Řešení

1) $y_p(t) = c_1(t) + c_2(t)e^{-t}$, $c_1(t) = \int \frac{e^t}{1+e^{2t}} dt = \arctan(e^t)$, $c_2(t) = \int -\frac{e^{2t}}{1+e^{2t}} dt = -1/2\ \ln(1+e^{2t})$.

2) $y_p(t) = c_1(t)\cos(t) + c_2(t)\sin(t)$, $c_1(t) = -\int \frac{dt}{\sin t} = -\ln|\tan(t/2)|$, $c_2(t) = \int \frac{\cos t}{\sin^2 t}\, dt = -\frac{1}{\sin t}$.

3) $y_p(t) = c_1(t) + c_2(t)e^{2t}$, $c_1(t) = \int 5/2\,\sqrt{t}\,(-3+4t)\, dt = -5\,t^{3/2} + 4\,t^{5/2}$, $c_2(t) = \int -5/2\,\sqrt{t}\,(-3+4t)\, e^{-2t} dt = 5\,\frac{t^{3/2}}{(e^t)^2}$, $y_p(t) = 4\,t^{5/2}$.

4) $y_p(t) = c_1(t) + c_2(t)e^{-3t}$, $c_1(t) = \int 1/3\,\frac{3t-1}{t^2} dt = 1/3\,t^{-1} + \ln(t)$, $c_2(t) = \int -1/3\,\frac{(3t-1)e^{3t}}{t^2} dt = -1/3\,\frac{e^{3t}}{t}$, $y_p(t) = \ln|t|$.

5) $y_p(t) = c_1(t) + c_2(t)e^t$, $c_1(t) = \int \frac{t+1}{t^2} dx = -t^{-1} + \ln(t)$, $c_2(t) = \int -\frac{(t+1)e^{-t}}{t^2} dt = \frac{e^{-t}}{t}$, $y_p(t) = \ln|t|$.

6) $y_p(t) = c_1(t)e^t + c_2(t)e^t t$, $c_1(t) = \int -1 dt = -t$, $c_2(t) = \int t^{-1} dt = \ln|t|$, $y_p(t) = t\ln|t|e^t$.

7) $y_p(t) = c_1(t)\cos(2t) + c_2(t)\sin(2t)$, $c_1(t) = \int -2 + 2\cos^2 t dx = -t + \sin(t)\cos(t)$, $c_2(t) = \int \sin(t)(-1+2\cos^2(t))/\cos(t) dt = -\cos^2(t) + \ln|\cos(t)|$,

8) $y_p(t) = c_1(t) + c_2(t)e^{-t}$ $c_1(t) = \int (1+e^t)^{-1}\, dt = -\ln(1+e^t) + t$, $c_2(t) = \int -\frac{e^t}{1+e^t} dt = -\ln(1+e^t)$.

9) $y_p(t) = c_1(t)e^t + c_2(t)e^{2t}$, $c_1(t) = \int -\frac{e^{-2t}}{e^{-x}+1} dt = e^{-t} + t - \ln(1+e^t)$, $c_2(t) = \int \frac{e^{-2t}}{1+e^t} dt = -\frac{1}{2}e^{-2t} + e^{-t} + t - \ln(1+e^t)$.

10) $y_p(t) = c_1(t)\sinh(t) + c_2(t)\cosh(t)$, $c_1(t) = \int -1/2\,(-1+e^{-2t})\, e^t dt = e^t/2 + e^{-t}/2$, $c_2(t) = \int -1/2\,\frac{\left(-1+e^{-2t}\right)^2 e^t}{1+e^{-2t}} dt = 2\arctan e^t - e^t/2 + e^{-t}/2$.

11) $y_p(t) = c_1(t)e^t\cos(3t) + c_2(t)e^t\sin(3t)$, $c_1(t) = \int -3\,\frac{\sin(3t)}{\cos(3t)} dt = \ln|\cos(3t)|$, $c_2(t) = \int 3dx = 3t$.

