infty = \lim_{t\to+\infty}\big(\frac{1}{2}\|v(t)\| + (k\circ A^{1/2}u)(t)\big) + \mathcal{E}_\infty$ which yields (iii). Statement (iv) follows immediately from definitions of I and J and (iii). □

Lemma 5.2. *Let $V(t) = H(t) - \mathcal{E}_\infty$. Then*

$$\frac{d}{dt}V(t) \le -c\left(\|v(t)\| + \|E_A'(u(t))\|_* + k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\| + (k^{1+\frac{1}{p}}\circ A^{1/2}u)(t)^{\frac{1}{2}}\right)^2. \tag{18}$$

Let $\sigma\in[0,1)$ be such that $\|A^{1/2}(u(t)-\phi)\|^2 \le C(1+t)^{-\sigma}$ and let $r\in(\frac{1-\sigma}{p},1)\cap[0,1)$ and

$$\gamma_0 = \min\left\{\frac{1}{1-\theta}, \frac{2(1-r)p}{1+(1-r)p}, 2-\frac{2}{p}\right\}. \tag{19}$$

Then for every $\gamma\in(1,\gamma_0]$ there exist C_γ, $T_\gamma>0$ such that

$$\begin{aligned} V(t) \le{}& C_\gamma\Big(\|v(t)\| + \|E_A'(u(t))\|_* + k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\| + (k^{1+\frac{1}{p}}\circ A^{1/2}u)(t)^{\frac{1}{2}} \\ &+ (1+t)^{-\frac{1}{\gamma}(1+\delta)}\Big)^\gamma \end{aligned} \tag{20}$$

holds for all $t\ge T_\gamma$.

Proof. Inequality (18) follows immediately from (17). To prove (20) we first get rid of the term $\int_t^{+\infty}\langle g,v\rangle$ in (16). By (17) we have

$$\int_t^{+\infty}\|v(s)\|^2 ds \le \frac{1}{c}(H(t)-\mathcal{E}_\infty) = \frac{1}{c}V(t),$$

and therefore

$$\int_t^{+\infty}\langle g(s),v(s)\rangle\,\mathrm{d}s \le \frac{c}{2}\int_t^{+\infty}\|v(s)\|^2 ds + C\int_t^{+\infty}\|g(s)\|^2 ds \le \frac{1}{2}V(t) + C\int_t^{+\infty}\|g(s)\|^2 ds.$$

Then, by (16) we have

$$\begin{aligned} V(t) \le{}& C\Big(\|v(t)\|^2 + E_A(u(t)) - E_\infty + \|E'(u(t))\|_*^2 + (k\circ A^{1/2}u)(t) \\ &+ (K_\infty - K(t))\|A^{1/2}u(t)\|^2 + \frac{1}{2}V(t) + C\int_t^{+\infty}\|g(s)\|^2 ds + \int_t^{+\infty}\|g(s)\|^2\,\mathrm{d}s\Big) \end{aligned}$$

and by subtracting $\frac{1}{2}V(t)$ we get

$$V(t) \leq C\Big(\|v(t)\|^2 + E_A(u(t)) - E_\infty + \|E'(u(t))\|_*^2 + (k \circ A^{1/2}u)(t) + (K_\infty - K(t))\|A^{1/2}u(t)\|^2 + \int_t^{+\infty} \|g(s)\|^2 \,\mathrm{d}s\Big).$$

Now, applying (g), (E), $\frac{1}{1-\theta} \leq 2$ and Lemma 3.1 we obtain

$$V(t) \leq C\Big(\|v(t)\|^2 + \|E_A'(u(t))\|_*^{\frac{1}{1-\theta}} + (k \circ A^{1/2}u)(t) + k(t)^{1-\frac{1}{p}}\|A^{1/2}u(t)\|^2 + (1+t)^{-1-\delta}\Big).$$

Further, we have

$$k(t)^{1-\frac{1}{p}}\|A^{1/2}u(t)\|^2 \leq Ck(t)^{1-\frac{1}{p}}\|A^{1/2}u(t)\|^{2-\frac{2}{p}} \leq C\left(k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\|\right)^{2-\frac{2}{p}}$$

and by Lemma 3.3

$$(k \circ A^{1/2}u)(t) \leq C(k^{1+\frac{1}{p}} \circ A^{1/2}u)(t)^{\frac{(1-r)p}{1+(1-r)p}}$$

for $r \in (\frac{1-\sigma}{p}, 1)$. So,

$$V(t) \leq C\left(\|v(t)\|^2 + \|E_A'(u(t))\|_*^{\frac{1}{1-\theta}} + \left((k \circ A^{1/2}u)(t)^{\frac{1}{2}}\right)^{\frac{2(1-r)p}{1+(1-r)p}} + \left(k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\|\right)^{2-\frac{2}{p}} + (1+t)^{-1-\delta}\right) \tag{21}$$

