Mathematics

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Preface

This textbook is intended for Economy students of the Faculty of Social Sciences CU as reference materials to the lecture Mathematics I–IV. We believe however that the text may also be useful for students of the Faculty of Maths and Physics in their first two years. The somewhat unusual order of the chapters is due to the needs of courses in microeconomics at FSS.

Theorems are numbered within chapters. If we wish to reference a theorem in the same chapter we simply use the number of the theorem (e.g. Theorem 8). If we reference a theorem in another chapter we also use the chapter number (e.g. Theorem 5.26). The symbol ■ (or ♣) designates the end of a proof (or an example).

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CHAPTER 1

Sets, statements and numerical sets

In this chapter we would like to demonstrate how mathematical claims are formulated and the basic methods of their proofs. Further we would like to specify some terms you have already met at secondary school, like a statement, real numbers and so on, more precisely. Initially it may seem that this is a little unnecessary, but already in the second chapter we will see that for example without an exact definition of what a real number is some important claims cannot be proven.

1.1. Sets

The aim of this section is to repeat notation commonly used for work with sets and to define the basic operations with sets. As we will tend to use very specific sets we will not spend time on the question what a general set is. This problem is at the boundary of mathematics and philosophy and therefore is not an easy one to answer. For our purposes we will suffice with the (somewhat imprecise) description: *A set is any collection of distinct objects (which we call elements) into a single whole.*

The fact that the element a **belongs** to the set A, will be represented as $a \in A$. We write $a \notin A$ to denote that a **does not belong** to A.

We may define a set by listing its elements or by specifying certain properties that the elements must possess. In the first case we use the notation

$$\{a, b, c, \dots\},$$

where $a, b, c, \dots$ are the elements of the set (e.g. $\{2, 3, 4, 5\}$), and in the second case we write

$$\{a \in M;\ a \text{ has the property } V\},$$

where M is some given set (e.g. $\{a \in \mathbb{N};\ a \text{ is smaller than } 6\}$, while the symbol $\mathbb{N}$ denotes the set of all natural numbers).

We say that a set A is **part of a set** B (or A is a **subset** of B), if all elements of A are also elements of B. We express this by writing $A \subset B$ and we call the relationship between sets an **inclusion**.

Two sets are **equal** ($A = B$), if they have the same elements, that is to say $A \subset B$ and $B \subset A$ both hold simultaneously.

The **empty set** is the set, which contains no elements. We denote it with the symbol $\emptyset$. The empty set is a subset of any other set.

We now define operations which create new sets from two (or more) other sets.

The **union** of the sets A and B is the set made of all elements that belong to at least one of the sets A or B. The union of the sets A and B is denoted by the symbol $A \cup B$. This definition can be generalized to use any system of sets A_α, where α is an index from some non-empty index set I (be it finite or infinite). We define $\bigcup_{\alpha\in I} A_\alpha$ as the set of all those elements, which belong to at least one of the sets A_α.

The **intersection** of two sets A and B is the set of those elements which belong to A and B simultaneously. (Notice that the intersection of two non-empty sets may also be an empty set.) If two sets have an empty intersection we call them **disjoint**. The intersection of the sets A and B is denoted as $A \cap B$. As before we can generalize the term intersection to any system of sets A_α, $\alpha \in I$, $I \neq \emptyset$. We define $\bigcap_{\alpha\in I} A_\alpha$ as the set of elements that belong to all of A_α.

The **difference** of two sets A and B (we write $A \setminus B$) is the set of elements which belong to A but not to B.

Let us define one more set operation: Let us have m sets $A_1, \dots, A_m$. The **Cartesian product** of $A_1 \times A_2 \times \cdots \times A_m$ is the set of all ordered m-tuples

$$\big\{[a_1, a_2, \dots, a_m];\ a_1 \in A_1, \dots, a_m \in A_m\big\}.$$

In the Cartesian product operation one cannot (in general) interchange the order of the sets because $A \times B \neq B \times A$, if $A \neq B$.

If A is a set and n is a natural number, then instead of $\underbrace{A \times \cdots \times A}_{n\text{-times}}$ we write A^n.

To finish the current section we will prove the following theorem.

Theorem 1 (de Morgan rules)**.** Let us have the sets S, A_α, $\alpha \in I$, where $I \neq \emptyset$. Then

$$S \setminus \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I} (S \setminus A_\alpha) \quad \text{and}$$
$$S \setminus \bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} (S \setminus A_\alpha).$$

Proof. Let us conduct a proof of the first of the given claims. Recall the definition of the equality of two sets. We must prove two inclusions, i.e.

$$S \setminus \bigcup_{\alpha\in I} A_\alpha \subset \bigcap_{\alpha\in I} (S \setminus A_\alpha) \text{ and simultaneously } \bigcap_{\alpha\in I} (S \setminus A_\alpha) \subset S \setminus \bigcup_{\alpha\in I} A_\alpha.$$

If $x \in S \setminus \bigcup_{\alpha\in I} A_\alpha$, then x belongs to S, but does not belong to the union $\bigcup_{\alpha\in I} A_\alpha$. Therefore $x \notin A_\alpha$ for each index $\alpha \in I$. This means for every $\alpha \in I$ that $x \in S \setminus A_\alpha$, and so $x \in \bigcap_{\alpha\in I}(S \setminus A_\alpha)$. Thus we prove the first inclusion.

Let $x \in S \setminus A_\alpha$ for each of the α. Then $x \in S$, but $x \notin A_\alpha$ for all α. therefore $x \notin \bigcup_{\alpha\in I} A_\alpha$. Thus we see $x \in S \setminus \bigcup_{\alpha\in I} A_\alpha$, which proves the second inclusion.

Try to prove the second of the de Morgan rules yourself. ■

1.2. Propositional calculus, mathematical proofs

A **statement** is any claim for which it makes sense to say that it either holds (is true), or does not hold (is false). An example of a true statement is "$2 < 3$", and example of a false statement is "The number 2 is odd." Not every grammatically legible sentence is a statement. For example the sentence "What will the weather be like tomorrow?" or "Let the natural number n be even." are not statements. Let us assign the number 1, to all true statements and the number 0 to untrue statements.

The **negation** of the statement A is the statement: "It is not true that A holds." We denote this new statement by $\mathrm{non}\, A$. The relationship between the truth of A and $\mathrm{non}\, A$ is given in the following table:

A	$\mathrm{non}\, A$
0	1
1	0

The **conjuctiuon** of the statements A and B is the statement: "Both A and B are true." This statement is true if both A and B are true; in all other cases it is false. We denote it as $A \;\&\; B$. The table of its values is

A	B	$A \;\&\; B$
0	0	0
0	1	0
1	0	0
1	1	1

The **disjunction** of A and B is the statement: "At least one of A or B holds." We denote this symbolically as $A \vee B$. The statement $A \vee B$ is true if at least of one A or B is true and is false if both A and B are false. The veracity of the statement is given by the table

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A very important statement is: "If A holds, then B also holds." This statement can also be understood as "The statement A implies B." This statement is always true, except if A is true and B is false. It is called an **implication**, is denoted as $A \Rightarrow B$ and its table is

A	B	$A \Rightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

The statement A in the implication is called the **assumption** (or also the **premise**), the statement B is called the **conclusion**. We can read implications as follows: "The veracity of statement A guarantees the veracity of B." The veracity of A is a **sufficient condition** for the veracity of B. The veracity of B is a **necessary condition** for the veracity of A.

We can use these basic operations to create more complicated statements. Let us give two important examples.

The statement $(A \Rightarrow B) \ \ \& \ \ (B \Rightarrow A)$ is true if and only if both statements $A \Rightarrow B$ and $B \Rightarrow A$ are true.

The table of values is

A	B	$(A \Rightarrow B) \ \& \ (B \Rightarrow A)$
0	0	1
0	1	0
1	0	0
1	1	1

The statement we have just described is denoted by the symbol $A \Leftrightarrow B$; we write $A \Leftrightarrow B$ instead of $(A \Rightarrow B) \ \ \& \ \ (B \Rightarrow A)$. The statement $A \Leftrightarrow B$ is called **equivalence**. We read it as follows: "A holds if and only if B holds" or "(The veracity of the statement) A is a necessary and sufficient condition for (the veracity of) B." Notice that $A \Leftrightarrow B$ is true exactly when both statements A and B are simultaneously true or both false.

Let us also consider the statement $(\operatorname{non} A) \vee B$. Now let us continue purely mechanically by filling in the table of values:

A	B	$\operatorname{non} A$	$(\operatorname{non} A) \vee B$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Because the table $(\operatorname{non} A) \vee B$ is the same as the table of the statement $A \Rightarrow B$, both of these statements are equivalent, that is to say $(A \Rightarrow B) \Leftrightarrow \big((\operatorname{non} A) \vee B\big)$ is true.

Example 2. Prove the equivalence of the statements $A \Rightarrow B$ and $(\operatorname{non} B) \Rightarrow (\operatorname{non} A)$.

Solution. The equivalence of the statements is obvious from the following table:

A	B	$\operatorname{non} B$	$\operatorname{non} A$	$A \Rightarrow B$	$(\operatorname{non} B) \Rightarrow (\operatorname{non} A)$
0	0	1	1	1	1
0	1	0	1	1	1
1	0	1	0	0	0
1	1	0	0	1	1

♣

A **predicate** is an expression, from which we retrieve a statement by substituting an element from a given set into the form as a variable. (The statement may depend on more than one variable – in this case the number of sets given must correspond to the number of variables taken by the form.) For example, the following expressions are prediactes:

$$\text{“}x < 10\text{”},\ x \in \{1, 2, 3, 4, 7, 8, 9, 10, 11\}, \tag{1}$$
$$\text{“}x < y\text{”},\ x \in \{-1, 0, 4, 6, 8\},\ y \in \{4, 5, 6, 7, 8, 9\}.$$

If we substitute $x = 1$ into the form (1) then we get a true statement, for $x = 10$ we get a false statement.

Generally we can write a prediacte as

$$A(x_1, x_2, \dots, x_m), \quad x_1 \in M_1, x_2 \in M_2, \dots, x_m \in M_m.$$

Now let $A(x)$, $x \in M$, be a prediacte. If we say that: “For all $x \in M$ it holds that $A(x)$.”, then we have a statement. (In case of the form (1) the statement is untrue.) We may write this as

$$\forall x \in M\colon A(x).$$

The symbol $\forall$ is called the **universal qualifier**.

If we say that: “There exists an $x \in M$ such that $A(x)$.”, Then we also retrieve a statement. We write this as

$$\exists x \in M\colon A(x).$$

In the case of the form (1) this is a true statement. The symbol $\exists$ is called the **existential quantifier**. Further we use the notation

$$\exists! x \in M\colon A(x),$$

Which we read as “There exists exactly one $x \in M$ such that $A(x)$.”

If a prediacte has several variables then we may relate them to one another using quantifiers. This way we get new prediacte with fewer variables or statements.

Let us have the prediacte $V(x,y)$, $x \in M_1$, $y \in M_2$. Now we can create new forms of a single variable$y \in M_2$ as follows:

$$\forall x \in M_1\colon V(x,y), \qquad \exists x \in M_1\colon V(x,y).$$

We can create statements from these forms using another quantifier as follows:

$$\begin{aligned} &\forall y \in M_2\colon \bigl(\forall x \in M_1\colon V(x,y)\bigr), &&\forall y \in M_2\colon \bigl(\exists x \in M_1\colon V(x,y)\bigr),\\ &\exists y \in M_2\colon \bigl(\forall x \in M_1\colon V(x,y)\bigr), &&\exists y \in M_2\colon \bigl(\exists x \in M_1\colon V(x,y)\bigr).\end{aligned}$$

We generally write the statements above as:

$$\begin{aligned} &\forall y \in M_2\ \forall x \in M_1\colon V(x,y), &&\forall y \in M_2\ \exists x \in M_1\colon V(x,y),\\ &\exists y \in M_2\ \forall x \in M_1\colon V(x,y), &&\exists y \in M_2\ \exists x \in M_1\colon V(x,y).\end{aligned}$$

Similarly we write the statements, which contain three or more quantifiers. Let us introduce some more notation. Let A and P be prediactes taking the variable $x \in M$. Then

$$\begin{aligned} &\forall x \in M, P(x)\colon A(x) &&\text{means the same as} &&\forall x \in M\colon \bigl(P(x) \Rightarrow A(x)\bigr),\\ &\exists x \in M, P(x)\colon A(x) &&\text{means the same as} &&\exists x \in M\colon \bigl(P(x)\ \ \&\ \ A(x)\bigr).\end{aligned}$$

The first statement is read as "For every $x \in M$ satisfying P the statement $A(x)$ holds." The second statement is read as "There exists an $x \in M$ satisfying P such that $A(x)$ holds." An example of statements of this type are the statements

$$\forall x \in \{1,3,20\}, x > 10\colon A(x) \qquad \text{or} \qquad \exists x \in \{1,3,20\}, x > 10\colon A(x),$$

where A is a prediacte taking the variable $x \in \{1,3,20\}$.

In order to negate a statement containing quantifiers we change universal quantifiers into existential and existential into universal a negate the prediacte. For example

$$\operatorname{non}\bigl(\forall x \in M_1\ \exists y \in M_2\ \forall z \in M_3\colon V(x,y,z)\bigr)$$

is the same as

$$\exists x \in M_1\ \forall y \in M_2\ \exists z \in M_3\colon\ \operatorname{non} V(x,y,z).$$

The claim above can be verified from the two obvious following rules. Let V be a prediacte of the variable $x \in M$, then

$$\begin{aligned} &\operatorname{non}\bigl(\forall x \in M\colon V(x)\bigr) &&\text{means the same as} &&\exists x \in M\colon\ \operatorname{non} V(x),\\ &\operatorname{non}\bigl(\exists x \in M\colon V(x)\bigr) &&\text{means the same as} &&\forall x \in M\colon\ \operatorname{non} V(x).\end{aligned}$$

The order of the quantifiers in some formulas can be interchanged at will as the following example shows Let $A(m,d)$ mean "The man m is the father of the child d.", while we take $m \in M$, $d \in D$, where M is the set of men and D is the set of children. An analysis of the statements

$$\begin{aligned} &\exists d \in D\, \forall m \in M\colon A(m,d), \qquad && \forall m \in M\, \exists d \in D\colon A(m,d),\\ &\exists m \in M\, \forall d \in D\colon A(m,d), \qquad && \forall d \in D\, \exists m \in M\colon A(m,d) \end{aligned}$$

shows that the order of the quantifiers $\exists$ and $\forall$ is important.

Mathematical theories are made of definitions, theorems and proofs. The definitions determine the new terms, theorems communicate the properties of these terms and the relationships between them. Mathematical theorems have assumptions and conclusions. The proofs of these theorems are sequences of considerations that lead to the conclusion of the theorem. In our considerations we use the assumptions and previously proven theorems. Only the existence of a logically correct proof will allow us to use a theorem in our later deductions.

At the same time a graduate of a course of mathematics should gain the ability to carry out considerations like: what could be said if a claim was true and what is still unknown. Considerations like these are irreplaceable, especially when looking for new results. It is necessary however to remain very critical, in order that one doesn't interchange one's wish for a claim to be true with certainty of the claim itself. One of the aims of this text is to cultivate this critical thinking of the reader towards his reasoning.

Now we will demonstrate a few of the basic ways to prove mathematical theorems. Let us assume that we have a mathematical theorem expressed as an implication, which is a very common scenario. We want to prove the implication $A \Rightarrow B$ for certain statements A and B.

Direct proof. By using the veracity of the statement A we show the veracity of the statement C_1, using C_1 we show the veracity of C_2, from which we show C_3, and so on until, using the veracity of C_n we show the statement B. We then have discovered the following chin of implications

$$A \Rightarrow C_1,\ C_1 \Rightarrow C_2,\ C_2 \Rightarrow C_3, \dots, C_{n-1} \Rightarrow C_n,\ C_n \Rightarrow B.$$

If we want to prove a theorem directly then we must find some appropriate intermediary terms $C_1, \dots, C_n$, which lead from the assumptions to the conclusion. Unfortunately there is no algorithm or general rule for how to find them in practice. Mathematics is a creative activity and without a certain degree of cunning one cannot prove any new theorem.

Remark. In the following examples we will work with natural, whole, rational and real numbers. So far we have sufficed with an intuitive understanding of them. In the section 1.4 we define real numbers more precisely. For now let us recall that a rational number is the quotient of a whole number with a natural number.

Example 3 (Cauchy inequality)**.** Let $a_1, \dots, a_n, b_1, \dots, b_n$ be real numbers. Then it holds that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2}$$

Proof. If all the numbers $a_1, \dots, a_n$ are equal to zero then we have equality in (2). Let us therefore assume that at least one of the numbers $a_1, \dots, a_n$ is non-zero. Let us consider the expression

$$\sum_{i=1}^{n} (a_i x + b_i)^2, \tag{3}$$

which can be written in the form

$$\left(\sum_{i=1}^{n} a_i^2\right) x^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right) x + \left(\sum_{i=1}^{n} b_i^2\right).$$

The x^2 coefficient is non-zero and the expression(3) is non-negative for any x real, therefore the equation

$$\left(\sum_{i=1}^{n} a_i^2\right) x^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right) x + \left(\sum_{i=1}^{n} b_i^2\right) = 0$$

has at most one real root. The discriminant D of this quadratic equation is less than or equal to zero. From here we see

$$D = 4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \le 0,$$

which very simply gives (2). ∎

Indirect proof. This type of proof is based on the equivalence of the statements $A \Rightarrow B$ and $\mathrm{non}\, B \Rightarrow \mathrm{non}\, A$ (see Example 2). If the second is true then so is the first. Therefore it suffices to find any proof of the second statement.

Proof by contradiction. This method is based on the equivalence of the statements $A \Rightarrow B$ and $\mathrm{non}(A \;\; \& \;\; \mathrm{non}\, B)$. In this method of proof we assume the veracity of $A \;\&\; \mathrm{non}\, B$. If we are able to deduce statement C, which we know to be false, then $A \;\&\; \mathrm{non}\, B$ must also be false (one cannot deduce a false statement from a true statement). It therefore holds that $\mathrm{non}(A \;\&\; \mathrm{non}\, B)$, or $A \Rightarrow B$.

Example 4. If the real number y solves the equation $y^2 = 2$, then y is not rational.

Proof. Let $y^2 = 2$ and let y be a rational number, i.e. $y = p/q$, where p is a whole number and q is a natural number. Further we may assume that the numbers p and q are indivisible. Our assumption implies that $p^2 = 2q^2$, and therefore p^2 can be divided by two. >From here we see that p can be divided by two and therefore p^2 can be divided by four. Since$p^2 = 4k$, where k is a whole number, and using our starting equation $p^2 = 2q^2$ we get that q^2, and therefore also q, is divisible by two. This is however a contradiction because we have proven that two indivisible numbers p and q have a common factor 2. The equation $y^2 = 2$ therefore has no solution in rational numbers. ■

Recall that the set of natural numbers is denoted by $\mathbb{N}$. In our text we will not consider 0 to be a natural number.