12) $y_p(t) = c_1(t)\cos(t) + c_2(t)\sin(t)$, $c_1(t) = \int \frac{\cos^2 t}{\sin t}\, dx = \ln|\tan(t/2)| + \cos t$, $c_2(t) = \int \frac{\cos^3 t}{\cos^2 t - 1} = \sin t + 1/\sin t$.

13) $y_p(t) = c_1(t)e^{-2t} + c_2(t)e^{-2t}t$, $c_1(t) = \int -\frac{t}{t+1} dt = -t + \ln|t+1|$, $c_2(t) = \int (t+1)^{-1}\, dt = \ln|t+1|$.

14) $y_p(t) = c_1(t)e^{2t} + c_2(t)e^{2t}t$, $c_1(t) = \int \frac{-2t}{t^2+1} dt = -\ln(t^2+1)$, $c_2(t) = \int 2\,(t^2+1)^{-1}\, dt = 2\,\arctan(t)$.

15) $y_p(t) = c_1(t)e^{-t}\cos(t) + c_2(t)e^{-t}\sin(t)$, $c_1(t) = \int -1 dt = -t$, $c_2(t) = \int \frac{\cos(t)}{\sin(t)} dt = \ln|\sin(t)|$.

16) $y_p(t) = c_1(t)e^{-t}\cos(2\,x) + c_2(t)e^{-t}\sin(2\,t)$, $c_1(t) = \int -\frac{\sin(2\,t)}{\cos(2\,t)}dt = 1/2\,\ln|\cos(2\,t)|$, $c_2(t) = \int 1dt = t$.

17) $y_p(t) = c_1(t) + c_2(t)e^{2\,t}$, $c_1(t) = \int(-1+2\,t+2\,\ln(t)t)/2tdt = -1/2\,\ln(t) + \ln(t)t$, $c_2(t) = -\int(-1+2\,t+2\,\ln(t)t)e^{2t}/2tdt = 1/2\,e^{-2\,t} + 1/2\,e^{-2\,t}\ln(t)$; $y_p(t) = \ln(t)t + 1/2$.

18) $y_p(t) = c_1(t)\sin t + c_2(t)\cos t + c_3(t) + c_4(t)\cdot t$, kde $c_1(t) = \int -\frac{\sin t}{\cos t}dt = \ln|\cos t|$, $c_2(t) = \int \frac{\sin^2 t}{\cos^2 t}dt = \operatorname{tg} t - t$, $c_3(t) = -\int t\frac{\sin t}{\cos^2 t}dt = -\frac{t}{\cos t} + \frac{1}{2}\ln|\frac{1+\sin t}{1-\sin t}|$, $c_4(t) = \int \frac{\sin t}{\cos^2 t}dt = \frac{1}{\cos t}$; řešení je definováno na intervalech $(-\pi/2 + k\pi, \pi/2 + k\pi)$.

19) $y_p = c_1(t)\ln t - c_2(t)\frac{1}{t}$, kde $c_1(t) = \frac{1}{2}t^2$, $c_2 = \frac{1}{3}t^3\ln t - \frac{1}{9}t^3$.

20) Po úpravách vyjde $x'' = -\frac{1}{t} - \ln t$, tj. $\lambda_{1,2} = 0$, partikulární řešení vyjde (pomocí metody variace konstant) $t + \frac{3}{4}t^2 - t\ln t - \frac{t^2}{2}\ln t$, tedy $x(t) = a + t(b+1) + \frac{3}{4}t^2 - t\ln t - \frac{t^2}{2}\ln t$ a následně $y(t) = b - 2a - t(1+2b) - \frac{3}{2}t^2 + (t+t^2)\ln t$, $t \in (0,\infty)$, a, $b \in \mathbb{R}$. Také je možno postupovat přímočařeji a z rovnice $x'' = -\frac{1}{t} - \ln t$ dostat integrováním rovnou výsledek, identický s výsledkem získaným pomocí metody variace konstant.