If γ satisfies

$$1 < \gamma \leq \min\left\{2, \frac{1}{1-\theta}, \frac{2(1-r)p}{1+(1-r)p}, 2-\frac{2}{p}\right\} = \min\left\{\frac{1}{1-\theta}, \frac{2(1-r)p}{1+(1-r)p}, 2-\frac{2}{p}\right\},$$

then (since the first four terms in the big brackets in (21) are bounded)

$$V(t) \leq C_\gamma\left(\|v(t)\|^\gamma + \|E_A'(u(t))\|_*^\gamma + \left((k \circ A^{1/2}u)(t)^{\frac{1}{2}}\right)^\gamma + \left(k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\|\right)^\gamma + (1+t)^{-1-\delta}\right)$$

Hence,

$$V(t) \leq C\Big(\|v(t)\| + \|E_A'(u(t))\|_* + (k \circ A^{1/2}u)(t)^{\frac{1}{2}} + k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\| + (1+t)^{-\frac{1}{\gamma}(1+\delta)}\Big)^\gamma$$

for any $\gamma \in (1, \gamma_0]$. □

Lemma 5.3. *There exists $\phi \in S$ such that* $\lim_{t\to+\infty} \|u(t)-\phi\|_1 = 0$.

Proof. Since $u(\cdot)$ is bounded in V, $\|A^{1/2}u\|^2 \le C(1+t)^{-\sigma}$ holds with $\sigma = 0$. Then we have (18) and (20) with any $r \in (\frac{1}{p}, 1)$. Take $r > 1 - \frac{1}{p} \in (\frac{1}{p}, 1)$. Then $\frac{2(1-r)p}{1+(1-r)p} > 1$, and therefore $\gamma_0 > 1$ (the other terms in (19) are obviously greater than 1). Let us take $\gamma \in (1, \gamma_0] \cap (1, 1+\delta)$ and denote

$$W(t) = \|v(t)\| + \|E'_A(u(t))\|_* + (k \circ A^{1/2}u)(t)^{\frac{1}{2}} + k(t)^{\frac{1}{2}}\|A^{1/2}u(t)\|.$$

Then

$$\begin{aligned}-\frac{d}{dt}V(t)^{1-\frac{1}{\gamma}} = &\left(1-\frac{1}{\gamma}\right)\frac{-\frac{d}{dt}V(t)}{V(t)^{\frac{1}{\gamma}}} \ge C\frac{W(t)^2}{W(t)+(1+t)^{-\frac{1}{\gamma}(1+\delta)}} \ge C\frac{W(t)^2 - \left((1+t)^{-\frac{1}{\gamma}(1+\delta)}\right)^2}{W(t)+(1+t)^{-\frac{1}{\gamma}(1+\delta)}}\\ =&C\left(W(t)-(1+t)^{-\frac{1}{\gamma}(1+\delta)}\right).\end{aligned}$$

Since the function

$$t \mapsto -\frac{d}{dt}V(t)^{1-\frac{1}{\gamma}} + C(1+t)^{-\frac{1}{\gamma}(1+\delta)}$$

is integrable on $\mathbb{R}_+$ (we have $\gamma < 1+\delta$), also $W \in L^1(\mathbb{R}_+)$, therefore (by definition of W) $v = \dot{u} \in L^1(\mathbb{R}_+, H)$, therefore u has a limit in H and due to precompactness in V, u has a limit in the norm of V as well. ☐

Before we prove the decay estimate (3) let us formulate two lemmas.

Lemma 5.4. *Let V be an arbitrary function satisfying* (18) *and* (20) *with a constant $\gamma < 2$. Then there exists $C > 0$ such that for all $t > 0$*

$$V(t) \le C(1+t)^{-\nu}, \tag{22}$$

where $\nu = \min\{\frac{\gamma}{2-\gamma}, 1+\delta\}$.