Mathematical induction. One can use this type of proof to show statements of the following sort

$$\forall n \in \mathbb{N}\colon V(n), \tag{4}$$

where $V(n)$, $n \in \mathbb{N}$ is a prediacte. In the first step of mathematical induction we show the veracity of the statement $V(1)$. In the second step we prove the statement

$$\forall n \in \mathbb{N}\colon V(n) \Rightarrow V(n+1),$$

that is we assume the veracity of $V(n)$ (the so called **induction hypothesis**) and deduce the veracity of $V(n+1)$. >From these two steps we get the veracity of the statement (4). The following example should shed light on the technique.

Example 5. Prove that for every $n \in \mathbb{N}$ it holds that

$$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1). \tag{5}$$

Proof. For $n = 1$ the left hand side is equal to $\sum_{j=1}^{1} j^2 = 1$ and the right hand side is $\frac{1}{6} \cdot 2 \cdot 3 = 1$. Therefore the statement holds for $n = 1$. Let us assume the veracity of the relationship (5) for some fixed n, i.e.

$$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1).$$

We want to prove

$$\sum_{j=1}^{n+1} j^2 = \frac{1}{6} (n+1)(n+2)(2n+3). \tag{6}$$

The left hand side of (6) can be written as

$$\sum_{j=1}^{n+1} j^2 = \left(\sum_{j=1}^{n} j^2 \right) + (n+1)^2.$$

but the sum on the right hand side can be calculated by the induction hypothesis. By conducting some algebraic alterations we get

$$\sum_{j=1}^{n+1} j^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3).$$

Thus the proof is concluded. ■

1.3. Numerical sets

We will devote the two following sections to numbers. First let briefly us say something about real numbers and make a short remark about complex numbers. In the section 1.4 we will give a precise definition of the set of real numbers by formulating three groups of properties which uniquely determine the set (the axiomatic approach to real numbers).

We will denote the set of whole numbers as $\mathbb{Z}$ and the set of rational numbers as $\mathbb{Q}$. The set $\mathbb{Q}$ is already rather broad. All numerical calculations conducted by computers are contained by it. Nevertheless there are still many good reasons to enlarge it. One of those is to get a positive answer to the existence of the n-th root of a non-negative number:

Does there exist, for any non-negative number x and any natural number n a non-negative number y satisfying $y^n = x$?

In Example 4 we have shown that the answer is negative in the realm of rational numbers since the solution to the equation $y^2 = 2$ cannot be a rational number.

A reasonable extension of the set $\mathbb{Q}$ is the set of real numbers, which we denote as $\mathbb{R}$. (The question of the existence of the n-th root of a non-negative number is in the affirmative.) There are many ways how to conduct such an enlargement – in the following section we assume the position that the extension has already been made and characterize the set of real numbers by their properties, which we call axioms.

If anyone is interested in knowing more about how to enrich the set $\mathbb{Q}$ into $\mathbb{R}$, then it can be found in *J. Kopáček: Matematická analýza nejen pro fyziky I*, Matfyzpress 2004. Because managing all such details is rather difficult we recommend that the reader accepts our approach, at least for the time being.

One of the motivations for further extending the set of reals sets, would be to find the roots of algebraic equations. Although the equation

$$x^2 + 1 = 0$$

does not have roots in the set of reals, it does have roots in the set of complex numbers, which are an extension of the reals.

The set of **complex numbers** $\mathbb{C}$ is defined as the set of all objects of the type $a = a_1 + a_2 i$, where a_1 a a_2 are real numbers and i is the so called **imaginary unit.** The number a_1 is called the **real part** of the number a (we write $\operatorname{Re} a$), and the number a_2 is the **imaginary part** (we write $\operatorname{Im} a$).

If $b = b_1 + b_2 i$, $b_1, b_2 \in \mathbb{R}$, we define:

$$\begin{aligned} a + b &= (a_1 + b_1) + (a_2 + b_2)i, \\ a \cdot b &= (a_1 \cdot b_1 - a_2 \cdot b_2) + (a_1 \cdot b_2 + a_2 \cdot b_1)i, \\ \overline{a} &= a_1 - a_2 i \quad \text{(the \textbf{complex conjugate} to the number } a\text{)}, \\ |a| &= \sqrt{a_1^2 + a_2^2} \quad \text{(\textbf{absolute value of the complex number})}. \end{aligned}$$

The precise introduction of the root is conducted in Theorem 14. We identify the set of all complex numbers with imaginary part equal to zero with the set of real numbers. The operations of addition and multiplication of complex numbers can be simplified into the multiplication and addition of real numbers. It follows from the definition of the multiplication of complex numbers that $i^2 = i \cdot i = -1$.

The following theorem shows the solubility of algebraic equations in the set of complex numbers. The proof of this theorem is not simple and so we will not include it.

Theorem 6 (fundamental theorem of algebra)**.** Let $n \in \mathbb{N}$, $a_0, \dots, a_n \in \mathbb{C}$, $a_n \neq 0$. Then the equation

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

In the following example we will summarize some of the properties of complex numbers.

Example 7. Prove that:

(i) $\forall z \in \mathbb{C}\colon |z|^2 = z\overline{z}$,
(ii) $\forall z \in \mathbb{C}\colon \overline{\overline{z}} = z$,
(iii) $\forall z_1 \in \mathbb{C}\, \forall z_2 \in \mathbb{C}\colon \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,
(iv) $\forall z_1 \in \mathbb{C}\, \forall z_2 \in \mathbb{C}\colon \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$,
(v) $\forall z \in \mathbb{C}\colon z \in \mathbb{R} \Leftrightarrow z = \overline{z}$.

Proof. (i) Let $z \in \mathbb{C}$. Then we have the numbers $a, b \in \mathbb{R}$ such that $z = a + bi$. Further it holds that

$$z\overline{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 = |z|^2 .$$

(ii) This claim is obvious.

(iii) Let $z_1, z_2 \in \mathbb{C}$. Then we have the numbers $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. It holds that

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{(a_1 + b_1 i) + (a_2 + b_2 i)} = \overline{(a_1 + a_2) + (b_1 + b_2) i} = \\ &= (a_1 + a_2) - (b_1 + b_2) i = a_1 - b_1 i + a_2 - b_2 i = \\ &= \overline{z_1} + \overline{z_2}.\end{aligned}$$

(iv) Let $z_1, z_2 \in \mathbb{C}$. Then we have the numbers $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. It holds that

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(a_1 + b_1 i)(a_2 + b_2 i)} = \overline{(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i} = \\ &= (a_1 a_2 - b_1 b_2) - (a_1 b_2 + a_2 b_1) i, \\ \overline{z_1} \cdot \overline{z_2} &= \overline{(a_1 + b_1 i)} \cdot \overline{(a_2 + b_2 i)} = (a_1 - b_1 i)(a_2 - b_2 i) = \\ &= a_1 a_2 - a_1 b_2 i - b_1 a_2 i - b_1 b_2 = (a_1 a_2 - b_1 b_2) - (a_1 b_2 + a_2 b_1) i,\end{aligned}$$

and the claim is proven.

(v) The claim is obvious.

■

1.4. The set of real numbers

The set of **real numbers** $\mathbb{R}$ can be described as the set on which we have defined the operations **addition** and **multiplication**, which we will represent with the usual symbols and the relation **order** ($\leq$), which satisfy the following three groups of properties.

I. Properties of addition and multiplication and their interdependence:

- $\forall x, y \in \mathbb{R} \colon x + y = y + x$ (commutativity of addition),[1]
- $\forall x, y, z \in \mathbb{R} \colon x + (y + z) = (x + y) + z$ (associativity of addition),
- there exists an element of $\mathbb{R}$ (which we denote as 0 and call **zero element**), such that for all $x \in \mathbb{R}$ we have $x + 0 = x$,
- $\forall x \in \mathbb{R}\ \exists! y \in \mathbb{R} \colon x + y = 0$ (y is the **opposite number** of x and we denote it as $-x$),
- $\forall x, y \in \mathbb{R} \colon x \cdot y = y \cdot x$ (commutativity of multiplication),
- $\forall x, y, z \in \mathbb{R} \colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity of multiplication),
- there exists a non-zero element (which we will denote as 1 and call the **identity element**) such that for all $x \in \mathbb{R}$ we have $1 \cdot x = x$,
- $\forall x \in \mathbb{R} \setminus \{0\}\ \exists! y \in \mathbb{R} \colon x \cdot y = 1$ (this y we denote as x^{-1} or also $\frac{1}{x}$),

[1] The symbol "$\forall x, y \in \mathbb{R}$" means the same as "$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}$". We will use this convention in the obvious way throughout the following.

- $\forall x, y, z \in \mathbb{R}\colon (x + y) \cdot z = x \cdot z + y \cdot z$ (distributivity).

II. The order relation and its relationship to addition and multiplication:

- $\forall x, y, z \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq z) \Rightarrow x \leq z$ (transitivity),
- $\forall x, y \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq x) \Rightarrow x = y$ (weak anti-symmetry),
- $\forall x, y \in \mathbb{R}\colon x \leq y \vee y \leq x$,
- $\forall x, y, z \in \mathbb{R}\colon x \leq y \Rightarrow x + z \leq y + z$,
- $\forall x, y \in \mathbb{R}\colon (0 \leq x \ \ \& \ \ 0 \leq y) \Rightarrow 0 \leq x \cdot y$.

Convention. When we write $x \geq y$ it is the same as writing $y \leq x$. By $x < y$ we imply the case where $x \leq y$, but $x \neq y$ (i.e. **sharp inequality**). Real numbers for which $x > 0$ (or $x < 0$), will be called **positive** (or **negative**). Real numbers for which $x \geq 0$ (or $x \leq 0$), will be called **non-negative** (or **non-positive**).

In the following, instead of writing $a \cdot b$ we will simply write ab and instead of $a + (-b)$ we will simply write $a - b$. Further we will use the usual notation $a^n = \underbrace{a \cdots a}_{n\text{-times}}$ a $a^{-n} = 1/a^n$, where $n \in \mathbb{N}$. For $a \in \mathbb{R}$, $a \neq 0$, we put $a^0 = 1$.

The above listed properties are also held by the set of rational numbers, $\mathbb{Q}$. In order to formulate the final properties of real numbers which distinguish $\mathbb{R}$ from $\mathbb{Q}$, we will need the following definition.

Definition. We say that the set $M \subset \mathbb{R}$ is **bounded from below**, if there exists a number $a \in \mathbb{R}$ such that for every $x \in M$ we have $x \geq a$. Such a number a is called a **lower bound** of the set M. Similarly we define a set as being **a set bounded from above** and an **upper bound**. We say that the set $M \subset \mathbb{R}$ is **bounded**, if it is bounded from above and below.

III. The infimum axiom:

Let $M \subset \mathbb{R}$ be a non-empty bounded set. Then there exists exactly one number $g \in \mathbb{R}$, which has the following properties:

(i) $\forall x \in M\colon x \geq g$,
(ii) $\forall g' \in \mathbb{R}, g' > g \ \exists x \in M\colon x < g'$.

We denote the number g with the symbol $\inf M$ and call it the **infimum** of M.

Remark. The infimum axiom says that one can always find a largest lower bound to a non-empty set bounded from below. This remarkable axiom does not hold in the set of rational numbers! More will be said about this in the section 1.5.

All the basic rules you learned at secondary school for making calculations with real numbers can be deduced from these axioms. Let us show this on a few examples.

Theorem 8.

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0$,
(ii) $\forall x \in \mathbb{R}\colon -x = (-1) \cdot x$,
(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0)$,
(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n$,
(v) $\forall x, y \in \mathbb{R}\colon (x > 0\ \ \&\ \ y > 0) \Rightarrow xy > 0$,
(vi) $\forall x \in \mathbb{R}, x \geq 0\ \forall y \in \mathbb{R}, y \geq 0\ \forall n \in \mathbb{N}\colon x < y \Leftrightarrow x^n < y^n$.

Proof. (i) It holds that

$$x = 1 \cdot x = (1 + 0) \cdot x = 1 \cdot x + 0 \cdot x = x + 0 \cdot x.$$

If we add to $x = x + 0 \cdot x$ the number $-x$ on both sides we get the required claim.

(ii) It holds that

$$0 = 0 \cdot x = (1 - 1) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x.$$

If in $0 = x + (-1) \cdot x$ we add $-x$ to both sides then we get the claim.

(iii) If $x = 0$, then we are done. If $x \neq 0$, then there exists$x^{-1} \in \mathbb{R}$ and it holds that

$$0 = x^{-1} \cdot 0 = x^{-1}(xy) = (x^{-1}x)y = 1 \cdot y = y.$$

(iv) Using the commutativity and associativity of multiplication we get

$$x^n \cdot (x^{-1})^n = x^n \cdot (\underbrace{x^{-1} \cdots x^{-1}}_{n\text{-times}}) = \underbrace{(x \cdot x^{-1}) \cdots (x \cdot x^{-1})}_{n\text{-times}} = 1 \cdots 1 = 1.$$

(v) From the final axiom of the second group we get that $xy \geq 0$. According to (iii) however, we see that $xy \neq 0$, otherwise x or y would be 0. Together we have $xy > 0$.

(vi) If $n = 1$, then the claim is obvious. If $n > 1$ then we can write

$$y^n - x^n = (y - x) \cdot (y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}).$$

If $y > x$, then both bracketed expressions are positive numbers and therefore it holds that $0 < y^n - x^n$, that is $x^n < y^n$.

Now let $y^n > x^n$. If $y = x$, then also $y^n = x^n$, which is a contradiction. In case $y < x$, we get a contradiction from $y^n < x^n$, which we have already proven. ∎

If we take a line, put a point on it, call it the origin, mark a unit of length and designate a direction as positive, then we can identify this line with $\mathbb{R}$. We often talk about real numbers as points of the **real axis**.

Let a, b be two real numbers with, $a \leq b$. The **open interval** (a, b) is the set $\{x \in \mathbb{R};\ a < x < b\}$. The **closed interval** $[a, b]$ is the set $\{x \in \mathbb{R};\ a \leq x \leq b\}$. Similarly we define **half-open intervals** $[a, b)$ and $(a, b]$.

The number a (or b) in the definitions of intervals are called the **left** (or **right**) **endpoint of the interval**. Endpoints may or may not be elements of the interval.

So the endpoint a belongs to the intervals $[a,b)$ and $[a,b]$, but not to the interval $(a,b]$, nor to (a,b). It can be seen from our definitions that $[a,a]$ is a set containing the single point $\{a\}$, while $(a,a)=\emptyset$. A point which belongs to an interval, but is not an endpoint is a so called **interior point of the interval**.

The intervals which we have just defined are bounded subsets of the real numbers. Further we define **unbounded intervals**

$$
\begin{aligned}
(a,+\infty) &= \{x\in\mathbb{R};\ x>a\},\\
[a,+\infty) &= \{x\in\mathbb{R};\ x\geq a\},\\
(-\infty,a) &= \{x\in\mathbb{R};\ x<a\},\\
(-\infty,a] &= \{x\in\mathbb{R};\ x\leq a\}
\end{aligned}
$$

and finally we put $(-\infty,+\infty)=\mathbb{R}$.

Since we have taken the stance that the real numbers $\mathbb{R}$ are specified by the axioms I–III, then we should verify that the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are its subsets and that the usual rules of arithmetic work there.

By repeatedly adding the real number 1 to 0 we get the set of naturals, $\mathbb{N}$. If we add zero and the opposite numbers of the the naturals to the naturals (the existence of the opposites are guaranteed by the fourth property in group I), then we get the set $\mathbb{Z}$ of all whole numbers. The products $p(q^{-1})$, where $p\in\mathbb{Z}$ and $q\in\mathbb{N}$, form the set $\mathbb{Q}$.

This idea is only a rough sketch of how to approach the proof of the claim, it is not a detailed proof itself. We avoided checking that if we transfer the arithmetic operations from $\mathbb{R}$ to those subsets then we get the usual operations on these smaller sets.

A real number that is not a rational number is called an **irrational number**. The set $\mathbb{R}\setminus\mathbb{Q}$ is called **the set of irrational numbers**.

Example 9. It holds that $\inf(0,1)=0$.

Proof. We have to prove that the number 0 has the properties (i) and (ii) of the infimum axiom, i.e. it is the greatest lower bound of the set $(0,1)$. By the definition of the interval $(0,1)$ the number 0 is a lower bound and therefore we have (i). We now choose any $g'\in\mathbb{R}$, $g'>0$. If we put $g'\geq 1$, then for $x=\frac{1}{2}$ it holds that $x\in(0,1)$ and $x<g'$. If $g'<1$, then we put $x=\frac{g'}{2}$. Because $x>0$ and $x<1$, we have that x belongs to the interval $(0,1)$ and $x<g'$. The number 0, therefore, also has the property (ii). Hence we get $\inf(0,1)=0$. ∎

1.5. Implications of the infimum axiom and further properties of $\mathbb{R}$

The infimum axiom (axiom III from section 1.4) can be used to show the existence of the smallest upper bound of a non-empty set bounded from above.

Definition. Let $M \subset \mathbb{R}$ and $G \in \mathbb{R}$. We say that the number G is the **supremum** of the set M, if:

(i) $\forall x \in M \colon x \leq G$,
(ii) $\forall G' \in \mathbb{R}, G' < G \ \exists x \in M \colon x > G'$.

Theorem 10 (on the supremum)**.** Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists exactly one supremum of the set M.

Proof. Given a set M we create the set $-M = \{-x;\ x \in M\}$. This set is non-empty and bounded from below. If $A \in \mathbb{R}$ is an upper bound of M, then $-A$ is a lower bound of $-M$. Therefore there exists $\inf(-M) = g$. We put $G = -g$ and show that G is the supremum of the set M.

If $x \in M$, then $-x \in -M$ and so $g \leq -x$ by the properties of the infimum. >From here we see that $x \leq -g = G$. The property (i) from the definition of the supremum is therefore satisfied. We now choose any $G' \in \mathbb{R}, G' < G$. Then $-G' > -G = g$ and therefore by the second property of the infimum we have a $y \in -M$ with $y < -G'$. For every number $y \in -M$ we have an $x \in M$, such that $-x = y$, and so $x = -y > G'$. So condition (ii) from the definition of the supremum holds too.

Now let us show uniqueness. Let $H \in \mathbb{R}$ have the properties (i) and (ii). If $H < G$ then by (ii) for G there exists $x \in M$, $x > H$. This is a contradiction of property (i) for H however. Similarly if $H > G$, then by (ii) for H there exists $x \in M, x > G$. This is a contradiction of property (i) for G. Therefore $H = G$. ■

Remark. The supremum of the set M is denoted as $\sup M$. Notice that we have observed above that $\sup M = -\inf(-M)$.

