Chapter 4. Self-adjoint semigroups on Hilbert spaces

For an unbounded linear operator A it may happen that A is a restriction of its adjoint, i.e. the adjoint operator A^* is almost equal to A but it has a larger domain. Such operators are called symmetric. If, moreover, the domains are equal, then the operator is called self-adjoint.

Definition. A linear operator $(A, D(A))$ on a Hilbert space H is

1. *symmetric* if $\langle Ax, y\rangle = \langle y, Ax\rangle$ for all $x, y \in D(A)$.
2. *self-adjoint* if A is symmetric and the following holds: if $y \in H$ is such that there exists $z \in H$ satisfying $\langle Ax, y\rangle = \langle x, z\rangle$ for all $x \in D(A)$, then $y \in D(A)$ and $z = Ay$.

Remark. 1. Every self-adjoint operator is closed.

2. For a multiplicative operator $A_m f = m \cdot f$ on L^2, the adjoint operator is equal to $A_{\bar{m}}$. So, A_m is self-adjoint if and only if m is real-valued. Since $\sigma(A_m)$ is essential range of m, a multiplicative operator A_m is self-adjoint if and only if $\sigma(a_m) \subset \mathrm{R}$.
3. For an example of a symmetric operator that is not self-adjoint see `https://en.wikipedia.org/wiki/Self-adjoint_operator`

Proposition 1. *1. If A is symmetric, then $\sigma(A) = \mathrm{C}$ or $\sigma(A) = \{\lambda \in \mathrm{C} : \Im\lambda \geq 0\}$ or $\sigma(A) = \{\lambda \in \mathrm{C} : \Im\lambda \leq 0\}$ or $\sigma(A) \subset \mathrm{R}$.*

2. A symmetric operator A is self-adjoint if and only if $\sigma(A) \subset \mathrm{R}$ if and only if $i - A$ is surjective if and only if $-i - A$ is surjective.

Proof. Observe that $\Im\langle(\lambda - A)x, x\rangle = -\Im\lambda\|x\|^2$. Hence, $\|(\lambda - A)x\| \geq \Im\lambda\|x\|$. It follows that $\lambda - A$ is injective whenever $\lambda \notin \mathrm{R}$ at it yields an estimate on resolvent $\|R(\lambda, A)\| \geq \frac{1}{|\Im\lambda|}$ provided $\lambda \in \rho(A) \setminus \mathrm{R}$. The resolvent estimate implies that if $\lambda_0 \in \rho(A) \setminus \mathrm{R}$, then the whole disc centered at λ_0 with radius $|\Im\lambda|$ is contained in $\rho(A)$. The first claim follows.
Part 2. The second and third equivalences are already proved. We let the first equivalence without proof. □

Definition. We say that $(A, D(A))$ has a *compact resolvent* if there exists $\lambda \in \rho(A)$ such that $R(\lambda, A)$ is compact.

Remark. 1. Of course, in infinite-dimensional spaces if A has compact resolvent, then it is necessarily unbounded (since I is not compact and $I = (\lambda - A)R(\lambda, A)$).

2. A multiplication operator A_m on L^2 is compact if and only the range of m is finite or countable with $H_m \cap \{z \in \mathrm{C} : |z| > \varepsilon\}$ being finite for every $\varepsilon > 0$. (not difficult to prove)

Proposition 2. *Let $(A, D(A))$ has a compact resolvent. Then*

1. $R(\lambda, A)$ is compact for every $\lambda \in \rho(A)$

2. $\sigma(A)$ is countable and if $\sigma(A) = \{\alpha_1, \alpha_2, \dots\}$ is infinite then $\lim_{n\to\infty} |\alpha_n| = +\infty$.

Proof. (i) follows immediately from the resolvent equation $R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$. Part (ii) follows immediately from spectral theory of compact operators and from the spectral mapping theorem for resolvents:

$$\sigma(R(\lambda_0, A)) \setminus \{0\} = \left\{ \frac{1}{\lambda_0 - \mu} : \mu \in \sigma(A) \right\}.$$

For the proof of the last statement, see IV.1.13 in [EN]. □

Remark. $R(\lambda, A)$ is compact if and only if the canonical embedding $(D(A), \|\cdot\|_A) \to H$ is compact (Proposition II.4.25 in [EN])

The following theorems say that every self-adjoint operator is similar to a multiplication operator on an L^2 space. This fact has a lot of consesquences. If you are interested in proofs, they can be found in E.B.Davies: Spectral theory of differential operators, Theorems 2.5.2 and 4.2.2 + Corollary. You probably know similar statements for bounded operators. For matrices it says that symmetric matrices (or normal matrices) are diagonalizable.

