Chapter 5. Analytic semigroups

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Initial remarks

- regularity of semigoup corresponds to regularity of solutions to DE
- initial value $x_0 \in D(A)$, then solution is differentiable for all $t \geq 0$
- initial value $x_0 \notin D(A)$, then solution is not differentiable for $t = 0$. It may and may no be differentiable for some $t > 0$.
- From PDE's: for every $x_0 \in X$ the solution to heat equation is smooth for all $t > 0$ ... high regularity of the heat semigroup.
- From PDE's: solution to transport equation preserves regularity of the initial value ... low regularity of the shift semigroup.
- If a solution is differentiable for $t = t_0$, then it is differentiable for all $t > t_0$ (since $T(t_0)x_0 \in D(A)$ implies $T(t)x_0 \in D(A)$ for all $t > 0$)
- If $x_0 \in D(A^n)$, then the solution is n-times differentiable for all $t \geq 0$.

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Types of regularity

We distinguish several types of regularity, the following implications hold

NORM CONTINUOUS $\Leftarrow$ DIFFFERENTIABLE $\Leftarrow$ ANALYTIC

Norm continuous means that the mapping $t \mapsto S(t)$ is continuous in the operator topology for all $t > 0$ (strictly!).

Differentiable means that the mapping $t \mapsto S(t)x$ is differentiable for all $x \in X$ and all $t > 0$ (strictly!).

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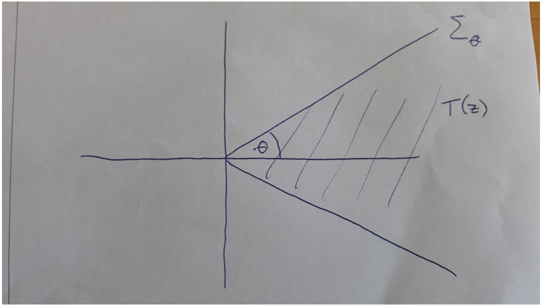
Analytic semigroups

Denote

$$\Sigma_\theta = \{z \in \mathbb{C} : \ z = 0 \text{ or } |\arg z| < \theta\}.$$



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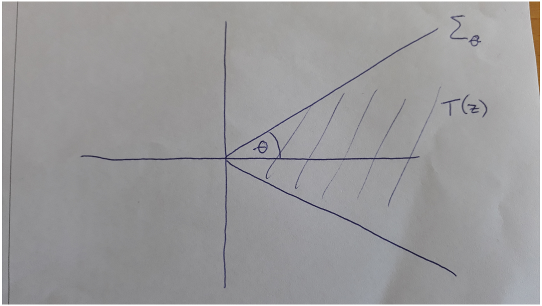


Analytic semigroups

Definition

A C_0-semigroup T is called analytic, if there exists $\theta > 0$ s.t. T has an analytic extension $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ satisfying

1. $\tilde{T}(z+w) = \tilde{T}(z)\tilde{T}(w)$ *for all* $z, w \in \Sigma_\theta$
2. $z \mapsto T(z)$ *is analytic in* $\Sigma_\theta \setminus \{0\}$
3. $\lim_{\Sigma_{\theta'} \ni z \to 0} T(z)x = x$ *for all* $x \in X$, $\theta' \in (0, \theta)$.

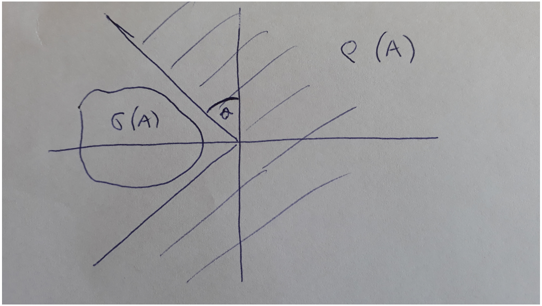
An analytic semigroup is called bounded analytic semigroup if it is bounded on each $\Sigma_{\theta'}$, $\theta' \in (0, \theta)$.

Sectorial operators

Definition

An operator $(A, D(A))$ *is called sectorial if there exists* $\delta \in (0, \frac{\pi}{2}]$ *such that*

- $\Sigma_{\frac{\pi}{2}+\delta} \setminus \{0\} \subset \rho(A)$
- *for every* $\varepsilon \in (0, \delta)$ *there exists* M_ε, $\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|}$ *for all* $\lambda \in \overline{\Sigma_{\frac{\pi}{2}+\delta-\varepsilon}} \setminus \{0\}$



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Remark 1

The resolvent estimate holds with $|\lambda|$ *instead of* $\Re\lambda$*.*

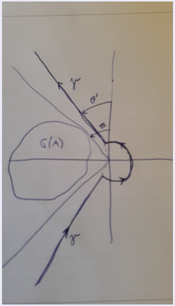
Generator of an analytic semigroup

Theorem 1

Let $(A, D(A))$ be a θ-sectorial operator. Then it generates a bounded analytic semigroup on Σ_θ given by

$$T(z) = \frac{1}{2\pi i}\int_\gamma e^{\mu z} R(\mu, A) d\mu$$

where γ is as follows wiht $\theta > \theta' > |\arg z|$.



Characterization of analytic semigroups

Theorem 2

Let $(A, D(A))$ is the generator of a C_0-semigroup T. Then the following assertions are equivalent

- *A is θ-sectorial*
- *T is a bounded analytic semigroup on Σ_θ.*
- *$T(t)X \subset D(A)$ and $\{\|tAT(t)\| : t \in (0,1]\}$ is bounded.*
- *$e^{\pm i\theta'}A$ generate bounded C_0-semigroups $T(e^{\pm i\theta'}t)$ for all $\theta' \in (0, \theta)$.*

parts of the proof in HW4

Multiplicative sectorial operators

Proposition 3

A multiplicative operator A_m on $L^2(\Omega)$ is θ-sectorial if and only if the essential range of m is contained in $\mathbb{C} \setminus \Sigma_{\frac{\pi}{2}+\theta} \cup \{0\}$.

Corollary 4

Every self-adjoint dissipative operator on a Hilbert space generates a bounded analytic semigroup on $\Sigma_{\frac{\pi}{2}}$.

Example

Dirichlet Laplacian on $L^2(\Omega)$, Ω bounded domain in $\mathbb{R}^n$ is self-adjoint and dissipative. Therefore, the heat semigroup is analytic and solutions to the corresponding heat equation are analytic for $t > 0$...?

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