Definition. Let $M \subset \mathbb{R}$. We say that a is the **greatest element** (**maximum**) of the set M (we write $\max M$), if $a \in M$ and a is an upper bound of the set M. Similarly we define the **smallest element** (**minimum**) of M, which we denote as $\min M$.

Remarks. 1. Notice that for every bounded non-empty set we have $\inf M \leq \sup M$. Equality holds if and only if the set M is a singleton.

2. If the set M has a maximum then $\sup M = \max M$, if it has a minimum then $\inf M = \min M$. For the interval $(0,1)$ it holds that $\inf(0,1) = 0$ and also $\sup(0,1) = 1$, but the numbers 0 and 1 do not belong to the interval $(0,1)$ and this set does not have a greatest or smallest element.

Lemma 11. Let $M \subset \mathbb{R}$ and let

$$\forall x, y \in M \; \forall z \in \mathbb{R}, x < z < y \colon z \in M.$$

Then M is an interval.

Proof. If $M = \emptyset$, then the claim is obvious. If M is not bounded from below or above then $M = \mathbb{R} = (-\infty, +\infty)$. If we take any number $z \in \mathbb{R}$, then there exists $x \in M$, $x < z$ (because M is not bounded from below) and also there exists $y \in M$, $y > z$ (because M is not bounded from above). By the assumption we have therefore that $z \in M$.

If M is bounded and non-empty then we put $G = \sup M$ and $g = \inf M$. Then we have $(g, G) \subset M$. So if $z \in (g, G)$, then by the definition of the infimum there exists an $x \in M$, such that $x < z$, and similarly by the definition of the supremum there exists a $y \in M$, $y > z$. By our hypothesis we have $z \in M$. Further $M \subset [g, G]$, because g is a lower bound on M and G is an upper bound on M. The set M is therefore the interval whose endpoints are g and G, and these endpoints may or may not belong to M.

The other cases (i.e. M is bounded only from below or only from above) are proven similarly. ■

Theorem 12. For every $r \in \mathbb{R}$ there exists an **integer part** r, i.e. the number $k \in \mathbb{Z}$ such that, $k \le r < k + 1$. The integer part of the number r is determined uniquely and we denote it as $[r]$.

Proof. Let $r \in \mathbb{R}$. Denote $M = \{n \in \mathbb{Z};\ n \le r\}$. The number r is an upper bound of the set M, and therefore M is bounded from above. Now we will prove that M is non-empty. Let us assume that this is not true. Then for every $n \in \mathbb{Z}$ it holds that $n > r$, and therefore the set $\mathbb{Z}$ is bounded from below. The set $\mathbb{Z}$ is non-empty and so there exists an infimum $g \in \mathbb{R}$ of the set $\mathbb{Z}$. Then for every $n \in \mathbb{Z}$ we have $n \ge g$. If $n \in \mathbb{Z}$, then also $n - 1 \in \mathbb{Z}$, and therefore $n - 1 \ge g$. For every $n \in \mathbb{Z}$ then it holds that $n \ge g + 1$. The element $g + 1$ is therefore a lower bound of the set $\mathbb{Z}$, which is a contradiction with the assumption that $g = \inf \mathbb{Z}$. The set M is therefore non-empty.

There exists therefore the supremum $G \in \mathbb{R}$ of the set M. By the definition of the supremum there exists $k \in M$ such that $G - 1 < k$. Then $k + 1 > G$, and so $k + 1 \notin M$. Using this and the fact that $k \in M$ we see $k \le r < k + 1$.

Uniqueness is not hard to show. ■

An interesting implication of the infimum axiom is the statement of the unboundedness of natural numbers from above. This statement is often referred to as Archimedes' axiom.

Theorem 13 (Archimedes' axiom)**.** For every $x \in \mathbb{R}$ there exists an $n \in \mathbb{N}$, for which $x < n$.

Proof. The claim follows immediately from the previous theorem since it suffices to put

$$n = \max\{1, [x] + 1\}. \qquad \blacksquare$$

Remark. An axiom is a claim that forms the basis of a theory and is something that is not proven. >From the point of view of our approach, where we deduced Theorem 13 from axiom III of the set $\mathbb{R}$, it would be more correct to speak about Archimedes' property. Let us note however that we can construct the same set $\mathbb{R}$ from a different set of axioms than those axioms I, II a III, and one of the axioms used could be exactly Archimedes' property.

Theorem 14 (on the n-th root)**.** For every $x \in [0, +\infty)$ and every $n \in \mathbb{N}$ there exists exactly one $y \in [0, +\infty)$ satisfying $y^n = x$.

Proof. If $n = 1$ or $x = 0$, the existence and uniqueness of y are clear. In the further we will therefore assume that $n > 1$ and $x > 0$. Call

$$S = \{z \in [0, +\infty);\ z^n \geq x\}.$$

This set is non-empty ($\max\{1, x\} \in S$) and bounded from below (0 is a lower bound). Call $y = \inf S$. By Lemma 11 and Theorem 8 (vi) the set S is an interval, and further we either have $S = [y, +\infty)$, or $S = (y, +\infty)$.

Assume that $y^n > x$. Then we have $y > 0$ and we can choose $h \in \mathbb{R}$ such that

$$0 < h < \min\left\{y, \frac{y^n - x}{ny^{n-1}}\right\}. \tag{7}$$

Then we have $y - h > 0$ and

$$\begin{aligned}(y-h)^n &= y^n - (y^n - (y-h)^n) = \\ &= y^n - h\left(y^{n-1} + y^{n-2}(y-h) + \cdots + y(y-h)^{n-2} + (y-h)^{n-1}\right) > \\ &> y^n - hny^{n-1} > y^n - (y^n - x) = x.\end{aligned}$$

It therefore holds that $y - h \in S$. Then, however, y is not a lower bound of the set S, which is a contradiction.

Now let us assume that $y^n < x$. Choose $h \in \mathbb{R}$ such that

$$0 < h < \min\left\{\frac{x - y^n}{n(y+1)^{n-1}}, 1\right\}. \tag{8}$$

Then we have

$$\begin{aligned}(y+h)^n &= y^n + ((y+h)^n - y^n) = \\ &= y^n + h\left((y+h)^{n-1} + (y+h)^{n-2}y + \cdots + (y+h)y^{n-2} + y^{n-1}\right) < \\ &< y^n + hn(y+1)^{n-1} < y^n + (x - y^n) = x.\end{aligned}$$

Then it holds that $y+h \notin S$. On the other hand $y+h \in S$, because $S = [y, +\infty)$, or $S = (y, +\infty)$. Thus we arrive at a contradiction. Therefore we must have $y^n = x$. (Notice that this implies that $S = [y, +\infty)$.)

Uniqueness is guaranteed by claim (vi) in Theorem 8. ∎

Remark. The reader may well be surprised by the choice of upper bounds for h in the expressions (7) and (8). Let us now try to explain how we came to the estimate (7). We want to choose the number h in such a way that $y - h > 0$ and also $(y - h)^n > x$. In order to satisfy the first of these requirements it suffices to have $h < y$. Satisfying the second is somewhat harder however because we do not see straight away from the required inequality$(y - h)^n > x$ what condition on h will be able to guarantee this. In the proof we estimate $(y - h)^n$ from below by the expression $y^n - hny^{n-1}$, for which we can easily find all h such that $y^n - hny^{n-1} > x$. This is satisfied if $h < \frac{y^n - x}{ny^{n-1}}$. For such h, obviously we will also have $(y - h)^n > x$.

Notice also that the inequality $(y - h)^n > x$ cannot be rewritten as $y - h > \sqrt[n]{x}$, because, as yet, we have not yet proven the existence of the n-th root so the expression $\sqrt[n]{x}$ cannot be used in the proof.

Further remarks. 1. The number y from Theorem 14 is called the **n-th root** of the number x and we write is as $\sqrt[n]{x}$. Instead of $\sqrt[2]{x}$ we usually write $\sqrt{x}$. It is not hard to deduce the well known rules for calculating with roots.

2. Let $x \in \mathbb{R}$, $x > 0$, and $p, q \in \mathbb{N}$. The expression $x^{p/q}$ represents the number $\sqrt[q]{x^p}$ and the expression $x^{-p/q}$ represents the number $1/x^{p/q}$. It can be shown that if $p', q' \in \mathbb{N}$ are such that $p'/q' = p/q$, then $x^{p'/q'} = x^{p/q}$.

3. If $x \in \mathbb{R}$, $x < 0$, and $n \in \mathbb{N}$ is odd, then we define $\sqrt[n]{x} = -\sqrt[n]{-x}$. Truly then $(\sqrt[n]{x})^n = (-\sqrt[n]{-x})^n = -(\sqrt[n]{-x})^n = -(-x) = x$.

Now we can easily see that the set of irrational numbers is non-empty. By Theorem 14 we have a number $\sqrt{2} \in \mathbb{R}$. We have proven in Example 4, there is no rational number satisfying $y^2 = 2$. Therefore we have $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

A very important implication of Theorem 13 is the theorem, that states that every real number can be approximated by a rational number with error as small as required. Notice that this is a result of the following fact.

Theorem 15 (on the density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$)**.** Let $a, b \in \mathbb{R}$, $a < b$. Then there exists $r \in \mathbb{Q}$ satisfying $a < r < b$ and $s \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $a < s < b$.

Proof. By Theorem 13 for every number $1/(b - a)$ there exists a number $n \in \mathbb{N}$ such that $1/(b - a) < n$. Therefore $na + 1 < nb$. It now suffices to put $r = ([na] + 1)/n$.

Further, using what we have already proven there exists a rational number $r' \in (r, b)$. Put $s = r + (r' - r)/\sqrt{2}$. ∎

Remark. >From the proof of Theorem 14 and from Theorem 15 it follows that

$$\sqrt{2} = \inf\{y \in [0,+\infty) \cap \mathbb{Q};\ y^2 \geq 2\}.$$

We know however that $\sqrt{2}$ is not rational and therefore the infimum axiom does not hold in the set of rational numbers.

Definition. For every $x \in \mathbb{R}$ we define the **absolute value** of x as

$$|x| = \max\{x, -x\}.$$

Therefore $|x| = x$, if $x \geq 0$, and $|x| = -x$, if $x \leq 0$.

Remarks. 1. For every $x \in \mathbb{R}$ we have:
(i) $|x| \geq 0$,
(ii) $|x| = 0 \Leftrightarrow x = 0$,
(iii) $|x| = |-x|$,
(iv) $\forall \lambda \in \mathbb{R}\colon\ |\lambda x| = |\lambda|\,|x|$.

2. Geometrically we can interpret $|x|$ as the **distance of the point x from the origin** on the real axis. The expression $|x-y|$ is called the **distance of the point x from the point y**.

3. Let $x \in \mathbb{R}$. Let us consider $\sqrt{x^2}$. First of all we see by Theorem 14 on the n-th root that $\sqrt{x^2}$ exists. It is a non-negative number and its second power is equal to x^2. Such a number is $|x|$, because $|x|^2 = |x|\,|x| = \left|x^2\right| = x^2$. This number is (again by the statement on the n-th root) uniquely determined. Therefore we get $\sqrt{x^2} = |x|$.

Remark. Notice that the set $M \subset \mathbb{R}$ is bounded if and only if there exists some $K \in \mathbb{R}$ such that for all $x \in M$ it holds that $|x| \leq K$.

Theorem 16 (triangle inequality)**.** For every $a, b \in \mathbb{R}$ it holds that

$$|a+b| \leq |a| + |b|\,. \tag{9}$$

Proof. Obviously $-|a| \leq a \leq |a|$ a $-|b| \leq b \leq |b|$. By adding the inequalities we get $-(|a|+|b|) \leq a+b \leq |a|+|b|$, which is equivalent to the inequality (9). ∎

Corollary 17. (i) For every $x, y \in \mathbb{R}$ we have

$$\big||x| - |y|\big| \leq |x-y|\,. \tag{10}$$

(ii) For every $x, y, z \in \mathbb{R}$ we have

$$|x-y| \leq |x-z| + |z-y|\,.$$

Proof. (i) In (9) put $a = y$, $b = x - y$ and subsequently put $a = x$, $b = y - x$. From which you can retrieve (10) independently.

(ii) This inequality is often also called the triangle inequality. The proof is easy, in (9) we put $a = x - z$, $b = z - y$. ∎

Solved exercises are intended to improve the reader's orientation in section 1 concerning the infimum and supremum of a set. Below you can find exercises which review secondary school knowledge and several exercises about the supremum and infimum.

Example 18. Find the supremum and infimum of the set

$$M = \left\{ \frac{m}{n};\ m \in \mathbb{N},\ n \in \mathbb{N},\ m < n \right\}.$$

Solution. We often try to solve exercises on finding the supremum and infimum of non-empty bounded sets of real numbers by "guessing" the answer and then later checking that we were right by verifying the properties of the infimum or supremum. There is no general way to find the supremum or infimum.

Let us consider the set M, and make the estimate that $\sup M = 1$ and $\inf M = 0$. Let us now prove this.

We want to prove that the number 1 has both of the properties from the definition of the supremum. The assumption that $m < n$, immediately gives that $m/n < 1$, and the number 1 is therefore an upper bound of the set M. Let us now choose any number $A < 1$. We want to prove that A is not an upper bound of the set M, that is that there exists an element $a \in M$, for which we have $a > A$. We look for our number in the form $n/(n+1)$. (All numbers of this form belong to the set M.) >From the requirement $n/(n+1) > A$ we get that the natural number n must satisfy the condition $n > A/(1-A)$. Such an n must exist however thanks to Archimedes' axiom. We have proven that the number 1 is the supremum of the set M. In our proof we have used Theorem 15.

The number 0 is a lower bound of the set M – all of its elements are positive. In order to show that the number $B > 0$ cannot be a lower bound of the set it suffices to realize that there exists a natural n such that $1/n < B$. The number $1/n$, however, is an element of M. $\clubsuit$

Example 19. Find the supremum and infimum of the set

$$M = \left\{ \frac{n+1}{n};\ n \in \mathbb{N} \right\}.$$

Solution. Looking at the elements of the set M for $n = 1, 2, 3, \ldots$, we notice that they are the numbers $2, 3/2, 4/3, \ldots$ It seems that $\sup M = 2$ a $\inf M = 1$. Let us check this.

The number 2 is an upper bound of the set M – the inequality $(n+1)/n \leq 2$ is equivalent with the inequality $1 \leq n$. If $A < 2$, then there exists an element of the set M, which is bigger than A – and that number is 2. Therefore $\sup M = \max M = 2$.

For every $n \in \mathbb{N}$ we have $(n+1)/n > 1$. In order to prove that the number 1 is the largest lower bound we choose $A > 1$ and try to find $n \in \mathbb{N}$ satisfying $(n +$

$1)/n < A$. We easily see that such an n exists by Archimedes' axiom. Therefore we have $\inf M = 1$. ♣

1.6. Exercises

In this section we assume secondary-school level knowledge of the reader on trigonometric functions, the natural logarithm (log),the decimal ($\log_{10}$) and the exponential function. These functions are introduced precisely in the section 4.3.

Solve the following equations and inequalities in the set of real numbers.

1. $\sqrt{6x+7} = 2x - 1$

2. $3 + \sqrt{x-1} > \sqrt{2x}$

3. $\left|\dfrac{3x-1}{x+5}\right| \leq 1$

4. $\dfrac{\log(3x-5)}{\log(x-1)} = 2$

5. $\cos x = -1/2$

6. $\sin 2x + \cos 2x = 1 + \operatorname{tg} x$

7. $3 \operatorname{tg}^2 x - 4\sqrt{3} \operatorname{tg} x \geq -3$

8. $\cos x + \operatorname{cotg} x = 1 + \sin x$

9. $\log_{10}(x+3) + \log_{10}(x-2) = 2 - \log_{10} 2$

10. Is there a $x \in \mathbb{Q}$ such that $2^{2x} \cdot 3^x = 144$?

Solve the following inequalities with the parameter $a \in \mathbb{R}$.

11. $\dfrac{x+a}{x} \leq x + 2$

12. $\sqrt{x+a} < x$

Solve the following exercises and represent their solutions graphically in $\mathbb{R}^2$.

13. $\sin y = \cos x$

14. $x^2 + y^2 \leq 4 \quad \& \quad x + y^2 > 1$

15. $\sqrt{x^2 + 4y^2 - 9} > x$

16. Let M denote the set of all men and W denote the set of all women. Consider the following predicate, where $m \in M$, $w \in W$:

$$\begin{aligned} S(m,w)&: \text{“The man } m \text{ is the husband of the woman } w\text{.”},\\ L_1(m,w)&: \text{“The man } m \text{ loves the woman } w\text{.”},\\ L_2(m,w)&: \text{“The woman } w \text{ loves the man } m\text{.”} \end{aligned}$$

Using quantifiers, logical conjunctions and the forms S, L_1, L_2 express the following claims:

(i) Every married man loves his wife.
(ii) There exists an unfaithful wife.
(iii) Every woman loves some man.

17. Find the supremum and infimum of the set
$$M = \left\{(-1)^n \cdot \frac{n}{n+1};\ n \in \mathbb{N}\right\}.$$

18. Find the supremum and infimum of the set $M = (0,1) \cup \{-1, 2\}$.

Solutions

1. $x = 3$ **2.** $x \in [1, 50)$ **3.** $x \in [-1, 3]$ **4.** $x = 3$ **5.** $x = 2\pi/3 + 2k\pi$, $x = -2\pi/3 + 2k\pi$, $k \in \mathbb{Z}$ **6.** $x = k\pi$, $x = \pi(4k+1)/8$, $k \in \mathbb{Z}$

7. $x \in (-\pi/2 + k\pi, \pi/6 + k\pi] \cup [\pi/3 + k\pi, \pi/2 + k\pi)$, $k \in \mathbb{Z}$

8. $x = \pi/4 + k\pi$, $x = 3\pi/2 + 2k\pi$, $k \in \mathbb{Z}$ **9.** $x = 7$ **10.** Ano, $x = 2$.

11.

$a \le -1/4$	$x \in (0, +\infty)$
$-1/4 < a < 0$	$x \in \left[\left(-1 - \sqrt{1+4a}\right)/2, \left(-1 + \sqrt{1+4a}\right)/2\right] \cup (0, +\infty)$
$a = 0$	$x \in [-1, 0) \cup (0, +\infty)$
$a > 0$	$x \in \left[\left(-1 - \sqrt{1+4a}\right)/2, 0\right) \cup \left[\left(-1 + \sqrt{1+4a}\right)/2, +\infty\right)$

12.

