Chapter 6. Asymptotic begavior of semigroups

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Growth of solutions to (ACP) is determined by the growth of the semigroup.

Is the growth of semigroups determined by the spectrum of its generator?

Definition

Let (A, D(A)) be the generator of a C_0 -semigroup T.

• The spectral bound of A is

 $s(A) = \sup\{\Re \lambda : \lambda \in \sigma(A)\}.$

• The growth bound of T is

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\omega(T) = \inf\{\omega \in \mathbb{R} : \exists M \ge 1 ||T(t)|| \le M e^{\omega t} \forall t \ge 0\}.
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Remark

1. By Proposition 4 (Chapter 1) we have $s(A) \leq \omega(T)$.

2. For a matrix A we have $s(A) = \omega(e^{tA})$. In fact,

 $\|e^{tA}\| \le Me^{ct}(1+t)^{n-1} \le \tilde{M}e^{(c+\varepsilon)t} \quad \forall t \ge 0$

where $c = \max{\{\Re \lambda : \lambda \in \sigma(A)\}}$ and n is the size of the biggest Jordan cell of A.

Does

$$\boldsymbol{s}(\boldsymbol{A}) = \boldsymbol{\omega}(\boldsymbol{T})$$

hold in general?

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Counterexample

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$$\|T(t)\| \stackrel{(1)}{=} r(T(t)) = \sup_{\lambda \in \sigma(T(t))} |\lambda| = \sup_{e^{\lambda t} \in \sigma(T(t))} |e^{\lambda t}| \stackrel{(2)}{=} \sup_{\lambda \in \sigma(A)} |e^{\lambda t}|$$

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Proposition 1

Let T be a C₀-semigroup. Then $\omega(T) = \frac{1}{t} \ln r(T(t))$ for each $t \ge 0$

Proof: see [EN], Proposition IV.2.2

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Spectral inclusion

Proposition 2

Let *T* be a C_0 -semigroup and *A* its generator. Then $e^{t\sigma(A)} \subset \sigma(T(t))$.

The other inclusion is more important.

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Definition

Let (A, D(A)) be a closed densely defined operator and $\lambda \in \sigma(A)$. We say that λ belongs to

- the point spectrum $P\sigma(A)$ if λA is not injective,
- the approximate spectrum Aσ(A) if λ A is not injective or range of λ – A is not closed,
- the residual spectrum $R\sigma(A)$ if λA is not dense,

Remark

1. Clearly, $P\sigma(A) \subset A\sigma(A)$.

2. It can be shown that $R\sigma(A) = P\sigma(A')$, where A' is the adjoint of A. 3. $\lambda \in A\sigma(A)$ if and only if there exists an approximate eigenvector $(x_n)_{n=1}^{\infty} \subset D(A)$ such that $||x_n|| = 1$ and $\lim_{n\to\infty} (\lambda - A)x_n = 0$.

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Spectral mapping theorem for point and residual spectrum

Theorem 3

Let (A, D(A)) be the generator of a C_0 -semigroup T. Then $P\sigma(T(t)) \setminus \{0\} = e^{tP\sigma(A)}$ and $R\sigma(T(t)) \setminus \{0\} = e^{tR\sigma(A)}$.

Proof can be found in [EN], Theorem IV.3.7.

Proposition 4

Let
$$\mu \in \sigma(T(t))$$
 and $\Lambda = \{\lambda \in \sigma(A) : e^{\lambda t} = \mu\}$. Then

$$\operatorname{Ker}(\mu - T(t)) = \overline{\operatorname{lin}} \bigcup_{\lambda \in \Lambda} \operatorname{Ker}(\lambda - A).$$

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Spectral mapping theorem for regular semigroups

Theorem 5

Let (A, D(A)) be the generator of a C_0 -semigroup T which is norm continuous for $t > t_0$. Then $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$.

Proof can be found in [EN], Theorem IV.3.10

Corollary 6 Let (A, D(A)) be the generator of an analytic semigroup. Then $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$.