Proof. By (18) and (20), we have for an appropriate constant C

$$C\frac{d}{dt}V(t) + V(t)^{\frac{2}{\gamma}} \le C(1+t)^{-\frac{2}{\gamma}(1+\delta)}.$$

Then we apply [9, Lemma 8] with $k = \frac{2}{\gamma} > 1$, $\lambda = \frac{2}{\gamma}(1+\delta)$ and obtain $V(t) \le C(1+t)^{-\nu}$, where $\nu = \min\{\frac{1}{k-1}, \frac{\lambda}{k}\} = \min\{\frac{\gamma}{2-\gamma}, 1+\delta\}$. ☐

Lemma 5.5. *Let* $\lim_{t\to+\infty} u(t) = \phi$ *and let V satisfies* (18) *and* (22) *with a constant $\nu > 1$. Then there exists $C > 0$ such that*

$$\|u(t)-\phi\| \le C(1+t)^{-\mu} \tag{23}$$

holds for all $t > 0$ with $\mu = \frac{\nu-1}{2}$.

Proof. We proceed as in [9]. We have

$$\|u(t)-\phi\| \le \int_t^{+\infty} \|v(s)\|\,\mathrm{d}s = \sum_{k=0}^{\infty} \int_{2^k t}^{2^{k+1}t} \|v(s)\|\,\mathrm{d}s,$$

by Hölder inequality

$$\|u(t)-\phi\| \le \sum_{k=0}^{\infty} \left(2^k t\right)^{1/2} \left(\int_{2^k t}^{2^{k+1}t} \|v(s)\|^2\,\mathrm{d}s\right)^{1/2},$$

and due to $\int_a^b \|v(s)\|^2\,\mathrm{d}s \le C(V(a)-V(b))$ we have for $t>1$

$$\begin{aligned}
\|u(t)-\phi\| &\le C\sum_{k=0}^{\infty} \left(2^k t\right)^{1/2} \left[\left(1+2^k t\right)^{-\nu/2} - \left(1+2^{k-1}t\right)^{-\nu/2}\right] \\
&\le C\sum_{k=0}^{\infty} \left(2^k t\right)^{1/2} \left[\left(2^k(1+t)\right)^{-\nu/2} - \left(\frac{1}{4}\cdot 2^k(t+1)\right)^{-\nu/2}\right] \\
&= C\left(\frac{3}{4}\right)^{-\nu/2} \sum_{k=0}^{\infty} \left(2^{\frac{1-\nu}{2}}\right)^k (1+t)^{\frac{1-\nu}{2}}.
\end{aligned}$$

Since $\nu>1$, the sum converges and we have $\|u(t)-\phi\| \le C(1+t)^{-\mu}$ with $\mu = \frac{\nu-1}{2}$ for $t>1$ and also for $t\in[0,1]$ if we change C appropriately. □

Proof of Theorem 2.1. By Lemma 5.3 we have $u(t)\to\phi$ in V for some ϕ. Further, we know that $\|A^{1/2}u\|^2 \le C(1+t)^{-\sigma}$ holds with $\sigma=0$, so we can take any $r\in(\frac{1}{p},1)$ in Lemma 5.2 and get (18) with $\gamma = \min\left\{\frac{1}{1-\theta}, \frac{2(1-r)p}{1+(1-r)p}, 2-\frac{2}{p}\right\}$. For such r we have $\frac{2(1-r)p}{1+(1-r)p} < 2-\frac{2}{p}$ and

$$\lim_{r\to\frac{1}{p}} \frac{2(1-r)p}{1+(1-r)p} = 2-\frac{2}{p}.$$

We distinguish two cases:

Case 1: $\frac{1}{1-\theta} < 2-\frac{2}{p}$. In this case, we can take $r\in(\frac{1}{p},1)$ so small that $\gamma = \frac{1}{1-\theta} < 2$. Then Lemma 5.4 gives (22) with $\nu = \min\left\{\frac{\gamma}{2-\gamma}, 1+\delta\right\} = \min\left\{\frac{1}{1-2\theta}, 1+\delta\right\} > 1$ and Lemma 5.5 yields (3) with $\mu = \frac{\nu-1}{2} = \min\left\{\frac{\theta}{1-2\theta}, \frac{\delta}{2}\right\}$.