$a < -1/4$	$x \in [-a, +\infty)$
$-1/4 \le a < 0$	$x \in \left[-a, \left(1 - \sqrt{1+4a}\right)/2\right) \cup \left(\left(1 + \sqrt{1+4a}\right)/2, +\infty\right)$
$a \ge 0$	$x \in \left(\left(1 + \sqrt{1+4a}\right)/2, +\infty\right)$

13. $y = -x + \pi/2 + 2k\pi$, $y = x + \pi/2 + 2k\pi$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$

16. (i) $\forall m \in M\ \forall w \in W\colon S(m,w) \Rightarrow L_1(m,w)$
(ii) $\exists w \in W\ \exists m_1, m_2 \in M\colon m_1 \ne m_2\ \ \&\ \ S(m_1,w)\ \ \&\ \ L_2(m_2,w)$
(iii) $\forall w \in W\ \exists m \in M\colon L_1(m,w)$

17. $\sup M = 1$, $\inf M = -1$ **18.** $\sup M = 2$, $\inf M = -1$

CHAPTER 2

Sequences of real numbers

2.1. Convergence of sequences

Definition. If to every natural number n we assign a real number a_n, then we call $\{a_n\}_{n=1}^{\infty}$ a **sequence** of real numbers. The number a_n is called the $\boldsymbol{n}$**-th term** of this sequence. The sequence $\{a_n\}_{n=1}^{\infty}$ equals the sequence $\{b_n\}_{n=1}^{\infty}$, if $a_n = b_n$ for every $n \in \mathbb{N}$. In the following we will sometimes use the notation $\{a_n\}$ instead of $\{a_n\}_{n=1}^{\infty}$.

The set of all terms of a sequence $\{a_n\}_{n=1}^{\infty}$ is the set

$$\{x \in \mathbb{R};\ \exists n \in \mathbb{N}\colon x = a_n\}.$$

Remark. It is necessary to distinguish between the sequences $\{a_n\}_{n=1}^{\infty}$ and the set of the elements of the sequence. For example the set of the elements of the sequence $\{(-1)^n\}$ is $\{-1, 1\}$.

Example 1. The sequences $\{n\}$, $\{1/n\}$, $\{(-1)^n\}$ are given explicitly. A sequence may also be specified as follows: $a_1 = 1$, $a_{n+1} = (n+1)a_n$, $n \in \mathbb{N}$. This is an example of a sequence defined **recurrently**. It can easily be checked by mathematical induction that $a_n = 1 \cdot 2 \cdots (n-1) \cdot n$. We denote this number as $n!$ (and read it as n factorial). It is advantageous to make the definition $0! = 1$.

Definition. We say that the sequence $\{a_n\}$ is

- **bounded from above** , if the set of its elements is bounded from above,
- **bounded from below**, if the set of its elements is bounded from below,
- **bounded**, if the set of the elements of this sequence is bounded.

Definition. We say that the sequence $\{a_n\}$ is

- **non-decreasing**, if $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$,
- **increasing**, if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
- **non-increasing**, if $a_n \geq a_{n+1}$ for every $n \in \mathbb{N}$,
- **decreasing**, if $a_n > a_{n+1}$ for every $n \in \mathbb{N}$.

The sequence $\{a_n\}$ is **monotone**, if it satisfies one of the previous conditions. The sequence $\{a_n\}$ is **strictly monotone**, if it is increasing or decreasing.

Realize that the fact that a sequence is not decreasing does not mean that it is non-decreasing.

One of the important aspects of sequences which is often investigated is their asymptotic behavior for n "growing above all bounds". Consider the sequence of real numbers $\{a_n\}$, where $a_n = 1 + (-2/3)^n$ for $n \in \mathbb{N}$. The following picture shows the behavior of some of the first terms of this sequence.

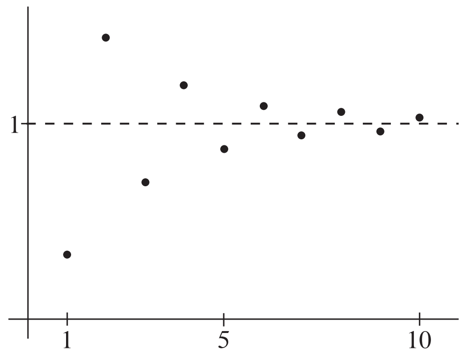


FIGURE 1.

It can easily be noticed that the terms of the sequence $\{a_n\}$ approaches the number 1 as n "grows". This intuitive idea can be expressed using exact mathematical terms.

Definition. Let $x_0 \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. The **neighborhood of the point** x_0 of radius ε denotes the set

$$B(x_0, \varepsilon) = \{x \in \mathbb{R};\ |x - x_0| < \varepsilon\}.$$

Remark. The neighborhood $B(x_0, \varepsilon)$ is actually the set of points on the real line whose distance from the point x_0 is smaller than the given positive number ε. It holds that

$$B(x_0, \varepsilon) = (x_0 - \varepsilon,\ x_0 + \varepsilon).$$

Definition. We say that the sequence $\{a_n\}$ has the **limit** equal to the real number A, if it holds that

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n \in B(A, \varepsilon).$$

Further wee will say that the sequence $\{a_n\}$ is **convergent**, if there exists an $A \in \mathbb{R}$, which is the limit of $\{a_n\}$.

In the previous definition the idea of the sequence $\{a_n\}$ "approaching" to the number A is understood as follows: whenever we choose a positive $\varepsilon \in \mathbb{R}$, then there must exist an index $n_0 \in \mathbb{N}$ such that, for all indexes $n \in \mathbb{N}$, which are greater or equal to n_0, it holds that a_n belongs to the interval $(A-\varepsilon, A+\varepsilon)$. The given n_0 will depend on the choice of ε, in general (see the following two pictures).

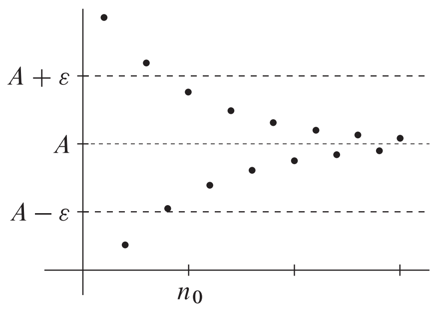


FIGURE 2.

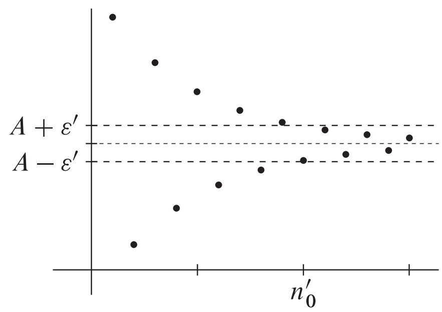


FIGURE 3.

Theorem 2 (uniqueness of the limit)**.** Every sequence has at most one limit.

Proof. Let $A, B \in \mathbb{R}$ be two limits of the same sequence $\{a_n\}$. Choose $\varepsilon > 0$. By the definition of the limit there exist natural numbers n_A, n_B such that for every index $n \in \mathbb{N}$, $n \geq n_A$, we have $|A - a_n| < \varepsilon/2$ and for every $n \in \mathbb{N}$, $n \geq n_B$, we have $|B - a_n| < \varepsilon/2$.

Choose $n_0 = \max\{n_A, n_B\}$. For indexes $n \geq n_0$ both of the prvious inequalities hold simultaneously, and therefore, by the triangle inequality, we have

$$0 \leq |A-B| \leq |A - a_{n_0}| + |B - a_{n_0}| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Because $|A-B| < \varepsilon$ for every ε, we see that $|A-B| = 0$, therefore $A = B$. ∎

The previous theorem allows us to introduce the following notation. If the sequence $\{a_n\}$ has a limit, then we denote it by the symbol $\lim\limits_{n\to\infty} a_n$ or simply $\lim a_n$.

Example 3. Prove that $\lim 1/n = 0$.

Proof. Choose $\varepsilon > 0$. We are to prove that there exists an n_0 such that for all $n \geq n_0$ we have $-\varepsilon < 1/n < \varepsilon$. The inequality on the left $-\varepsilon < 1/n$ is satisfied for all n natural. The right inequality $1/n < \varepsilon$ is satisfied if and only if $n > 1/\varepsilon$. It suffices, therefore, to choose n_0 as any natural number greater than $1/\varepsilon$, because for $n \geq n_0$ we have$1/n \leq 1/n_0 < \varepsilon$. Such a natural number n_0 must exist by Archimedes' axiom (Theorem 1.13). ∎

Example 4. Prove that the sequence $\{(-1)^n\}$ is not convergent.

Proof. Let us conduct our proof by contradiction. Let us assume that $\lim(-1)^n = A \in \mathbb{R}$. Choose $\varepsilon = 1/4$. Bn the definition of the limit there exists an $n_0 \in \mathbb{N}$ such that for all natural numbers $n \geq n_0$ we have $|A - (-1)^n| < 1/4$. By using the triangle inequality we get

$$2 = \left|(-1)^{n_0} - (-1)^{n_0+1}\right| \leq |(-1)^{n_0} - A| + \left|A - (-1)^{n_0+1}\right| < \\ < 1/4 + 1/4 = 1/2,$$

which is a contradiction. ∎

Remark. In Example 3 we verified the limit of the number 0, which we "guessed" is in fact the limit of the sequence $\{1/n\}$. In the following example we used the definition to prove that the limit of the sequence $\{(-1)^n\}$ cannot exist. The definition does not, however, tell us how we should go about finding the limit itself. We will therefore now try, in the following sections, to build a simple theory, whose theorems will clarify the basic properties of the limit and will sometimes be useful in calculating the limit of specific examples.

Example 5. Prove that $\lim a_n = a$, if and only if $\lim |a_n - a| = 0$.

Proof. By the definition of the limit it holds that $\lim a_n = a$, if and only if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon\ |a_n - a| < \varepsilon,$$

and $\lim |a_n - a| = 0$, if and only if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon\ \big||a_n - a| - 0\big| < \varepsilon.$$

Now the properties of the absolute value imply the desired equivalence. ∎

Theorem 6. Every convergent sequence is bounded.

Proof. Denote $\lim a_n = A \in \mathbb{R}$. Then there exists an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n > n_0$, we have $A - 1 < a_n < A + 1$. Put $h_1 = \max\{|A - 1|, |A + 1|\}$. It holds that

$$\forall n \in \mathbb{N}, n > n_0 \colon\ |a_n| \leq h_1. \tag{1}$$

Let $h_2 = \max\{|a_1|, |a_2|, \ldots, |a_{n_0}|\}$. It holds that

$$\forall n \in \{1, 2, \ldots, n_0\} \colon\ |a_n| \leq h_2. \tag{2}$$

For $h = \max\{h_1, h_2\}$ it follows from (1) and (2):

$$\forall n \in \mathbb{N} \colon\ |a_n| \leq h$$

and the sequence $\{a_n\}$ is bounded. ∎

Remark. A bounded sequence may non-convergent. An example is the sequence $\{(-1)^n\}$.

Definition. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence of natural numbers, then $\{a_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence** of $\{a_n\}_{n=1}^{\infty}$ or also **chosen sequence** of the sequence $\{a_n\}_{n=1}^{\infty}$.

Theorem 7. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$. If $\lim\limits_{n\to\infty} a_n = A \in \mathbb{R}$, then also $\lim\limits_{k\to\infty} a_{n_k} = A$.

Proof. Choose $\varepsilon > 0$. Then there exists an $n' \in \mathbb{N}$ such that
$$\forall n \in \mathbb{N}, n \geq n' \colon \ |A - a_n| < \varepsilon.$$
Then for all $k \geq n'$ we have $n_k \geq n'$, and therefore $|a_{n_k} - A| < \varepsilon$ and the theorem has been proven. ■

Remark. The previous theorem allows us to prove the claim about the non-existence of the limit $\lim(-1)^n$ differently than we did in Example 4. If we denote $a_n = (-1)^n$, $n \in \mathbb{N}$, Then $\lim\limits_{k\to\infty} a_{2k} = 1$ and also $\lim\limits_{k\to\infty} a_{2k+1} = -1$. This means that if $\lim(-1)^n = A \in \mathbb{R}$, then $1 = A$ and $-1 = A$, which is a contradiction.

Remark. Let $m_0 \in \mathbb{N}$ a $\lim a_n = A \in \mathbb{R}$. If we have, for a sequence $\{b_n\}$, that $a_n = b_n$ for all $n \in \mathbb{N}$, $n \geq m_0$, then also $\lim b_n = A$. This can be easily seen from the definition of the limit. If we choose $\varepsilon > 0$, then we can find the corresponding $n_0 \in \mathbb{N}$ satisfying
$$\forall n \in \mathbb{N}, n \geq n_0 \colon \ |a_n - A| < \varepsilon.$$
For all natural $n \geq \max\{m_0, n_0\}$ then it holds that
$$|b_n - A| = |a_n - A| < \varepsilon.$$
The above statement can be formulated also as follows: if we change a finite number of terms of a convergent sequence then the newly formed sequence will have the same limit since the two sequences equal each other after a certain given index.

Definition. Let $\{a_n\}$ and $\{b_n\}$ be two given sequences. We define

- the **sum** (or **difference**) of these sequences as
$$\{a_n\} \pm \{b_n\} = \{a_n \pm b_n\},$$
- the **product** of there sequences as
$$\{a_n\} \cdot \{b_n\} = \{a_n \cdot b_n\},$$
- the **quotient** of these sequences as
$$\{a_n\}/\{b_n\} = \{a_n/b_n\},$$

where the last is only defined if $b_n \neq 0$ for all $n \in \mathbb{N}$.

Theorem 8 (arithmetic of the lmit)**.** Let $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$. Then

(i) $\lim (a_n + b_n) = A + B$,
(ii) $\lim (a_n \cdot b_n) = A \cdot B$,
(iii) if $B \neq 0$ and $b_n \neq 0$ then for all $n \in \mathbb{N}$, we have $\lim(a_n/b_n) = A/B$.

Remark. The theorem says that *if its hypothesis are satisfied*, the limit of the sum (product, quotient) of two sequences equals the sum (product, quotient) of the limit of these sequences.

Proof of Theorem 8 (iii). Let us restrict ourselves to the proof of the statement about the limit of the quotient, which is the most complicated. We have to prove that

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists m \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq m\colon\ \left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \varepsilon. \tag{3}$$

We can use the estimates on the difference $|a_n - A|$ and $|b_n - B|$ – and we know these are small for large $n \in \mathbb{N}$. Our task then is to find an estimate of the absolute value in (3) by a suitable expression in which we can use the smallness of the differences $|a_n - A|$ a $|b_n - B|$. By putting the expression over a common denominator and adding zero in a clever way, i.e. $(0 = -AB + AB)$ we get

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \left|\frac{a_nB - b_nA}{b_nB}\right| = \left|\frac{a_nB - AB + AB - Ab_n}{b_nB}\right|.$$

Now we use the triangle inequality on the inequality we got. We get

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{A}{B}\right| &\leq \left|\frac{a_nB - AB}{b_nB}\right| + \left|\frac{AB - Ab_n}{b_nB}\right| = \\ &= \frac{1}{|b_n|}\,|a_n - A| + \frac{|A|}{|b_n|\,|B|}\,|b_n - B|\,.\end{aligned} \tag{4}$$

The right hand side of this equation contains "small" differences $|a_n - A|$ and $|b_n - B|$, the question now is whether the factors $1/|b_n|$ and $|A|/(|b_n|\,|B|)$ cannot "spoil" this in some way. But because $|A|/|B|$ is a real number, it suffices to think how the factor $1/|b_n|$ behaves as n grows . By the definition of the limit and from the inequality (i) from Corollary 1.17 we have, for the number $|B|/2$, an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that

$$|B| - |b_n| \leq \big||B| - |b_n|\big| \leq |B - b_n| < |B|/2.$$

>From here it follows for $n \geq n_0$ that $1/|b_n| < 2/|B|$. We use this estimate in the inequality (4) and if we put $K = \max\{2/|B|, 2|A|/|B^2|\}$, then we have

$$\begin{aligned}\forall n \in \mathbb{N}, n \geq n_0\colon\ \left|\frac{a_n}{b_n} - \frac{A}{B}\right| &\leq \frac{2}{|B|}\,|a_n - A| + \frac{2\,|A|}{|B^2|}\,|b_n - B| \leq \\ &\leq K \cdot (|a_n - A| + |b_n - B|).\end{aligned} \tag{5}$$

Let now $\varepsilon > 0$. There exist numbers $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, n \geq n_1\colon\ |a_n - A| < \frac{\varepsilon}{2K}, \tag{6}$$

$$\forall n \in \mathbb{N}, n \geq n_2\colon\ |b_n - B| < \frac{\varepsilon}{2K}. \tag{7}$$

Put $m = \max\{n_0, n_1, n_2\}$. For $n \geq m$ we have that the inequalities (5), (6) and (7) hold simultaneously. Therefore we have

$$\forall n \in \mathbb{N}, n \geq m\colon \left|\frac{a_n}{b_n} - \frac{A}{B}\right| < K \cdot \left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon.$$

Thereby the statement (iii) of Theorem 8 proven. ■

Remark. We have conducted the previous proof very thoroughly – which we will not always do, sometimes we will omit the proof completely. Mostly this will be for proofs that the reader could “think up” by themselves. If we leave out a harder proof, based on deeper ideas we will always draw attention to this fact. Try to prove statements (i) and (ii) from the previous theorem yourself.

Example 9. Theorem 8 answers the question of the limit of the product only when both sequences are convergent. There is no true general claim about the case where one or both of the sequences diverge. Let us look at some specific examples:

- For $\{a_n\} = \{(-1)^n\}$, $\{b_n\} = \{(-1)^n\}$ it holds that $\lim a_n b_n = 1$.
- For $\{a_n\} = \{n^2\}$ and $\{b_n\} = \{1/n\}$ it holds that$\lim a_n b_n = \lim n$ is not equal to any real number. This can be seen immediately from Theorem 6, because the sequence $\{n\}$ is not bounded.
- For $\{a_n\} = \{n\}$ and $\{b_n\} = \{1/n^2\}$ it holds that $\lim a_n b_n = \lim 1/n = 0$.

Think about similar examples for other arithmetic operations and try and construct some similar examples.

Example 10. Calculate $\lim \dfrac{n^3 + n}{2n^3 + 1}$.