Theorem 3 (Spectral theorem)**.** *Let $(A, D(A))$ be self-adjoint on a separable Hilbert space H. Then there exists a space if finite measure (Y, Σ, μ) and a unitary operator $U : H \to L^2(Y, \Sigma, \mu)$ and a measurable function $m : Y \to \mathrm{R}$ such that $UAU^{-1} = A_m$, where A_m is the multiplication operator $(A_m f)(x) = m(x)f(x)$, $D(A_m) = \{f \in L^2(Y, \Sigma, \mu) : \; m \cdot f \in L^2(Y, \Sigma, \mu)\}$.*

Theorem 4 (Spectral theorem II)**.** *Let $(A, D(A))$ be self-adjoint with compact resolvent on a separable Hilbert space H. Then there exists a sequence $(\alpha_n)_{n=1}^{\infty}$ of real numbers and a unitary operator $U : H \to l^2$ such that $UAU^{-1} = A_\alpha$, where A_α is the multiplication operator $(x_n) \mapsto (\alpha_n x_n)$, $D(A_\alpha) = \{x \in l^2 : \; (\alpha_n x_n) \in l^2\}$. Moreover, there exists an orthonormal basis $(\phi_n)_{n=1}^{\infty}$ consisting of eigenvectors of A and $Ax = \sum_{n=1}^{\infty} \alpha_n \langle x, \phi_n\rangle \phi_n$, $D(A) = \{x \in H : (\alpha_n \langle x, \phi_n\rangle) \in l^2\}$.*

One consequence is the following theorem.

Theorem 5. *Let $(A, D(A))$ be self-adjoint dissipative operator on a separable Hilbert space H. Then A generates a C_0-semigroup of contractions T. Moreover, $T(t)$, $t \geq 0$ are self-adjoint operators and if A has compact resolvent, then $T(t)$ are compact for $t > 0$.*

Proof. By spectral theorems, A is similar to a multiplicative operator associated with a function m that attains only non-positive real values. Then operators $T(t)$ are similar to multiplication operators associated with $e^{m(\cdot)t}$ which are self-adjoint. The rest follows from characterization of compact multiplication operators (see a remark above). □

Many more results on self-adjoint operators follow easily from the corresponding properties of multiplication operators. Moreover, we can write a formula for the semigroup if we know the unitary operator as the following examples show.

Example. Dirichlet Laplacian on $[0, 1]$. Let $D(A) = \{f \in H^2(0, 1) : \; f(0) = f(1) = 0\}$, $Af = \Delta f$. (the details are to work out in HW3)

Example. Dirichlet Laplacian on R^n. Let $D(A) = H^2(\mathrm{R}^n)$, $Af = f''$. The Fourier transform $(\mathcal{F}f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathrm{R}^n} e^{-ix\cdot y} f(y)dy$ is known to be a unitary operator from L^2 to L^2. Moreover, $\mathcal{F}(\frac{\partial}{\partial x_i} f)(x) = -x_i f(x)$, so $\mathcal{F} : \Delta f \mapsto -\sum x_i^2 f$. Hence, $\mathcal{F}$ is the unitary operator and the multiplication operator similar to A is A_m with $m(x) = -\|x\|_2^2$, $x \in \mathrm{R}^n$. In this case, A does not have a compact resolvent. As a consequence

$$(T(t)f)(x) = \mathcal{F}^{-1}(e^{-\|z\|_2^2 t}(\mathcal{F}f)(z))(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathrm{R}^n} e^{ixz} e^{-\|z\|_2^2 t} \int\limits_{\mathrm{R}^n} e^{-izy} f(y)dydz,$$

which leads to the Gaussian kernel.

