Exam test (sample)

for Mathematics 2, SS 2017/18

1. (15 points) Find the rank of the following matrix depending on the parameters $x, y \in \mathrm{R}$:

$$\begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 20+x & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 40+y \end{pmatrix}.$$

2. (15 points) Compute the antiderivative

$$\int \frac{x+7}{(x+1)^2(x^2+2x+5)} dx.$$

3. (20 points) Find supremum and infimum (and maximum and minimum if they exist) of the function f on the set M, where

$$f(x,y,z) = x, \qquad M = \{[x,y,z] \in \mathrm{R}^3 : \ x^2 + y^2 + 2z^2 = 4, \ xz \geq 1\}.$$

Solution.

1. Let us apply elementary row and column operations

$$\begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 20+x & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 40+y \end{pmatrix} \sim \begin{pmatrix} 11 & 12 & 13 & 14 \\ -1 & -4+x & -3 & -4 \\ -2 & -4 & -6 & -8 \\ -3 & -6 & -9 & -16+y \end{pmatrix} \sim \begin{pmatrix} 1 & 4-x & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 16-y \\ 11 & 12 & 13 & 14 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 4-x & 3 & 4 \\ 0 & -4+2x & 0 & 0 \\ 0 & -6+3x & 0 & 4-y \\ 0 & -32+11x & -20 & -30 \end{pmatrix} \sim \begin{pmatrix} 1 & 4-x & 3 & 4 \\ 0 & -32+11x & -20 & -30 \\ 0 & -6+3x & 0 & 4-y \\ 0 & -4+2x & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & 4-x \\ 0 & -20 & -30 & -32+11x \\ 0 & 0 & 4-y & -6+3x \\ 0 & 0 & 0 & -4+2x \end{pmatrix}.$$

We now distinguish several cases:

1. $x \neq 2$ and $y \neq 4$, then rank of the matrix is 4,
2. $x = 2$ and $y \neq 4$, then rank of the matrix is 3,
3. $x \neq 2$ and $y = 4$, then the last matrix is not in the row echelon form and the last row can be subtracted to zero, so rank of the matrix is 3,
4. $x = 2$ and $y = 4$, then rank of the matrix is 2.

Grading: 9 pts - transformation of the matrix, 6 pts - discussion.

Remark: If you first subtract the first row from each of the others, then you get a more elegant solution.

2. The function is continuous on $(-\infty, -1)$, $(-1, +\infty)$, so the antiderivative is defined on these intervals.
We compute the partial fractions:

$$\frac{x+7}{(x+1)^2(x^2+2x+5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+5},$$

which yields the equality

$$x + 7 = A(x+1)(x^2+2x+5) + B(x^2+2x+5) + (Cx+d)(x+1)^2.$$

Insert $x = -1$ and obtain $B = 3/2$. Compare the coefficient at 1, x, and x^3, we get

$$7 \quad = \quad 5A + 5B + D = 5A + D + 15/2$$

$$
\begin{aligned}
1 &= 7A + 2B + C + 2D = 7A + C + 2D + 3 \\
0 &= A + C.
\end{aligned}
$$

Subtract the third equation from the second one we obtain

$$
\begin{aligned}
-1/2 &= 5A + D \\
-2 &= 6A + 2D.
\end{aligned}
$$

Solution of the system is $A = 1/4$, $C = -1/4$, $D = -7/4$.
We have the equality

$$\int \frac{x+7}{(x+1)^2(x^2+2x+5)}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{x+1}\,\mathrm{d}x + \frac{3}{2}\int \frac{1}{(x+1)^2}\,\mathrm{d}x - \frac{1}{4}\int \frac{x+7}{x^2+2x+5}\,\mathrm{d}x.$$

The first and second integrals are easy:

$$\int \frac{1}{x+1}\,\mathrm{d}x \overset{c}{=} \ln|x+1|, \qquad \int \frac{1}{(x+1)^2}\,\mathrm{d}x \overset{c}{=} -\frac{1}{x+1}.$$

We split the third integral into two parts

$$\int \frac{x+7}{x^2+2x+5}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,\mathrm{d}x + 6\int \frac{1}{x^2+2x+5}\,\mathrm{d}x.$$

Here the first integral is (substitution $t = x^2 + 2x + 5$)

$$\int \frac{2x+2}{x^2+2x+5}\,\mathrm{d}x \overset{c}{=} \ln(x^2+2x+5).$$

In the second integral we use the standard procedure to complete the square first and we get

$$\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{(x+1)^2+2^2}dx \overset{c}{=} \frac{1}{2}\operatorname{arctg}\left(\frac{x+1}{2}\right).$$

Putting everything together we have

$$\int \frac{x+7}{(x+1)^2(x^2+2x+5)}\,\mathrm{d}x \overset{c}{=} \frac{1}{4}\ln|x+1| - \frac{3}{2}\cdot\frac{1}{x+1} - \frac{1}{8}\ln(x^2+2x+5) - \frac{3}{4}\operatorname{arctg}\left(\frac{x+1}{2}\right).$$

Grading: 2 pts - domain of the antiderivative, 6 pts - parcial fractions, 2 pts - two simple integrals, 3 pts - the third integral, 2 pts - final result.