Solution. The sequences $\{n^3 + n\}$ and $\{2n^3 + 1\}$ are not convergent because they are not bounded. Therefore we cannot use the theorem on the limit of the quotient directly. For large n the number n^3 will be much larger than n, in the expression $n^3 + n$ the “dominant role” will therefore be played by the term n^3. Similarly, in the expression $2n^3 + 1$ the “dominant role” will be played by the term $2n^3$. We will use this observation in the following calculation.

$$\lim \frac{n^3 + n}{2n^3 + 1} = \lim \frac{n^3\bigl(1 + \frac{1}{n^2}\bigr)}{n^3\bigl(2 + \frac{1}{n^3}\bigr)} = \lim \frac{1 + \frac{1}{n^2}}{2 + \frac{1}{n^3}} = \frac{1}{2}.$$

The final equality follows from Theorem 8 and from the Example 3. ♣

Sometimes the following result is useful.

Theorem 11. Let $\lim a_n = 0$ and let the sequence $\{b_n\}$ be bounded. Then it holds that $\lim a_n b_n = 0$.

Proof. Because $\{b_n\}$ is bounded there exists a positive number $K \in \mathbb{R}$, such that $|b_n| < K$ for all natural n. Choose $\varepsilon > 0$. Because $\lim a_n = 0$, given ε/K we can find a number $n_0 \in \mathbb{N}$ such that fir all natural $n \geq n_0$ we have $|a_n| = |a_n - 0| < \varepsilon/K$. For all $n \geq n_0$ therefore

$$|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| < (\varepsilon/K) \cdot K = \varepsilon.$$

We have proven that for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every natural $n \geq n_0$ we have $|a_n b_n - 0| < \varepsilon$. That means, by the very definition of the limit, that $\lim a_n b_n = 0$. ∎

Example 12. By the previous theorem it holds that $\lim \frac{\sin n}{n} = \lim \left(\frac{1}{n} \cdot \sin n\right) = 0$.

The previous theorems have put into context the limit and operation on the real numbers. Now we will look at the relationship between the limit and the order relation.

Theorem 13. Let $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$.

(i) Let there exist an $n_0 \in \mathbb{N}$ such that for all natural $n \geq n_0$ it holds that $a_n \geq b_n$. Then $A \geq B$.

(ii) Let $A < B$. Then there exists an $n_0 \in \mathbb{N}$ such that for all natural $n \geq n_0$ we have $a_n < b_n$.

Proof. Statement (i) is a corollary of statement (ii).

In order to prove (ii), we choose $\varepsilon = (B - A)/2$. >From the definition of the limit it follows that there exists a natural number n_0, for which we have

$$\forall n \in \mathbb{N}, n \geq n_0 \colon a_n < A + (B - A)/2 = B - (B - A)/2 < b_n.$$

∎

Theorem 14 (squeeze theorem)**.** Let $\{a_n\}$, $\{b_n\}$ be two convergent sequences and $\{c_n\}$ be such a sequence that:

(i) there exists an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$, it holds that $a_n \leq c_n \leq b_n$,

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ exists and it holds that $\lim c_n = \lim a_n$.

Proof. Denote $\lim a_n = A$. For any given number $\varepsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ it holds that

$$A - \varepsilon < a_n < A + \varepsilon \quad \text{and} \quad A - \varepsilon < b_n < A + \varepsilon.$$

>From here and from hypothesis (i) it follows for $n \geq \max\{n_0,\ n_1\}$

$$A - \varepsilon < a_n \leq c_n \leq b_n < A + \varepsilon,$$

therefore $c_n \in B(A, \varepsilon)$. ∎

2.2. Infinite limits

Besides the case where the terms of a sequence approach a real number A, it makes sense to consider other possible behavior of sequences.

Definition. We say that the limit of the sequence $\{a_n\}$ is $+\infty$ (we read **plus infinity**), if

$$\forall L \in \mathbb{R}\, \exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

We say that the limit of the sequence $\{a_n\}$ is $-\infty$ (we read **minus infinity**), if

$$\forall K \in \mathbb{R}\, \exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}, n \geq n_0 \colon a_n < K.$$

The theorem on the uniqueness of the limit holds also in the case that the limit may be plus or minus infinity.

Theorem 15 (uniqueness of the limit of sequence)**.** Every sequence has at most one limit.

Proof. Let us have the sequence $\{a_n\}$. We have shown that at most one real number can be the limit of the sequence $\{a_n\}$. It suffices to show that none of the following cases can occur:

(i) the sequence $\{a_n\}$ is convergent and simultaneously has the limit $+\infty$,
(ii) the sequence $\{a_n\}$ is convergent and simultaneously has the limit $-\infty$,
(iii) the sequence $\{a_n\}$ has the limit $+\infty$ and simultaneously $-\infty$.

Let us assume case (i) occurs. According to Theorem 6 there exists an $L \in \mathbb{R}$ such that $a_n \leq L$ for every $n \in \mathbb{N}$. Because the sequence $\{a_n\}$ has the limit $+\infty$, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $a_n > L$, which is a contradiction. Check yourself that none of the remaining possibilities can occur. ■

Remark. We also use the symbol $\lim a_n$ to denote the value of the limit of the sequence (if it exists). We can differentiate the behavior of a given sequence as follows:

$$\lim a_n \begin{cases} \text{exists and is } \begin{cases} \textbf{real}\text{, that is equal to a real number,} \\ \textbf{infinite}\text{, that is equal to } +\infty \text{ or } -\infty, \end{cases} \\ \text{does not exist.} \end{cases}$$

Example 16. Let us show that $\lim n^2 = +\infty$. Choose $L \in \mathbb{R}$. According to Theorem 1.13 there exists $n_0 \in \mathbb{N}$ such that $n_0 > L$. Then for every $n \in \mathbb{N}$ such that $n \geq n_0$ we have $n^2 \geq n_0^2 \geq n_0 > L$.

Now let us conduct the extension of the real axis with the elements $+\infty$ and $-\infty$ and state how the operations of addition and multiplication can be defined on these new elements. The **extended real axis**, i.e. the set $\mathbb{R} \cup \{+\infty, -\infty\}$, and we will denote it as $\mathbb{R}^*$. We extend the original order of the set $\mathbb{R}$ onto the elements $+\infty$ and $-\infty$ such that $-\infty < +\infty$ and for any $x \in \mathbb{R}$ it holds that $-\infty < x$ and $+\infty > x$.

Addition and multiplication on $\mathbb{R}^*$ is defined as:

- $\forall a \in \mathbb{R}^* \setminus \{+\infty\}\colon\ -\infty + a = a + (-\infty) = -\infty$,
- $\forall a \in \mathbb{R}^* \setminus \{-\infty\}\colon\ +\infty + a = a + (+\infty) = +\infty$,
- $\forall a \in \mathbb{R}^*, a > 0\colon a \cdot (\pm\infty) = \pm\infty$,
- $\forall a \in \mathbb{R}^*, a < 0\colon a \cdot (\pm\infty) = \mp\infty$,
- $\forall a \in \mathbb{R}\colon \frac{a}{\pm\infty} = 0$.

It can be seen from the above list that the operations addition and multiplication are not defined for all pairs of $\mathbb{R}^*$. For example the expressions $+\infty + (-\infty)$ or $0 \cdot (+\infty)$ are not defined. Why are some expressions defined and others not? The extension of the operations onto $\mathbb{R}^*$ have been conducted so that the following theorem could hold, whose proof is similar to Theorem 8.

Theorem 17 (arithmetic of the limit)**.** Let $\lim a_n = A \in \mathbb{R}^*$ and $\lim b_n = B \in \mathbb{R}^*$. Then it holds that:

(i) $\lim (a_n + b_n) = A + B$, if the right hand side is defined,
(ii) $\lim (a_n \cdot b_n) = A \cdot B$, if the right hand side is defined,
(iii) if $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\lim(a_n/b_n) = A/B$, if the right hand side is defined.

The expression $+\infty + (-\infty)$ is not defined because nothing can be extracted about $\lim(a_n + b_n)$ from the hypothesis that $\lim a_n = +\infty$ and $\lim b_n = -\infty$. In this context consider the following examples:

- if $\{a_n\} = \{n\}, \{b_n\} = \{-n\}$, then $\lim a_n = +\infty, \lim b_n = -\infty$ and $\lim(a_n + b_n) = 0$;
- if $\{a_n\} = \{n\}, \{b_n\} = \{-2n\}$, then $\lim a_n = +\infty, \lim b_n = -\infty$ and $\lim(a_n + b_n) = -\infty$;
- if $\{a_n\} = \{2n\}, \{b_n\} = \{-n\}$, then $\lim a_n = +\infty, \lim b_n = -\infty$ and $\lim(a_n + b_n) = +\infty$.

Similarly one can rationalize why some other expressions are not defined. One of these is the expression $A/0$. Therefore we cannot use Theorem 17 to calculate limits of the type "$A/0$", nevertheless the following variant of the theorem on the limit of quotients holds.

Theorem 18. Let $\lim a_n = A \in \mathbb{R}^*$, $A > 0$, $\lim b_n = 0$ and let there exist an $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$, it holds that $b_n > 0$. Then $\lim a_n/b_n = +\infty$.

Proof. We consider two cases.

1. Let us assume that $A \in \mathbb{R}$. We take $L \in \mathbb{R}$, $L > 0$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq n_1 \colon a_n > A - \frac{1}{2}A = \frac{1}{2}A,$$
$$\forall n \in \mathbb{N},\ n \geq n_2 \colon b_n < \frac{A}{2L}.$$

Then for all $n \in \mathbb{N}$, $n \geq \max\{n_0, n_1, n_2\}$, it holds that

$$\frac{a_n}{b_n} > \frac{\frac{1}{2}A}{\frac{A}{2L}} = L.$$

2. The case $A = +\infty$ still remains. Again we take $L \in \mathbb{R}$, $L > 0$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq n_1 \colon a_n > 1,$$
$$\forall n \in \mathbb{N},\ n \geq n_2 \colon b_n < \frac{1}{L}.$$

Then for every $n \in \mathbb{N}$, $n \geq \max\{n_0, n_1, n_2\}$, we have

$$\frac{a_n}{b_n} > \frac{1}{\frac{1}{L}} = L.$$

■

For infinite limits the following squeeze theorem holds.

Theorem 19. Let $\{a_n\}$ and $\{c_n\}$ be sequences satisfying:

(i) there exists an $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$, we have $a_n \leq c_n$,
(ii) $\lim a_n = +\infty$.

Then it holds that $\lim c_n = +\infty$.

Proof. Choose $L \in \mathbb{R}$. Then we find $n_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_1$, we have $a_n > L$. For every natural $n \geq \max\{n_0, n_1\}$, we have that $c_n \geq a_n > L$, by which we have proven that $\lim c_n = +\infty$. ■

According to the previous theorem it suffices to find only one "squeezing" sequence to prove the claim $\lim c_n = +\infty$. The following theorem is an obvious analogy for the previous theorem for the limit equal to $-\infty$.

Theorem 20. Let the sequence $\{a_n\}$ and $\{c_n\}$ satisfy the following conditions:

(i) there exists an $n_0 \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $n \geq n_0$, it holds that $a_n \geq c_n$,
(ii) $\lim a_n = -\infty$.

Then $\lim c_n = -\infty$.

Remark. It is not hard to show that the theorem on the limit of subsequences (Theorem 7) and the theorem on the limit and order (Theorem 13) also hold for infinite limits.

So far we have defined the term supremum for non-empty subsets of $\mathbb{R}$ bounded from above and the infimum for non-empty subsets of $\mathbb{R}$ bounded from below. Now we introduce the term suprema and infima also for unbounded sets of real numbers.

Definition. Let $M \subset \mathbb{R}$ be a non-empty set. If M is not bounded from above then we define $\sup M = +\infty$. If M is not bounded from below then we define $\inf M = -\infty$.

Remark. Notice that this definition is in agreement with the claim that the supremum is the smallest upper bound of the set. Let $M \subset \mathbb{R}$. If we look at M as a subset of $\mathbb{R}^*$, then by the definition of the order $+\infty$ is an upper bound of M. If the set M is not bounded from above, then there is no upper bound for M in the real numbers, and so the element $+\infty$ is the smallest upper bound M.

The following claim shows an interconnection between the terms limit of a sequence and supremum of a set.

Lemma 21. Let $M \subset \mathbb{R}$ be a non-empty set, $G \in \mathbb{R}^*$. Then the following claims are equivalent:

(i) $G = \sup M$.
(ii) The element G is an upper bound of M and there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in M, for which $\lim x_n = G$.

Proof. If $G = \sup M$, then G is obviously an upper bound of M.

If $G = +\infty$, then M is not bounded from above and therefore for every $n \in \mathbb{N}$ there exists an $x_n \in M$, $x_n > n$. According to Theorem 19 we have $\lim x_n = +\infty = G$.

In the case that $G \in \mathbb{R}$, then according to the definition of the supremum, for every $n \in \mathbb{N}$ we can find an $x_n \in M$, $x_n > G - 1/n$. Because G is an upper bound M, is automatically $x_n \leq G$ for every $n \in \mathbb{N}$. According to Theorem 14 we have that $\lim x_n = G$.

Now let us prove the opposite implication. Because G is an upper bound of the non-empty set it holds that $G \in \mathbb{R} \cup \{+\infty\}$. If $G = +\infty$, then the sequence $\{x_n\}$ is not bounded from above by the definition of the limit. The set M is therefore not bounded from above and therefore $\sup M = +\infty$.

Now assume that $G \in \mathbb{R}$. Condition (i) from the definition of the supremum is satisfied by our hypothesis. In order to check condition (ii) we choose any number $G' \in \mathbb{R}$, $G' < G$. Then from the definition of the limit is follows that there exists an $n_0 \in \mathbb{N}$ for which $x_{n_0} > G'$ (it suffices to take $\varepsilon = G - G'$). We have found an element in M, which is greater then G', and so we have verified condition (ii) from the definition of the supremum. ■

Remark. An analogical claim holds for the infimum of course.

In the following examples we will show how the previous theorems can be used.

Example 22. It holds that

$$\lim_{n\to\infty} q^n = \begin{cases} +\infty, & \text{if } q > 1, \\ 1, & \text{if } q = 1, \\ 0, & \text{if } q \in (-1, 1), \\ \text{does not exist}, & \text{if } q \leq -1. \end{cases}$$

Proof. Firstly let us consider the case $q > 1$. Here we can write $q = 1 + h$, where $h > 0$. By the binomial theorem we have $q^n = (1+h)^n \geq 1 + hn$ for every $n \in \mathbb{N}$. Of course $\lim\limits_{n\to\infty}(1 + hn) = +\infty$, and according to Theorem 19 we have $\lim q^n = +\infty$.

If $q \in (0, 1)$, then by the previous paragraph $\lim (q^{-1})^n = +\infty$. By applying Theorem 17 we get

$$\lim q^n = \lim \frac{1}{(q^{-1})^n} = 0.$$

The cases $q = 0$, $q = 1$ and $q = -1$ are completely obvious. If $q \in (-1, 0)$, then $\lim |q^n| = \lim |q|^n = 0$, and therefore $\lim q^n = 0$ (Example 5).

Finally, if $q < -1$, then we have $\lim q^{2n} = \lim (q^2)^n = +\infty$ and conversely $\lim q^{2n+1} = \lim q(q^2)^n = -\infty$. We have found two subsequences with differing limits and therefore $\lim q^n$ does not exist. ■

Example 23. Let $\{a_n\}$ be a sequence of non-negative numbers having the limit $A \in \mathbb{R}$. Then $\lim \sqrt{a_n} = \sqrt{A}$.

Proof. Let us assume first that $A = 0$. We choose $\varepsilon > 0$ arbitrarily. Then there is an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$, it holds that $|a_n| < \varepsilon^2$. For every $n \in \mathbb{N}$ it holds that $\left|\sqrt{a_n}\right| < \varepsilon$. This consideration shows that in the case $A = 0$ we have $\lim \sqrt{a_n} = \sqrt{A}$. Now assume that $A > 0$. Then for every $n \in \mathbb{N}$ it holds that

$$\left|\sqrt{a_n} - \sqrt{A}\right| = \left|\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}\right| \leq \frac{1}{\sqrt{A}} \left|a_n - A\right|.$$

>From here and from the fact that $\lim \frac{1}{\sqrt{A}} |a_n - A| = 0$ it follows that $\lim \left|\sqrt{a_n} - \sqrt{A}\right| = 0$ by Theorem 14. Therefore $\lim \sqrt{a_n} = \sqrt{A}$.

The claim of the example can be extended as follows. Let $k \in \mathbb{N}$ and $\{a_n\}$ is a sequence of non-negative numbers with the limit equal to $A \in \mathbb{R}^*$. Then

$$\lim \sqrt[k]{a_n} = \begin{cases} \sqrt[k]{A}, & \text{if } A \in \mathbb{R}, \\ +\infty, & \text{if } A = +\infty. \end{cases}$$

We can prove this using Theorem 4.16, the notes on page 67 and Example 4.32. ■

Example 24. It holds that $\lim \sqrt[n]{n} = 1$.

Proof. With respect to the fact that $\sqrt[n]{n} \geq 1$, we can write $\sqrt[n]{n} = 1 + \eta_n$, where $\eta_n \geq 0$. For $n \in \mathbb{N}$, $n \geq 2$, it holds that

$$n = (1+\eta_n)^n \geq 1 + n\eta_n + \frac{n(n-1)}{2}\eta_n^2 \geq \frac{n(n-1)}{2}\eta_n^2 \geq \frac{n^2}{4}\eta_n^2.$$

>From here we get for $n \geq 2$ the inequality $\frac{2}{\sqrt{n}} \geq \eta_n$. According to the previous example and Theorem 8 we get $\lim \frac{2}{\sqrt{n}} = 0$, so according to Theorem 14 we have $\lim \eta_n = 0$. >From whence we get our claim. ■

Example 25. Calculate $\lim\bigl(\sqrt{4n^2 - n} - 2n\bigr)$.

Solution. We cannot use the theorems on the quotient of limits because

$$\lim \sqrt{4n^2 - n} = \lim \sqrt{n(4n-1)} = +\infty \qquad \text{and} \qquad \lim 2n = +\infty.$$

We will try to alter it into a form where we can directly use the theorem on the arithmetic of limits.

First we multiply the n-th term of our sequence by one, but written in the following form

$$\frac{\sqrt{4n^2-n}+2n}{\sqrt{4n^2-n}+2n}$$

and we use the expression $(a-b)(a+b) = a^2 - b^2$, where we put $a = \sqrt{4n^2 - n}$, $b = 2n$. Thus we get

$$\left(\sqrt{4n^2-n} - 2n\right) \cdot \frac{\sqrt{4n^2-n}+2n}{\sqrt{4n^2-n}+2n} = \frac{-n}{\sqrt{4n^2-n}+2n}.$$

We multiply on both sides by $1/n$ and get

$$\sqrt{4n^2-n} - 2n = \frac{-1}{\sqrt{4-\frac{1}{n}}+2}.$$

Because the last equality for all $n \in \mathbb{N}$, by Theorem 17 and Examples 23 we get

$$\lim\bigl(\sqrt{4n^2-n} - 2n\bigr) = \lim \frac{-1}{\sqrt{4-\frac{1}{n}}+2} = -1/4.$$

♣

Example 26. Calculate $\lim \dfrac{n\sqrt{2n+5} - 3\sqrt[3]{2n}}{\sqrt{n^3+2} + \sqrt[3]{n^4}}$.