Further, we can define a functional calculus on self-adjoint operators. For a real or complex function f we can define $f(A)$ by the formula $f(A) = U^{-1} A_{f\circ m} U$ as we did for the exponential function $f(x) = e^{tx}$. If A is dissipative, then m is real valued and non-positive and we can define fractional powers of $-A$, e.g. $\sqrt{-A} := U^{-1} A_{\sqrt{-m}} U$.

In the second part of this lecture we show that self-adjoint generators are associated with bilinear forms.

Definition. Let V, H be complex Hilbert spaces, V densely embedded into H. A mapping $a : V \times V \to \mathrm{C}$ is called *sesquilinear form* if $a(\lambda x, y) = \lambda a(x, y)$, $a(x, \lambda y) = \bar{\lambda} a(x, y)$, $a(x + z, y) = a(x, y) + a(z, y)$ and $a(x, y + z) = a(x, y) + a(x, z)$ for all x, y, $z \in V$, $\lambda \in \mathrm{C}$. We say that a is *symmetric* if $a(x, y) = \overline{a(y, x)}$ for all x, $y \in V$. We say that a is *positive* if $a(x, x) \geq 0$ for all $x \in V$ and *H-elliptic* if there exists $\omega \in \mathrm{R}$ and $\alpha > 0$ such that $\Re a(x, x) + \omega\|x\|_H \geq \alpha\|x\|_V$

Remark. a is continuous if there exists $c > 0$ such that $|a(x,y)| \le c\|x\|_V\|y\|_V$ for all x, $y \in V$.

Example. $a(f,g) = \int_0^1 f'(x)\overline{g'(x)}dx$ with $H = L^2(0,1)$ and $V = H^1(0,1)$ or $H_0^1(0,1)$.

Theorem 6. *Let $a : V \times V \to \mathrm{C}$ is a continuous, positive, symmetric, H-elliptic sesquilinear form. Define $(A, D(A))$ by*

$$D(A) = \{x \in V : \ \exists y \in H, \ a(x,\phi) = \langle y, \phi\rangle_H \ \forall \phi \in V\}, \qquad Ax = -y. \tag{1}$$

Then A is dissipative and self-adjoint, and therefore it generates a C_0-semigroup of contractions.

Proof. We show that A is well-defined, i.e. y in (**??**) is unique. If y_1, y_2 satisfy (**??**), then $\langle y_1 - y_2, \phi\rangle = 0$ for all $\phi \in V$ and V is dense, hence $y_1 = y_2$. It is easy to show that A is symmetric and dissipative, the latter follows from $\langle Ax, x\rangle = -a(x,x) \le 0$. It remains to show that $\omega \in \rho(A)$, in particular that $\omega x - Ax = y$ has a solution for each $y \in H$. For fixed $y \in H$ let us define $F_y(\phi) = \langle \phi, y\rangle_H$. Then F_y is a bounded functional on V, so there exists a unique $x \in V$ such that $F_y(\phi) = a_1(\phi, x)$ where $a_1(x,y) := a(x,y) + \omega\langle x, y\rangle_H$ (it follows from the fact that a_1 is due to H-ellipticity an equivalent inner product on V). Hence, $a(\phi,x) = a_1(\phi,x) - \omega\langle\phi,x\rangle = \langle \phi, y - \omega x\rangle_H$. So, $x \in D(A)$ and $Ax = -(y - \omega x)$. ☐

Remark. Also the converse is true: To each self-adjoint dissipative operator $(A, D(A))$ there exists a unique continuous positive symmetric H-elliptic sesquilinear form such that (**??**) holds.

Example. If $a_{ij} \in W^{1,\infty}(\Omega)$ are real functions such that $a_{ij} = a_{ji}$ and $\Re \sum_{i,j=1}^n a_{ij}(x)\xi_i\bar{\xi}_j \ge \alpha|\xi|_2^2$ for all $x \in \Omega$ and $\xi \in \mathrm{C}^n$. We can define the form

$$a(f,g) = \int\limits_\Omega \sum_{ij} a_{ij}(x)\partial_i f(x)\overline{\partial g_j(x)}dx$$

on $V_1 = H_0^1(\Omega)$ resp. $V_2 = H^1(\Omega)$. Then a is continuous, symmetric, positive, H-elliptic form associated with the elliptic operators $Av = \sum_{ij}\partial_j(a_{ij}\partial_i v)$ with Dirichlet resp. Neumann boundary conditions.