3. The set M is closed, since it is an intersection of two sets which are closed by Theorem 11: $\{[x,y,z]:\ x^2+y^2+2z^2=4\}$ and $\{[x,y,z]:\ xz \geq 1\}$. The set is bounded, since $|x|$, $|y|$, $|z| \leq 2$. So, M is compact. Since f is continuous, M attains its maximum and minimum on M. We find suspected points first on $M_1 = \{[x,y,z]:\ x^2+y^2+2z^2=4, xz>1\}$ and then on $M_2 = \{[x,y,z]:\ x^2+y^2+2z^2=4, xz=1\}$.

In M_1: By the Lagrange multiplier theorem we either have $\nabla g = o$, or $\nabla f + \lambda \nabla g = o$, where $g(x,y,z) = x^2 + y^2 + 2z^2 - 4$. Since $\nabla g = (2x, 2y, 4z)$, we have $\nabla g = o$ only if $x = y = z = 0$, and $[0,0,0] \notin M_1$. The second possibility leads to the system

$$\begin{aligned} 1 + 2\lambda x &= 0 & (1)\\ 2\lambda y &= 0 & (2)\\ 4\lambda z &= 0 & (3)\\ x^2 + y^2 + 2z^2 &= 4. & (4) \end{aligned}$$

From the third equation we have either $\lambda = 0$ (contradiction with the first equation) or $z = 0$ (then $xz > 1$ cannot be true). So, there are no suspected points in M_1.
In M_2: By the Lagrange multipliers theorem, either ∇g_1, ∇g_2 are linearly dependend or $\nabla f + a\nabla g_1 + b\nabla g_2 = o$, where $g_1(x,y,z) = x^2 + y^2 + 2z^2 - 4$ and $g_2(x,y,z) = xz - 1$. The vectors $(2x, 2y, 4z)$ and $(z, 0, x)$ are linearly dependend only if $y = 0$, $2x = cz$ and $4z = cx$, i.e. $c = \pm 2\sqrt{2}$ and $x = \pm\sqrt{2}z$. Since $xz = 1$ we have $x = \pm\sqrt[4]{2}$ and $z = \pm 1/\sqrt[4]{2}$. However, this contradicts $x^2 + y^2 + 2z^2 = 4$.
It remains to solve the system

$$\begin{aligned} 1 + 2ax + bz &= 0 & (5)\\ 2ay &= 0 & (6)\\ 4az + bx &= 0 & (7)\\ x^2 + y^2 + 2z^2 &= 4 & (8)\\ xz &= 1. & (9) \end{aligned}$$

From the second equation we have either $a = 0$ or $y = 0$. If $a = 0$, the third equation yields $b = 0$ (contradiction with the first equation) or $x = 0$ (contradiction with the fifth equation). So, $y = 0$. Then we have $z = \frac{1}{x}$ and inserting this to the fourth equation we obtain

$$x^2 + \frac{2}{x^2} = 4.$$

This leads (multiply by $x^2 \neq 0$) to $x^4 - 4x^2 + 2 = 0$, which gives four solutions

$$x = \pm\sqrt{2 \pm \sqrt{2}}.$$

So, we have suspected points

$$\left[\pm\sqrt{2 \pm \sqrt{2}}, 0, \frac{1}{\pm\sqrt{2 \pm \sqrt{2}}}\right].$$

Obviously, $f(x) = x$ is maximal in

$$\left[\sqrt{2 + \sqrt{2}}, 0, \frac{1}{\sqrt{2 + \sqrt{2}}}\right]$$

and minimal in

$$\left[-\sqrt{2+\sqrt{2}}, 0, -\frac{1}{\sqrt{2+\sqrt{2}}}\right].$$

The values of maximum resp. minimum are $\sqrt{2+\sqrt{2}}$, resp. $-\sqrt{2+\sqrt{2}}$.

Grading: 5 pts — existence of extrema (including some reasoning), 4 pts — M1 ($\nabla g = o$ 1 pt, system 3 pts), 9 pts — M2 (linear independence 4 pts, system 5 pts), 2 pts — final answer.