Solution. In order to understand the quotient better let us try a "dry-run" of the calculations. The numbers 5 and 2, which appear under the roots are small in comparison with n, which grows above all bounds. The following terms appear in the quotient n: $n^{3/2}$, $n^{1/3}$ and $n^{4/3}$. The highest exponent is $3/2$. Now we will continue similarly as in Example 10. Thus we get

$$\frac{n\sqrt{2n+5}-3\sqrt[3]{2n}}{\sqrt{n^3+2}+\sqrt[3]{n^4}}=\frac{n^{3/2}\big(\sqrt{2+5/n}-3\sqrt[6]{4/n^7}\,\big)}{n^{3/2}\big(\sqrt{1+2/n^3}+\sqrt[6]{1/n}\,\big)}.$$

The limit of the last expression can be calculated easily using Theorem 17 on the arithmetic of the limit and Example 23:

$$\lim\frac{\sqrt{2+5/n}-3\sqrt[6]{4/n^7}}{\sqrt{1+2/n^3}+\sqrt[6]{1/n}}=\sqrt{2}.$$

♣

2.3. Deeper theorems on limits

Let us realize the following important property of monotone sequences.

Theorem 27. Every monotone sequence has a limit. If $\{a_n\}$ is non-decreasing then $\lim a_n = \sup\{a_n;\ n\in\mathbb{N}\}$. If $\{a_n\}$ is non-increasing then $\lim a_n = \inf\{a_n;\ n\in\mathbb{N}\}$.

Proof. Let the sequence $\{a_n\}$ be non-decreasing. Let us firstly assume that the sequence is not bounded from above, which means that $\sup\{a_n;\ n\in\mathbb{N}\}=+\infty$. Let us therefore prove that $\lim a_n = +\infty$. Let us choose the number $L\in\mathbb{R}$. Because $\{a_n\}$ is not bounded from above we can surely find an index n_0 such that $a_{n_0} > L$. Because $\{a_n\}$ is non-decreasing, however, it holds that $a_n \ge a_{n_0} > L$ for every $n\ge n_0$. Thus it has been proven that $\lim a_n = +\infty$.

If $\{a_n\}$ is bounded from above then we put $A=\sup\{a_n;\ n\in\mathbb{N}\}\in\mathbb{R}$. We will prove that the number A is the limit of our sequence. Let us choose some $\varepsilon>0$. Because $A-\varepsilon<A$, there exists an element of $\{a_n;\ n\in\mathbb{N}\}$, call it a_{n_0}, such that $A-\varepsilon<a_{n_0}$. On the other hand $\{a_n\}$ is non-decreasing and $A-\varepsilon<a_{n_0}\le a_n$ for all $n\ge n_0$. The inequality $a_n<A+\varepsilon$ holds for all $n\in\mathbb{N}$, because A is an upper bound for the entire set of elements of the sequence $\{a_n\}$. For a chosen $\varepsilon>0$ we have found an $n_0\in\mathbb{N}$ such that

$$\forall n\in\mathbb{N}, n\ge n_0\colon\ A-\varepsilon<a_n<A+\varepsilon.$$

Thus it has been proven that the number A is the limit of the sequence $\{a_n\}$.

The claim for non-increasing sequences can been proven similarly. The other possibility is to consider the sequence $\{-a_n\}$ and use the already proven claim for non-decreasing sequences and Theorem17. ■

Remark. The importance of the previous theorem lies in the fact that it allows one to show the existence of a limit of a sequence without needing to calculate its value. The information of the existence of the limit is sometimes useful in itself. In some cases this information allows us to calculate the exact value of the limit in further calculations, see, for instance, the following example.

Example 28. Choose the number $c > 0$ and examine the limit of the sequence $\{a_n\}$, which is determined in the following way:

$$a_1 = \sqrt{c}, \qquad a_{n+1} = \sqrt{a_n + c} \quad \text{for every } n \in \mathbb{N}. \tag{8}$$

Solution. First we realize that the sequence $\{a_n\}$ is well defined. The first term is defined explicitly and is non-negative. If we assume that a_n is defined and is non-negative then a_{n+1} is also defined and non-negative. According to the principle of mathematical induction, the sequence $\{a_n\}$ is defined and is non-negative.

Let us assume for a moment that the sequence $\{a_n\}$ has a real limit. We denote the limit by the letter A. From (8) it follows that for all $n \in \mathbb{N}$ we have $a_{n+1}^2 = a_n + c$. According to Theorem 7 on the limit of subsequences we have $\lim a_{n+1} = A$ and according to Theorem 8 about the arithmetic of the limit we get $\lim a_{n+1}^2 = A^2$ and $\lim(a_n + c) = A + c$. Then we have $A^2 = A + c$, from here we calculate that A is either equal to $\bigl(1+\sqrt{1+4c}\bigr)/2$, or $\bigl(1-\sqrt{1+4c}\bigr)/2$. The second of those values is negative and as such cannot be the limit of the sequence, since all its terms are non-negative. It would be a contradiction with Theorem 13. If the sequence $\{a_n\}$ has a real limit then the value must be $\bigl(1+\sqrt{1+4c}\bigr)/2$.

It remains to prove that our hypothesis on the existence of the limit is indeed satisfied. To do this we use Theorem 27. The sequence $\{a_n\}$ is increasing. Since $a_1 < a_2$ and if $a_{n-1} < a_n$, then also

$$a_n = \sqrt{a_{n-1} + c} < \sqrt{a_n + c} = a_{n+1},$$

so we see that our sequence is increasing by the principle of mathematical induction. According to Theorem 27 the sequence $\{a_n\}$ has a limit $\lim a_n = \sup\{a_n;\ n \in \mathbb{N}\}$.

The sequence $\{a_n\}$ is bounded from above. We have $a_1 < \sqrt{c} + 1$ and if $a_n < \sqrt{c} + 1$, then

$$\begin{aligned} a_{n+1} = \sqrt{a_n + c} &< \sqrt{\sqrt{c} + 1 + c} < \\ &< \sqrt{c + 2\sqrt{c} + 1} = \sqrt{\bigl(\sqrt{c}+1\bigr)^2} = \sqrt{c} + 1. \end{aligned}$$

>From the principle of mathematical induction it follows that for all $n \in \mathbb{N}$ we have $a_n < \sqrt{c}+1$, and therefore $\{a_n\}$ is bounded from above, so $\sup\{a_n;\ n \in \mathbb{N}\} \in \mathbb{R}$. Therefore we also have $\lim a_n \in \mathbb{R}$. ♣

Example 29. Determine for which real numbers x the sequence $\{\bigl(x^3/(3x-2)\bigr)^n\}$ is monotone.

Solution. In order for the quotient to make sense we have to eliminate $x = 2/3$. For all other values of $x \in \mathbb{R}$ it is simply a geometric sequence. Note that the sequence $\{q^n\}$ is monotone if and only if $q \geq 0$. If $q = 0$ or $q = 1$, the sequence is **constant** (i.e. the set of elements of the terms of the sequence has only one element), for $0 < q < 1$ the sequence is decreasing and for $q > 1$ it is increasing.

It is therefore necessary to conduct a discussion and find for which $x \in \mathbb{R}$ the values of the quotient lie in the interval $(0,1)$, or in the interval $(1,+\infty)$, and for which x the value is equal to 0 or 1. By solving the corresponding inequalities we get that the sequence $\{\bigl(x^3/(3x-2)\bigr)^n\}$ is increasing for $x \in (-\infty,-2) \cup (2/3,1) \cup (1,+\infty)$ and decreasing for $x \in (-2,0)$. For the values $x = -2$, $x = 0$ and $x = 1$ the sequence is constant. For the remaining real numbers x the sequence is not monotone. ♣

We now include an important property of bounded sequences, whose meaning will become apparent in the following chapters.

Theorem 30 (Bolzano-Weierstraß theorem)**.** Let $\{x_n\}$ be a bounded sequence of real numbers. Then there exists a convergent subsequence $\{x_{n_k}\}$.

Proof. The sequence $\{x_n\}$ is bounded and therefore there exists an interval $[A,B]$ containing all the terms of the sequence $\{x_n\}$. We construct the intervals $\{[a_k,b_k]\}_{k=1}^{\infty}$ satisfying the following two conditions for all $k \in \mathbb{N}$:

(i) the set $I_k = \{n \in \mathbb{N};\ x_n \in [a_k,b_k]\}$ is infinite,
(ii) the interval $[a_{k+1},b_{k+1}]$ is either equal to $[a_k,(a_k+b_k)/2]$, or $[(a_k+b_k)/2,b_k]$.

For $k = 1$ it suffices to put $a_1 = A$ and $b_1 = B$. Let us have, for some $k \in \mathbb{N}$, the interval $[a_k,b_k]$ satisfying condition (i). If the set of indexes $\{n \in \mathbb{N};\ x_n \in [a_k,(a_k+b_k)/2]\}$ is infinite then we put $a_{k+1} = a_k$ and $b_{k+1} = (a_k+b_k)/2$. If this is not so then from the property (i) we get that there is an infinite set of indexes $\{n \in \mathbb{N};\ x_n \in [(a_k+b_k)/2,b_k]\}$. In this case we put $a_{k+1} = (a_k+b_k)/2$ and $b_{k+1} = b_k$. By this the convergence of the sequence of intervals is achieved.

Now (again by induction) we construct an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $x_{n_k} \in [a_k,b_k]$ for every $k \in \mathbb{N}$. The element n_k can be chosen arbitrarily from the infinite (and so non-empty) set $I_k \setminus \{1,2,\ldots,n_{k-1}\}$.

Further we see by (ii) that the sequence $\{a_k\}_{k=1}^{\infty}$ is non-decreasing and bounded and the sequence $\{b_k\}_{k=1}^{\infty}$ is non-increasing and is bounded. By the theorem on the limit of monotone sequences (Theorem 27) it holds that both sequences have a real limit. Denote $\lim_{k\to\infty} a_k = \alpha$ and $\lim_{k\to\infty} b_k = \beta$. Further we have $b_k - a_k = 2^{-(k-1)}(B - A)$ for every $k \in \mathbb{N}$, and so $\lim(b_k - a_k) = 0$, which by Theorem 8 means that $\alpha = \beta$. We also know that $a_k \le x_{n_k} \le b_k$ for every $k \in \mathbb{N}$. The squeeze theorem (Theorem 14) gives that the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to α. ■

2.4. Exercises

Decide whether the following sequences are monotone.

1. $\{2n + (-1)^n\}$

2. $\left\{(n+1)/\sqrt{n^2 + 2n - 2}\right\}$

Calculate the following limits.

3. $\lim \dfrac{3n^2 + 5n}{-n^2 + 4n + 1}$

4. $\lim \dfrac{3^n + 5^n + 10^n}{(-2)^{n+1} + 5^{n+1} + 10^{n+1}}$

5. $\lim \left(\dfrac{4 + (-1)^n}{-7}\right)^n$

6. $\lim \dfrac{1}{n} \cdot \sin n^2$

7. $\lim \cos^2(n\pi/4)$

8. $\lim(n + \cos n)$

9. $\lim \dfrac{1 + 2 + \cdots + n}{\sqrt[3]{8n^6 - n}}$

10. $\lim \dfrac{2^n + (-2)^n}{3^n}$

11. $\lim \left(\sqrt{n+5} - \sqrt{n+1}\right)$

12. $\lim \left(\sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}\right)$

Solutions

1. The sequence is non-decreasing. **2.** The sequence is decreasing. **3.** -3
4. $1/10$ **5.** 0 **6.** 0 **7.** Does not exist. **8.** $+\infty$ **9.** $1/4$ **10.** 0
11. 0 **12.** 0

CHAPTER 3

Mappings

The aim of this chapter is to introduce the concept of a mapping and other closely related terms. A more abstract approach allows us to include a wide range of specific cases (functions of one real variable, functions of multiple variables, linear mappings), which we will be interested in the course of the text.

Definition. Let A and B be a pair of sets. **A mapping f from the set A into the set B** assigns every element x of the set A one element y from the set B. The element y is denoted as $f(x)$.

If $x \in A$, $y \in B$ and it holds that $y = f(x)$, then the element y is called the **image** of the element x and the element x is called the **pre-image** of the element y.

Let us introduce a few more notations:

- The symbol $f\colon A \to B$ means that the mapping f is a mapping of the set A into the set B.
- The symbol $f\colon x \mapsto f(x)$ means that the mapping f assigns the element$f(x)$ to the element x.
- The set A from the definition is called the **domain** of the mapping f and we also use the symbol D_f to denote this set.

Example 1. 1. Let us consider formula $x \mapsto x^2$. This makes sense for all $x \in \mathbb{R}$, therefore we can use it to define a mapping $f\colon \mathbb{R} \to \mathbb{R}$. the set which we are mapping into can be taken to be $[0, +\infty)$. Here we firstly took the allocation and then asked for what x it is defined. The domain of the mapping was determined as the largest possible set on which the formula made sense. We will use this same approach in the following as well. Notice that the target set B from the definition does not have to be covered entirely by the values $f(x)$.

2. Let us consider the formula $x \mapsto \sqrt{x}$. The domain of the corresponding mapping is$D_f = [0, +\infty)$ (this follows from Theorem 1.14 on the n-th root), and we can take the target set to be $\mathbb{R}$ for example.

3. Notice that sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ are actually mappings from the set $\mathbb{N}$ into the set $\mathbb{R}$. More generally, **sequences of elements of a set** M are mappings from the set $\mathbb{N}$ into the set M.

4. Let A be the set of all convergent sequences of real numbers $\{a_n\}$, $B = \mathbb{R}$, $f\colon \{a_n\} \mapsto \lim a_n$. The mapping f allocates each convergent sequence its limit.

5. Let A be the set of all non-empty subsets of $\mathbb{R}$ bounded from above, $B = \mathbb{R}$, $f\colon M \mapsto \sup M$. The mapping f allocates each set of real numbers bounded from above the supremum of the set.

6. Let $A = \mathbb{R} \times (0, +\infty)$, and B be the set of all open and bounded intervals and $f\colon [a, \varepsilon] \mapsto (a - \varepsilon, a + \varepsilon)$.

Definition. Let $f\colon A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A,\ y = f(x)\}$ of the Cartesian product $A \times B$ is called the **graph of the mapping** f.
- Let $M \subset A$. The **image** $f(M)$ of the **set** M in the mapping f is the set
$$\{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$
- The set $f(A)$ is called the **range** of the mapping f. We denote it with the symbol R_f.
- Let $W \subset B$. The **pre-image** $f_{-1}(W)$ of a **set** W in the mapping f is the set $\{x \in A;\ f(x) \in W\}$.

Example 2. Let $f\colon A \to B$ be a mapping, $X, Y \subset A$ and $U, V \subset B$. Then it holds that:

- $f_{-1}(U \cup V) = f_{-1}(U) \cup f_{-1}(V)$ (the pre-image of the union of two sets is the union of the pre-images of their sets),
- $f_{-1}(U \cap V) = f_{-1}(U) \cap f_{-1}(V)$ (the pre-image of the intersection of two sets is the intersection their pre-images),
- $f(X \cup Y) = f(X) \cup f(Y)$ (The image of the union of two sets is the union of the image of the sets),
- $f(X \cap Y) \subset f(X) \cap f(Y)$.

The proofs of these claims is not hard to prove. Work out that the opposite inclusion in the final claim is generally not true.

Later we will use the following terms.

Definition. Let A, B, C be sets, $C \subset A$ and $f\colon A \to B$ is a mapping of the set A into the set B. The **restriction** of the **mapping** f onto the set C is the mapping $\tilde{f}\colon C \to B$ defined by the formula $\tilde{f}(x) = f(x)$ for all $x \in C$. The restriction of f onto C is denoted as $f|_C$.

Definition. Let $f\colon A \to B$ and $g\colon B \to C$ be two mappings. The symbol $g \circ f$ denotes the mapping of the set A into the set C defined by the formula
$$g \circ f\colon x \mapsto g\bigl(f(x)\bigr).$$
A mapping defined this way is called a **compound mapping**.

Remark. Let $f\colon A \to B$ and $g\colon C \to D$. If $B \subset C$, then the compound mapping $g \circ f$ is defined as above since we can interpret f as a mapping into C. If it does not hold that $B \subset C$, then the compound mapping $g \circ f$ will be understood to by the mapping $g \circ (f|_{\tilde{A}})$, where $\tilde{A} = \{a \in A;\ f(a) \in C\}$.

The terms in the following definition are very important.

Definition. We say that the mapping $f\colon A \to B$

- maps the set A **onto the set B**, if $f(A) = B$, i.e. for every $y \in B$ there exists an $x \in A$ such that $f(x) = y$,
- is **injective** (also **one-to-one**), if every pair of distinct points has a pair of mutually distinct images i.e.
$$\forall x_1, x_2 \in A\colon x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$
- is a **bijection A onto B** (or also a **one-to-one correspondance**), if it is one-to-one and maps A onto B.

Remark. Many problems in mathematics and its applications can be formulated as the following problem. For a given mapping $f\colon A \to B$ and $y \in B$ find the (or a) solution to the equation
$$f(x) = y, \quad x \in A. \tag{1}$$
We want the elements x, such that the value in the mapping $f(x)$ is equal to a given right hand side y. Such an x is then called a solution of the equation (1) with the right hand side y. The following two questions are important:

(i) Does there *exist* a solution for every right hand side y from the set B?
(ii) *How many* for a given right hand side can the equation (1) permit?

If f maps the set A onto the set B, then the first (existential) question is answered in the positive. If the mapping f is one-to-one, then the equation (1) has at most one solution for each right hand side y– i.e. it either has no solution or only one.

If we can show that the mapping f is injective and **onto** (i.e. f maps A onto B), then the equation (1) has exactly one solution for every right hand side, and thus we answer questions (i) and (ii).

On the basis of what has just been said about an injective mapping of A onto B, it is clear that we can associate a mapping that assigns each right hand side y of the equation (1) its solution x. Let us make this definition in more detail.

Definition. Let $f\colon A \to B$ be an injective mapping. To every element $y \in R_f$ we assign the element $x \in A$ satisfying $f(x) = y$. (We know that there is exactly one such element.) We denote this element x with the symbol $f^{-1}(y)$. Thus we have defined the mapping $f^{-1}\colon R_f \to A$, which is called the **inverse mapping** to the mapping f.

The inverse mapping f^{-1} is an injective mapping of the set R_f onto the set A. For every two elements $x \in A, y \in R_f$ it holds that $y = f(x)$, if and only if $x = f^{-1}(y)$.