21) Vlastní čísla jsou $\lambda_1 = 0$ a $\lambda_2 = 2$, příslušné vlastní vektory jsou $v_1 = (2,1)$, $v_2 = (1,1)$, tedy fundamentální systém je

$$\begin{pmatrix}2\\1\end{pmatrix},\ e^{2t}\begin{pmatrix}1\\1\end{pmatrix}.$$

Partikulární řešení (metodou variace konstant) je

$$\frac{1}{2}\begin{pmatrix}3+8t\\3+4t\end{pmatrix} + \begin{pmatrix}-3\\-3\end{pmatrix}(e^t + te^{2t}) + \ln(1+e^t)\left[\begin{pmatrix}-4\\-2\end{pmatrix} + \begin{pmatrix}3\\3\end{pmatrix}e^{2t}\right].$$

22) Řešení čekáme ve tvaru

$$\begin{pmatrix}y(t)\\z(t)\end{pmatrix} = c(t)\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix} + d(t)\begin{pmatrix}te^{2t}\\e^{2t}(t+1)\end{pmatrix}.$$

Vyjde $c'(t) = t + t^2 - t\ln t$ a $d'(t) = \ln t - t$, odtud $c(t) = \frac{3}{4}t^2 + \frac{1}{2}t^3 - \frac{1}{2}t^t\ln t$ a $d(t) = t\ln t - t - \frac{1}{2}t^2$. Řešení je definováno na $(0,+\infty)$.

23) $y(t) = (c+d)e^{2t}\cos 3t + (c-d)e^{2t}\sin 3t + e^{2t}\operatorname{tg} 3t(\sin 3t + \cos 3t) + \frac{e^{2t}}{2\cos^2 3t}(\sin 3t - \cos 3t)$, $z(t) = ce^{2t}\sin 3t + de^{2t}\cos 3t - \frac{e^{2t}\cos 6t}{\cos 3t}$.

24) $y(t) = c + dt - \frac{8}{15}t^{5/2}$, $z(t) = 2c + d(t-1/2) - \frac{16}{15}t^{5/2} + \frac{2}{3}t^{3/2}$.

25) $y(t) = 2ce^{-2t} + de^{-t} - 2e^{-2t} - e^{-2t}\ln(1+e^{-2t}) - 2e^{-t}\operatorname{arctg} e^t$, $z(t) = 5ce^{-2t} + 3de^{-t} - \frac{5}{2}e^{-2t}\ln(1+e^{-2t}) - 6e^{-t}\operatorname{arctg} e^t$.

26) $x(t) = -\frac{4}{35}t^{7/2} - \frac{2}{3}t^{5/2} + ct + d$, $y(t) = \frac{12}{35}t^{7/2} + \frac{4}{5}t^{5/2} - 3ct + e$, $z(t) = \frac{8}{35}t^{7/2} + \frac{8}{15}t^{5/2} + \frac{2}{3}t^{3/2} - 2ct + d - c + e$,

27) $x(t) = a(1/2 + 1/2\sin t + 1/2\cos t) + b(-1/2 + 1/2\cos t - 3/2\sin t) + c(2\sin t) + 1/2\ln(\cos t) - 1/2 + 1/2\sin t\ln((1+\sin t)/\cos t) + 3/2\cos t\ln((1+\sin t)/\cos t)$, $y(t) = a(-1/2 + 1/2\sin t + 1/2\cos t) + b(-1/2 + 1/2\cos t - 3/2\sin t) + c(2\sin t) - 1/2\ln(\cos t) - 1/2 + 1/2\sin t\ln((1+\sin t)/\cos t) + 3/2\cos t\ln((1+\sin t)/\cos t)$, $z(t) = a(-1/2 + 1/2\cos t) + b(1/2 - \sin t - 1/2\cos t) + c(\sin t + \cos t) - 1/2\ln(\cos t) + 1/2 - 1/2\sin t\ln((1+\sin t)/\cos t) + \cos t\ln((1+\sin t)/\cos t)$.