Case 2: $\frac{1}{1-\theta} \ge 2-\frac{2}{p}$. Then for any $r\in(\frac{1}{p},1)$ we have $\gamma = \frac{2(1-r)p}{1+(1-r)p} < 2$ and Lemma 5.4 implies (22) with $\nu = \min\{p-rp, 1+\delta\}$. If $1+\delta < p-1$, we can take r close to $\frac{1}{p}$ such that $p-rp > 1+\delta$. Then $\nu = 1+\delta > 1$ and Lemma 5.5 yields (3) with $\mu = \frac{\nu-1}{2} = \frac{\delta}{2}$. If, on the other hand, $1+\delta \ge p-1$, we take $r = \frac{1+\varepsilon}{p}$ with $\varepsilon>0$ small enough. Then we have (22) with $\nu = p-1-\varepsilon > 1$ (if ε is small enough) and Lemma 5.5 yields (3) with $\mu = \frac{\nu-1}{2} = \frac{p-2-\varepsilon}{2}$.

Thus, in any case the decay estimate holds with $\mu = \min\left\{\frac{\theta}{1-2\theta}, \frac{p-2-\varepsilon}{2}, \frac{\delta}{2}\right\}$ and the proof is finished. □

Proof of Proposition 2.2. Let us first observe that boundedness of $\|Au(t)\|$ implies (4). In fact, from $\|u(t)-\phi\|_1 \to 0$ we have $\|Au(t)-A\phi\|_* \to 0$, and consequently $A\phi \in H$ (a ball in H is weakly sequentially compact, so $Au(t_n) \to x$ weakly in H for some $t_n \nearrow +\infty$, therefore weakly in V^*, so $x = A\phi$ and $x \in H$). Then

$$\|A^{1/2}(u(t)-\phi)\|^2 \leq \|Au(t)-A\phi\|\|u(t)-\phi\| \leq C\|u(t)-\phi\| \leq C(1+t)^{-\mu},$$

where the last inequality follows from Theorem 2.1. So, (4) holds with $\omega = \frac{\mu}{2} > 0$.

Now, let us assume that (4) holds. Then we can take $r = \frac{1}{p}$ in Lemma 5.2 and obtain (20) with $\gamma = \min\left\{\frac{1}{1-\theta}, 2-\frac{2}{p}\right\} < 2$. Then Lemma 5.4 yields $V(t) \leq C(1+t)^{-\nu}$ with $\nu = \min\left\{\frac{1}{1-2\theta}, p-1, 1+\delta\right\}$ and Lemma 5.5 gives $\mu = \frac{\nu-1}{2} = \min\{\frac{\theta}{1-2\theta}, \frac{p-2}{2}, \frac{\delta}{2}\}$. □

References

[1] F. Alabau-Boussouira, P. Cannarsa, D. Sforza, *Decay estimates for second order evolution equations with memory*, J. Funct. Anal. **254** (2008), no. 5, 1342–1372.

[2] T. Bárta, *Polynomial decay for solutions of hyperbolic integrodifferential equations*, Glasgow Math. J. **50** (2008), 575–581.

[3] T. Bárta and E. Fašangová, *Convergence to equilibrium for solutions of an abstract wave equation with general damping function*, J. Differential Equations **260** (2016), no. 3, 2259–2274.

[4] R. Chill, E. Fašangová, *Convergence to steady states of solutions of semilinear evolutionary integral equations*, Calc. Var. Partial Differential Equations **22** (2005), no. 3, 321–342.

[5] R. Chill, A. Haraux, and M. A. Jendoubi, *Applications of the Lojasiewicz-Simon gradient inequality to gradient-like evolution equations*, Anal. Appl. **7** (2009), 351–372.

[6] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 297–308.

[7] M. Fabrizio, C. Giorgi, V. Pata, *A new approach to equations with memory*, Arch. Ration. Mech. Anal. **198** (2010), no. 1, 189–232.

[8] M. Renardy, W. J. Hrusa, J. A. Nohel, Mathematical problems in viscoelasticity. Pitman Monographs and Surveys in Pure and Applied Mathematics, 35. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987.

[9] H. Yassine, *Convergence to equilibrium of solutions to a nonautonomous semilinear viscoelastic equation with finite or infinite memory*, J. Differential Equations **263** (2017), no. 11, 7322–7351.

[10] R. Zacher, *Convergence to equilibrium for second-order differential equations with weak damping of memory type*, Adv. Differential Equations **14** (2009), no. 7-8, 749–770.