Remark. The symbol Id_M means the mapping defined on the set M using the formula $\mathrm{Id}_M \colon x \mapsto x$. We call it the **identity mapping on the set M**. If $f \colon A \to B$ is a bijection then one can easily see that the following equalitlies hold:

$$f^{-1} \circ f = \mathrm{Id}_A, \qquad f \circ f^{-1} = \mathrm{Id}_B\,.$$

Directly from the definition of the inverse mapping it follows that $(f^{-1})^{-1} = f$ for any injective mapping f.

The concept of the inverse mapping is very important but one has to be somewhat careful in its application.

Example 3. Let us consider the mapping $f \colon \mathbb{R} \to [0, +\infty)$, $f(x) = x^2$. This mapping is onto the set $[0, +\infty)$, but it is not injective on its domain. Its inverse mapping therefore does not exist. Let us use the symbol f_1 to denote the restriction of the mapping f onto the set $[0, +\infty)$, that is the mapping

$$\begin{aligned} &f_1 \colon [0, +\infty) \to [0, +\infty),\\ &f_1 \colon x \mapsto x^2. \end{aligned}$$

The mapping f_1 is injective from$[0, +\infty)$ onto $[0, +\infty)$, and $f_1(x) = y$, if and only if $x = \sqrt{y}$. Therefore

$$\begin{aligned} &f_1^{-1} \colon [0, +\infty) \to [0, +\infty),\\ &f_1^{-1} \colon y \mapsto \sqrt{y}. \end{aligned}$$

Similarly let

$$\begin{aligned} &f_2 \colon (-\infty, 0] \to [0, +\infty),\\ &f_2 \colon x \mapsto x^2, \end{aligned}$$

we easily get

$$\begin{aligned} &f_2^{-1} \colon [0, +\infty) \to (-\infty, 0],\\ &f_2^{-1} \colon y \mapsto -\sqrt{y}. \end{aligned}$$

Remark. Let $A \subset \mathbb{R}$, $B \subset \mathbb{R}$ and $f \colon A \to B$ be a bijection. We recall the definition of the graph, it is

$$\begin{aligned} G_f &= \{[x, y] \in \mathbb{R}^2;\ x \in A, y = f(x)\},\\ G_{f^{-1}} &= \{[u, v] \in \mathbb{R}^2;\ u \in B, v = f^{-1}(u)\}. \end{aligned}$$

Let us now draw both graphs in one picture on the plane, then G_f is symmetrical with $G_{f^{-1}}$ with respect to the axis of the first quadrant.

Example 4. Let A be the set of all lines which can be written in point-slope form (i.e. in the form $p = \{[x, y];\ x \in \mathbb{R}, y = kx + q\}$, where $k \in \mathbb{R}, q \in \mathbb{R}$). Is the set $G = \{[p, k];\ p \in A, k \text{ is the gradient of } p\} \subset A \times \mathbb{R}$ the graph of a mapping?

Solution. The set G is the graph of a mapping because for every p from A there exists a gradient k and if $[p, k_1] \in G, [p, k_2] \in G$, then $k_1 = k_2$, because each line in A has only one gradient. The mapping defined by the set G is the mapping $f\colon A \to \mathbb{R}$, which for every line in point-slope form assigns its gradient k, thus $f\colon p \mapsto k$.

The mapping f is a mapping from the set A onto the set $\mathbb{R}$, because for every real number k there exists in A a line p, whose gradient is the number k (for example $p = \{[x, y] \in \mathbb{R}^2;\ x \in \mathbb{R}, y = kx\}$). The mapping f is not injective however because one can find two distinct lines $p_1 \in A$ and $p_2 \in A$, such that $f(p_1) = f(p_2)$, because they have the same gradient. They are for example the lines $p_1 = \{[x, y];\ x \in \mathbb{R}, y = x\}$, $p_2 = \{[x, y];\ x \in \mathbb{R}, y = x + 1\}$. $\clubsuit$

Example 5. Let us define compound mappings $f \circ g$ and $g \circ f$, where the mappings f and g are given by the formulas $f\colon x \mapsto \operatorname{tg} x$, $g\colon x \mapsto \sqrt{x}$.[1]

Solution. Concerning $f \circ g$, we have to find all the $x \in \mathbb{R}$ for which $\sqrt{x}$ is in the domain of tg. >From the domain of the mapping g, which is the interval $[0, +\infty)$, we must subtract those points $\big((2k-1)\pi/2\big)^2$, where $k \in \mathbb{N}$. The domain of the compound function $f \circ g\colon x \mapsto \operatorname{tg}\big(\sqrt{x}\big)$ will therefore be the set

$$A = \big[0, (\pi/2)^2\big) \cup \bigcup_{k\in\mathbb{N}} \Big(\big((2k-1)\pi/2\big)^2, \big((2k+1)\pi/2\big)^2\Big).$$

The compound mapping $g \circ f$ will be investigated more quickly. The domain is the set of real numbers from the domain of the mapping tg, whose values in the mapping tg are non-negative. The domain therefore is the set

$$B = \bigcup_{k\in\mathbb{Z}} [k\pi, k\pi + \pi/2).$$ $\clubsuit$

Example 6. Let us have the formula

$$f\colon [x, y] \mapsto \sqrt{\frac{x^2 + 2x + y^2}{x^2 - 2x + y^2}}.$$

determine the domain of f and the pre-images of the singleton-sets $\{0\}$ and $\{1\}$.

Solution. The domain is the set

$$D_f = \left\{[x, y] \in \mathbb{R}^2;\ \frac{x^2 + 2x + y^2}{x^2 - 2x + y^2} \geq 0\right\}.$$

[1] To solve this problem we will suffice with secondary school knowledge of the function tg. The exact definition will be given in section 4.3.

>From the solution to the system of equations of two unknowns can be found that the set D_f is the intersection of the complement of two disks and that disks without their bounding circles $\{[x,y] \in \mathbb{R}^2;\ (x+1)^2 + y^2 < 1\}$ and the disk $\{[x,y] \in \mathbb{R}^2;\ (x-1)^2 + y^2 \leq 1\}$. Now

$$f_{-1}(\{0\}) = \{\}[x,y] \in D_f;\ \sqrt{\frac{x^2+2x+y^2}{x^2-2x+y^2}} = 0.$$

Therefore $f_{-1}(\{0\})$ is a circle with its center at $[-1,0]$ and radius 1 without the point $[0,0]$.

The pre-image of the set $\{1\}$ is

$$f_{-1}(\{1\}) = \{\}[x,y] \in D_f;\ \sqrt{\frac{x^2+2x+y^2}{x^2-2x+y^2}} = 1 = \{[x,y] \in D_f;\ x = 0\}.$$

The set $f_{-1}(\{1\})$ is therefore the y axis without the point $[0,0]$. ♣

We will use the following notation:

$$\sup_M f = \sup f(M) = \sup \{f(x);\ x \in M\} \quad \text{and}$$
$$\inf_M f = \inf f(M) = \inf \{f(x);\ x \in M\}.$$

Example 7. Let M be a non-empty set, $f\colon M \to \mathbb{R}$, $g\colon M \to \mathbb{R}$ are mappings and $f(x) \leq g(x)$ for every $x \in M$. Then it holds that

$$\sup_M f \leq \sup_M g \quad \text{and} \quad \inf_M f \leq \inf_M g.$$

Proof. For every $x \in M$ it holds that $g(x) \leq \sup_M g$, and therefore also $f(x) \leq \sup_M g$. The number $\sup_M g$ is an upper bound of the set $f(M)$, therefore the inequality $\sup_M f \leq \sup_M g$ holds. The relationship for the infimum is proven similarly. ■

Example 8. Let M be a non-empty set and $f, g\colon M \to \mathbb{R}$. Then it holds that

$$\sup_M (f+g) \leq \sup_M f + \sup_M g \quad \text{and}$$
$$\inf_M (f+g) \geq \inf_M f + \inf_M g.$$

Proof. Let us choose any $y \in M$. Then it holds that

$$f(y) + g(y) \leq \sup_M f + \sup_M g.$$

The number $\sup_M f + \sup_M g$ is therefore an upper bound of the set $\{f(y) + g(y);\ y \in M\}$, and from this the required inequality follows.

The inequality for the infimum is proven similarly. ■

Example 9. Let M be a non-empty set, $f\colon M \to \mathbb{R}$ a $C \in \mathbb{R}$. If $|f(a) - f(b)| \leq C$ for every $a, b \in M$, then $\sup\limits_M f - \inf\limits_M f \leq C$.

Proof. Let us choose $b \in M$. Then for all $a \in M$ it holds that $f(a) \leq C + f(b)$, and therefore $\sup_M f \leq C + f(b)$. >From here we see that $\sup_M f - C \leq f(b)$ for every $b \in M$, which gives the necessary inequality. ∎

Example 10. Let M be a non-empty set and $f\colon M \to \mathbb{R}$. Then it holds that

$$\sup_M |f| - \inf_M |f| \leq \sup_M f - \inf_M f.$$

Proof. Put $C = \sup_M f - \inf_M f$. For every $a, b \in M$ it obviously holds that $f(a) - f(b) \leq C$ and $f(b) - f(a) \leq C$. From here it follows that $|f(a) - f(b)| \leq C$. >From the inequality in Corollary 1.17 (i) we have $\big||f(a)| - |f(b)|\big| \leq |f(a) - f(b)| \leq C$ for every $a, b \in M$. >From Example 9 used on the mapping $|f|$ the required inequality holds. ∎

3.1. Exercises

1. Determine the set $D \subset \mathbb{R}^2$ of all $[x, y] \in \mathbb{R}^2$, for which the following formula is defined

$$f\colon [x, y] \mapsto \sqrt{\frac{1}{y} - x^2}.$$

Determine the pre-images of the sets $\{0\}$ and $\{1\}$ in the mapping $f\colon D \to \mathbb{R}$. Draw them.

2. Let the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula

$$f\colon [x, y] \mapsto \max\{x, y\}.$$

Determine the pre-images of the sets $\{0\}, \{1\}$. Represent it graphically.

3. Find the compound mappings $f \circ g$ and $g \circ f$, determine their domain and ranges if

(i) $f\colon \mathbb{R} \to \mathbb{R}$, $f\colon x \mapsto x^2$; $g\colon (0, +\infty) \to \mathbb{R}$, $g\colon x \mapsto \log x$,
(ii) $f\colon \mathbb{R} \to \mathbb{C}$, $f(x) = x + ix$; $g\colon \mathbb{C} \to \mathbb{R}$, $g(x) = |x|$.

4. Determine whether the following mappings have an inverse and if they do then determine their domains and ranges and the formulas that define them.

(i) $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^2 - 1$,
(ii) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + x + 1$,
(iii) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x + 4$.

Solutions

1. $D = \{[x,y] \in \mathbb{R}^2;\ x^2y \leq 1, y > 0\}$, $f_{-1}(\{0\}) = \{[x, 1/x^2];\ x \in \mathbb{R} \setminus \{0\}\}$, $f_{-1}(\{1\}) = \{[x, 1/(1+x^2)];\ x \in \mathbb{R}\}$

2. $f_{-1}(\{0\}) = \{[0,y] \in \mathbb{R}^2;\ y \leq 0\} \cup \{[x,0] \in \mathbb{R}^2;\ x \leq 0\}$, $f_{-1}(\{1\}) = \{[1,y] \in \mathbb{R}^2;\ y \leq 1\} \cup \{[x,1] \in \mathbb{R}^2;\ x \leq 1\}$

3. (i) $(f \circ g)(x) = \log^2 x$, $D_{f\circ g} = (0, +\infty)$, $R_{f\circ g} = [0, +\infty)$; $(g \circ f)(x) = \log x^2$, $D_{g\circ f} = \mathbb{R} \setminus \{0\}$, $R_{g\circ f} = \mathbb{R}$
(ii) $(f \circ g)(x) = |x| + i\,|x|$, $D_{f\circ g} = \mathbb{C}$, $R_{f\circ g} = \{a + ia;\ a \in \mathbb{R}, a \geq 0\}$; $g \circ f = |x + ix| = \sqrt{2}\,|x|$, $D_{g\circ f} = \mathbb{R}$, $R_{g\circ f} = [0, +\infty)$

4. (i) $D_{f^{-1}} = [-1, +\infty)$, $f^{-1}(x) = -\sqrt{x+1}$
(ii) The inverse mapping does not exist because f is not an injective mapping. for example $f(0) = f(-1) = 1$.
(iii) $D_{f^{-1}} = (4, +\infty)$, $f^{-1}(x) = \log_2(x - 4)$

CHAPTER 4

Functions of one real variable

4.1. Limit of a Function

A real function f **of one real variable** (in the following simply a **function**) is a mapping $f\colon M \to \mathbb{R}$, where M is a subset of the set of the real numbers.

Definition. A function $f\colon J \to \mathbb{R}$ is **increasing** on the interval J, if for every pair $x_1, x_2 \in J$, $x_1 < x_2$, the following inequality holds $f(x_1) < f(x_2)$. Similarly we define the term of a **decreasing** function, **non-decreasing** and **non-increasing** on the interval J.

Definition. A monotone function (respectively **strictly monotone function**) on the interval J is a function that is non-decreasing or non-increasing (respectively increasing or decreasing) on J.

Definition. Let f be a function and $M \subset D_f$. We say that the function f is

- **bounded from above** on M, if the set $f(M)$ is bounded from above,
- **bounded from below** on M, if the set $f(M)$ is bounded from below,
- **bounded** on M, if the set $f(M)$ is bounded,
- **constant** on M, if for every $x, y \in M$ it holds that $f(x) = f(y)$,
- **odd**, if for every $x \in D_f$ it holds that $-x \in D_f$ and $f(-x) = -f(x)$,
- **even**, if for every $x \in D_f$ it holds that $-x \in D_f$ a $f(-x) = f(x)$,
- **periodic with the period** $\boldsymbol{a}$, where $a \in \mathbb{R}$, $a > 0$, if for every $x \in D_f$ it holds that $x + a \in D_f$, $x - a \in D_f$ and $f(x + a) = f(x)$.

Example 1. Let $a, b \in \mathbb{R}$. We define the function

$$f(x) = ax + b, \quad x \in \mathbb{R}.$$

For $a = 0$ the function f is constant and $R_f = \{b\}$. If $a > 0$, then f is increasing on $\mathbb{R}$ and is not bounded from above nor below and $R_f = \mathbb{R}$. Prove these claims in detail and consider how it would be if $a < 0$. A function defined like f is called an **affine function**. If $b = 0$, we say that f is **linear**. Here we define the term linear function differently than is usual at secondary school because this is the definition used in advanced mathematical books.

In the following pictures we have the graphs of three different functions. On the first picture it appears that as the values of x approach the point 1, the values of $f(x)$ approach the value of f at the point 1. In the second picture something similar can be seen but $f(1)$ is different to the value that $f(x)$ approach as x nears to 1. In the third picture we see that for x approaching 1 the functional values $f(x)$ do not approach any one value.

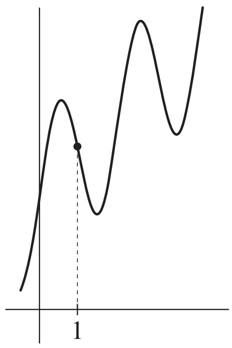


FIGURE 1.

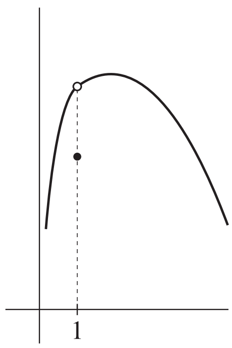


FIGURE 2.

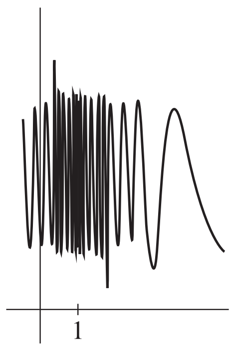


FIGURE 3.

If we want to express that which is common for the first two pictures then we will ignore the value of f at the point 1, but rather focus on the fact that the values $f(x)$ "approach a value", as x approaches 1. This is appropriately formulated in the following definitions.

Definition. **Deleted neighborhood of a point** (also known as a **punctured neighborhood of a point**) $x_0 \in \mathbb{R}$ with radius $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, is the set

$$P(x_0, \varepsilon) = \{x \in \mathbb{R};\ 0 < |x - x_0| < \varepsilon\},$$

or

$$P(x_0, \varepsilon) = (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon) = B(x_0, \varepsilon) \setminus \{x_0\}.$$

Definition. We say that the number $A \in \mathbb{R}$ is the **limit of the function f at the point** $c \in \mathbb{R}$, if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P(c, \delta)\colon\ f(x) \in B(A, \varepsilon).$$

Try to prove the following theorem yourself using the ideas of the proof of Theorem 2.2.

Theorem 2 (uniqueness of the limit)**.** A function has at most one limit $A \in \mathbb{R}$ at any given point.

Similarly to sequences, the previous theorem allows us to introduce the following notation. If a function f has the limit $A \in \mathbb{R}$ at the point $c \in \mathbb{R}$ then we write $\lim_{x \to c} f(x) = A$.

Remark. Picture 1 shows the case where the graph of the function f is not "broken" at the point 1. The limit of this function at the point 1 is the same as the functional value at the point 1. This situation is so important that it deserves its own definition.

Definition. We say that the function f is **continuous at the point** $c \in \mathbb{R}$, if it holds that

$$\lim_{x \to c} f(x) = f(c).$$

Remark. Consider the fact that a function f is continuous at the point c, if and only if it holds that

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in B(c, \delta) \colon f(x) \in B(f(c), \varepsilon).$$

So far we have defined the term real limit of a function at the point $c \in \mathbb{R}$. We will also want to describe the situation where a "function grows above all bounds in the neighborhood of a point". We will also be interested in the behavior of the function for "very large values of the variable". In order to avoid having to give different definitions for different cases let us extend the definition of a neighborhood and a deleted neighborhood also for the points $+\infty$ and $-\infty$.

Definition. Let $\varepsilon > 0$. Then we define

$$\begin{aligned} P(+\infty, \varepsilon) &= B(+\infty, \varepsilon) = (1/\varepsilon, +\infty), \\ P(-\infty, \varepsilon) &= B(-\infty, \varepsilon) = (-\infty, -1/\varepsilon). \end{aligned}$$

Remark. It can be seen that in the definition above the number $1/\varepsilon$ increases as ε decreases. The corresponding neighborhood (or deleted neighborhood) gets smaller. Notice that in the case of $+\infty$ and $-\infty$ that a neighborhood and deleted neighborhood are the same set.

Definition. We say that $A \in \mathbb{R}^*$ is the **limit of the function f at the point** $c \in \mathbb{R}^*$, if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in P(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

With respect to Theorem 2 for $c \in \mathbb{R}^*$ and $A \in \mathbb{R}^*$ we can use the notation $\lim_{x \to c} f(x) = A$, if the limit of f at the point c is A.

Remarks. 1. Let $\lim\limits_{x\to c} f(x) = A$, where $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$. Then we can distinguish the two cases:

$$\text{the limit} \begin{cases} \textbf{at a real point}\text{, i.e. } c \in \mathbb{R}\text{, and} \begin{cases} A \in \mathbb{R} \text{ (the limit limit is } \textbf{real}\text{)}, \\ A = +\infty \text{ (the limit is infinity)}, \\ A = -\infty \text{ (the limit is minus infinity)}, \end{cases} \\ \textbf{at infinity}\text{, i.e. } c = \pm\infty\text{, and} \begin{cases} A \in \mathbb{R} \text{ (the limit is real)}, \\ A = +\infty \text{ (the limit is infinity)}, \\ A = -\infty \text{ (the limit is minus infinity)}. \end{cases} \end{cases}$$

2. Notice that if, for example $A = +\infty$, then the previous definition can be formulated equivalently as

$$\forall L \in \mathbb{R}\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall x \in P(c, \delta)\colon\ f(x) > L.$$

Example 3. Show that $\lim\limits_{x\to 0} 1/x^2 = +\infty$.

Solution. Let $\varepsilon > 0$. Then it suffices to put $\delta = \sqrt{\varepsilon}$. Immediately we see that for every $x \in P(0, \sqrt{\varepsilon})$ it holds that $1/x^2 > 1/\varepsilon$, and so $1/x^2 \in B(+\infty, \varepsilon)$. ♣

Example 4. Let us show that $\lim\limits_{x\to +\infty} 1/(1 + x) = 0$.

Solution. Choose $\varepsilon > 0$ and take $\delta = \varepsilon$. For $x \in P(+\infty, \delta)$ it holds that $0 < 1/(1 + x) < 1/x < \varepsilon$, and therefore $1/(1 + x) \in B(0, \varepsilon)$. ♣

Remark. We can see that the following statement holds from the definition of the limit. If the function f and g are equal on a certain deleted neighborhood of a point $a \in \mathbb{R}^*$, then if one of the limits $\lim\limits_{x\to a} f(x)$, $\lim\limits_{x\to a} g(x)$, exists then the other also exists and they are equal.

If we want to define the term right-handed limit (or left-handed), then we will need the concept of right (and left) handed neighborhoods of a point. These terms are defined as follows.

Definition. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Then we define

- **the right-hand neighborhood of a point** c as $B^+(c, \varepsilon) = [c, c + \varepsilon)$,
- **the left-hand neighborhood of a point** c as $B^-(c, \varepsilon) = (c - \varepsilon, c]$,
- **the right-hand deleted neighborhood of a point** c as $P^+(c, \varepsilon) = (c, c + \varepsilon)$,
- **the left-hand deleted neighborhood of a point** c as $P^-(c, \varepsilon) = (c - \varepsilon, c)$.

We further define

- **the left-hand neighborhood of the point** $+\infty$ as $B^-(+\infty, \varepsilon) = (1/\varepsilon, +\infty)$,
- **the right-hand neighborhood of the point** $-\infty$ as $B^+(-\infty, \varepsilon) = (-\infty, -1/\varepsilon)$,
- **left-hand deleted neighborhood of the point** $+\infty$ as $P^-(+\infty, \varepsilon) = B^-(+\infty, \varepsilon)$,
- **right-hand deleted neighborhood of the point** $-\infty$ as $P^+(-\infty, \varepsilon) = B^+(-\infty, \varepsilon)$.

Definition. Let $A \in \mathbb{R}^*$, $c \in \mathbb{R} \cup \{-\infty\}$. We say that the function f has a **right-handed limit** at the point c equal to A (we denote $\lim\limits_{x \to c+} f(x) = A$)[1], if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall x \in P^+(c, \delta) \colon f(x) \in B(A, \varepsilon).$$

Similarly we define the term **left-handed limit** at the point $c \in \mathbb{R} \cup \{+\infty\}$. For the left-handed limit of the function f at the point c we use the symbol $\lim\limits_{x \to c-} f(x)$.

Remark. Consider that the function f has a limit at the point $c \in \mathbb{R}$ if and only if it has both a left and a right-handed limit at c and that these one-sided limits are equal.

Definition. Let $c \in \mathbb{R}$. We say that the function f is **right-continuous** at the point c (or **left-continuous**), if $\lim\limits_{x \to c+} f(x) = f(c)$ (or $\lim\limits_{x \to c-} f(x) = f(c)$).

Example 5. Let us define

$$f(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

We call this function the **sign function** and denote it as sgn. We can easily see that

$$\lim_{x \to 0+} \operatorname{sgn} x = 1 \qquad \text{and} \qquad \lim_{x \to 0-} \operatorname{sgn} x = -1.$$

The function sgn is therefore not continuous at the point 0.

Example 6. An affine function $f \colon x \mapsto ax + b$ is continuous at all points $c \in \mathbb{R}$. We can show this directly from the definition. Let us do it in detail for the case that $a > 0$. Let us have $c \in \mathbb{R}$. Then for $x \in (c - \delta, c + \delta)$ it holds that

$$f(c) - a\delta < f(x) = f(c) + a(x - c) < f(c) + a\delta.$$

Let $\varepsilon > 0$. If we put $\delta = \varepsilon / a$, then it follows from the previous inequality that for every $x \in (c - \delta, c + \delta)$ it holds that

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon.$$

Thus we have proved that $\lim\limits_{x \to c} f(x) = f(c)$.

Let us note that the definition of the limit does not include instructions on how to find the limit itself (or how to prove that the limit does not exist). Similarly to the case of sequences we will build a humble theory which allows us to calculate some limits.

[1] It is possible to prove uniqueness and therefore that the notation is correct; similarly for the left-handed limits.

Theorem 7 (real limits and boundedness)**.** If a function f has a real limit at the point $c \in \mathbb{R}^*$ then there exists some $P(c,\delta)$ such that f is bounded on $P(c,\delta)$.

Proof. Let us denote $\lim\limits_{x\to c} f(x) = A \in \mathbb{R}$. According to the definition of the limit there exists an $\delta \in \mathbb{R}$, $\delta > 0$, such that $f(x) \in B(A,1)$ for $x \in P(c,\delta)$. For this x we have

$$A - 1 < f(x) < A + 1. \qquad \blacksquare$$

Theorem 8 (arithmetics of the limits of functions)**.** Let $c \in \mathbb{R}^*$. If $\lim\limits_{x\to c} f(x) = A \in \mathbb{R}^*$ and $\lim\limits_{x\to c} g(x) = B \in \mathbb{R}^*$, then it holds that:

(i) $\lim\limits_{x\to c}\big(f(x) + g(x)\big) = A + B$,
(ii) $\lim\limits_{x\to c}\big(f(x)g(x)\big) = AB$,
(iii) $\lim\limits_{x\to c}\big(f(x)/g(x)\big) = A/B$,

if the right-hand side is defined.

Proof. We will only show the claim about the quotients of two functions for $A \in \mathbb{R}$ and $B \in \mathbb{R}$, $B > 0$. The techniques used in the other proofs are similar and we leave their execution to the reader.

Our aim is to prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\forall x \in P(c,\delta)\colon \left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| < \varepsilon. \tag{1}$$

>From the definition of the limit it follows that for the number $B/2$ there exists $\eta > 0$ such that for $x \in P(c,\eta)$ it holds that $g(x) \in (B - B/2, B + B/2)$, and so $g(x) > B/2 > 0$, therefore the expression $f(x)/g(x)$ is defined for all $x \in P(c,\eta)$. For $x \in P(c,\eta)$ by using the inequality $1/g(x) < 2/B$ we estimate

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| &= \frac{1}{|g(x)|\,B}\,|f(x)B - g(x)A| = \\
&= \frac{1}{|g(x)|\,B}\,|f(x)B - AB + AB - g(x)A| \le \\
&\le \frac{1}{|g(x)|\,B}\big(B\,|f(x) - A| + |A|\,|B - g(x)|\big) \le \\
&\le M\big(|f(x) - A\,| + |g(x) - B|\big),
\end{aligned} \tag{2}$$

where $M = \max\left\{\frac{2}{B}, \frac{2|A|}{B^2}\right\}$. Let us now choose $\varepsilon > 0$. From the hypothesis of the theorem it follows that for the number $\frac{\varepsilon}{2M}$ there exists a $\delta_1 > 0$ and $\delta_2 > 0$ such that it holds that

$$\forall x \in P(c,\delta_1)\colon\ |f(x)-A| < \frac{\varepsilon}{2M}, \tag{3}$$

$$\forall x \in P(c,\delta_2)\colon\ |g(x)-B| < \frac{\varepsilon}{2M}. \tag{4}$$

Now if δ is the smallest of the three positive numbers η, δ_1 and δ_2, then the inequalities (2), (3) and (4) simultaneously hold for $x \in P(c,\delta)$. >From here we see that (1) follows. ■

The expression "$A/0$" is not defined, nevertheless this theorem holds.

Theorem 9. Let $c \in \mathbb{R}^*$, $\lim\limits_{x\to c} g(x) = 0$, $\lim\limits_{x\to c} f(x) = A \in \mathbb{R}^*$ and $A > 0$. If there exists some $\eta > 0$ such that g is positive on $P(c,\eta)$, then $\lim\limits_{x\to c}\big(f(x)/g(x)\big) = +\infty$.

Proof. Let us consider two cases. Firstly we assume that $A \in \mathbb{R}$. Let us choose $L \in \mathbb{R}$ arbitrarily. We find $\delta_1 > 0$ such that for all $x \in P(c,\delta_1)$ it holds that $f(x) \in (A-A/2, A+A/2)$. Further we find $\delta_2 > 0$ such that for all $x \in P(c,\delta_2)$ it holds that $|g(x)| < \frac{A}{2(|L|+1)}$. We put $\delta_3 = \min\{\delta_1,\delta_2,\eta\}$. Then fro all $x \in P(c,\delta_3)$ we have $0 < g(x) < \frac{A}{2(|L|+1)}$, and so

$$\frac{f(x)}{g(x)} > \frac{\frac{A}{2}}{\frac{A}{2(|L|+1)}} = |L|+1 > L.$$

Thus we have proved the claim for $A \in \mathbb{R}$.

Now let us assume that $A = +\infty$. Again let us choose $L \in \mathbb{R}$ arbitrarily. We find $\delta_1 > 0$ such that for all $x \in P(c,\delta_1)$ it holds that $f(x) > 1$. Further we find $\delta_2 > 0$ such that for all $x \in P(c,\delta_2)$ it holds that $|g(x)| < \frac{1}{|L|+1}$. We put $\delta_3 = \min\{\delta_1,\delta_2,\eta\}$. Then for all $x \in P(c,\delta_3)$ we have $0 < g(x) < 1/(|L|+1)$, and so

$$\frac{f(x)}{g(x)} > \frac{1}{1/(|L|+1)} = |L|+1 > L.$$ ■

Remark. The previous theorems have variations also for one-sided limits. For example if $c \in \mathbb{R}\cup\{-\infty\}$, $\lim\limits_{x\to c+} g(x) = 0$, $\lim\limits_{x\to c+} f(x) = A \in \mathbb{R}^*$, $A > 0$ and there exists an $\eta > 0$ such that the function g is positive on $P^+(c,\eta)$, then $\lim\limits_{x\to c+}\big(f(x)/g(x)\big) = +\infty$.

We will illustrate the use of Theorem 8 on the following example.

Example 10. Calculate $\lim\limits_{x\to -1} \dfrac{x^2+3x+2}{x^3+2x^2-x-2}$.

Solution. Because the limit of the numerator and the denominator at the point -1 is zero by Theorem 8 (i), (ii) and Example 6, we cannot use the quotient rule of Theorem 8 (iii). Nevertheless

$$\frac{x^2+3x+2}{x^3+2x^2-x-2} = \frac{(x+1)(x+2)}{(x+1)(x^2+x-2)} = \frac{x+2}{x^2+x-2}$$

Everywhere where the denominator is different from zero. The functions

$$x \mapsto \frac{x^2+3x+2}{x^3+2x^2-x-2}, \qquad x \mapsto \frac{x+2}{x^2+x-2}$$

are equal on a certain deleted neighborhood of the point -1, for example $P(-1,1)$. According to the remark on page 54 it suffices to calculate the limit of the second function at -1. Thus we have

$$\lim_{x\to -1} \frac{x^2+3x+2}{x^3+2x^2-x-2} = \lim_{x\to -1} \frac{x+2}{x^2+x-2} = -1/2. \qquad \clubsuit$$

Remark. >From Theorem 8 it immediately follows that if the functions f and g are continuous at the point $c \in \mathbb{R}$, then the functions $f+g$ and fg are also continuous at the point c. Further if $g(c) \neq 0$, the the function f/g is continuous at c.

Example 11. We already know that the function $f(x) = x$ is continuous at all points $c \in \mathbb{R}$. According to the previous remark the functions $x \mapsto x^2$, $x \mapsto x^3, \ldots$ are also continuous at all points in $\mathbb{R}$.

Definition. A **polynomial** is a function P having the form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R}, \tag{5}$$

where $n \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$. The numbers $a_0, \ldots, a_n$ are called the **coefficients of the polynomial** of P.

Example 12. Let P be a polynomial in the form $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $n \geq 1$ and $a_n \neq 0$. Then

$$\lim_{x\to+\infty} P(x) = \begin{cases} +\infty, & \text{if } a_n > 0, \\ -\infty, & \text{if } a_n < 0. \end{cases}$$

Solution. It can be seen immediately from the definition that $\lim\limits_{x\to+\infty} x = +\infty$. From here, by repeatedly using Theorem 8, we get

$$\lim_{x\to+\infty} P(x) = \lim_{x\to+\infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + \frac{a_{n-1}}{x} + a_n\right) \cdot x^n = a_n \cdot (+\infty).$$

$\clubsuit$

Remark. Let $n, m \in \mathbb{N} \cup \{0\}$ and

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad x \in \mathbb{R},$$
$$Q(x) = b_0 + b_1 x + \cdots + b_m x^m, \quad x \in \mathbb{R},$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$, $b_0, b_1, \ldots, b_m \in \mathbb{R}$, $b_m \neq 0$. If the polynomials P and Q are equal (i.e. $P(x) = Q(x)$ for all $x \in \mathbb{R}$), then

$$\lim_{x \to +\infty} \bigl(P(x) - Q(x)\bigr) = \lim_{x \to +\infty} 0 = 0,$$

which according to Example 12 is only possible if $n = m$ and $a_0 = b_0, \ldots, a_n = b_n$.

With respect to this observation we know that the following definition is correct.

Definition. Let P be a polynomial in the form (5). We say that P is a polynomial of **degree $\boldsymbol{n}$**, if $a_n \neq 0$. The degree of the **zero polynomial** (i.e. the constant zero function defined on $\mathbb{R}$) is defined as -1.

Once again it can be seen immediately from Theorem 8, that polynomials are continuous at all points in $\mathbb{R}$. If P, Q are two polynomials and Q is not the zero polynomial, then the function $F = P/Q$, which is called a **rational** function, is defined at all points of the real line except those where the polynomial Q takes the values zero. There are at most m such points, where m is the degree of the polynomial Q (see Theorem **??**). According to the last claim of Theorem 8 the function F is continuous at all points, where it is defined.

Theorem 13 (limit and order)**.** Let $c \in \mathbb{R}^*$ and let $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

(i) If

$$\lim_{x \to c} f(x) > \lim_{x \to c} g(x),$$

then there exists a $\delta > 0$ such that

$$\forall x \in P(c, \delta)\colon f(x) > g(x).$$

(ii) If there exists some $\delta > 0$ such that

$$\forall x \in P(c, \delta)\colon f(x) \leq g(x),$$

then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

(iii) (squeeze) Let there exist some $\eta > 0$ such that

$$\forall x \in P(c, \eta)\colon f(x) \leq h(x) \leq g(x).$$

If moreover $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = A \in \mathbb{R}^*$, then also $\lim_{x \to c} h(x)$ exists and equals A.

Proof. Let us prove claim (iii) for $A \in \mathbb{R}$. Try to prove claims (i) and (ii) yourself. Let us choose $\varepsilon > 0$. To this number there exists some $\delta_1 > 0$ such that for all $x \in P(c, \delta_1)$ it holds that

$$A - \varepsilon < f(x) < A + \varepsilon, \qquad A - \varepsilon < g(x) < A + \varepsilon.$$

Now let $\delta = \min\{\delta_1, \eta\}$. If $x \in P(c, \delta)$, then

$$A - \varepsilon < f(x) \le h(x) \le g(x) < A + \varepsilon,$$

and therefore $h(x) \in (A - \varepsilon, A + \varepsilon)$. For every $\varepsilon > 0$, therefore, there exists a $\delta > 0$ such that

$$\forall x \in P(c, \delta)\colon h(x) \in (A - \varepsilon, A + \varepsilon),$$

and therefore $\lim\limits_{x \to c} h(x) = A$. ■

Remark. If the function f at the point c is continuous and $f(c) \neq 0$, then there exists a $\delta > 0$ such that f is non-zero on $B(c, \delta)$. Think through how this claim follows in part from (i) of the previous theorem (instead of g take the zero function).

Remark. In the chapter about sequences we proved variations on the squeeze theorem also for infinite limits (Theorem 2.19 and 2.20). It is similar here in the case of infinite limits of functions too. We include the version for limits equal to $+\infty$.

Theorem. Let there exist an $\eta > 0$ such that for all $x \in P(c, \eta)$ we have $f(x) \le h(x)$. Further let us assume that $\lim\limits_{x \to c} f(x) = +\infty$. Then also $\lim\limits_{x \to c} h(x)$ exists and equals $+\infty$.

Example 14. Calculate $\lim\limits_{x \to 1} \dfrac{(x-1)^2}{2 + \sin \frac{1}{x-1}}$.

Solution. For every $x \in \mathbb{R} \setminus \{1\}$ it holds that $2 + \sin \frac{1}{x-1} \ge 1$, and so

$$0 \le \frac{(x-1)^2}{2 + \sin \frac{1}{x-1}} \le (x-1)^2.$$

It holds that $\lim\limits_{x \to 1} 0 = 0$ and $\lim\limits_{x \to 1} (x-1)^2 = 0$. So also

$$\lim_{x \to 1} \frac{(x-1)^2}{2 + \sin \frac{1}{x-1}} = 0$$

by (iii) from Theorem 13. ♣

Limits are also often calculated by using the following theorem, whose proof can easily be conducted using claim (iii) from Theorem 13.

Theorem 15. Let $c \in \mathbb{R}^*$, $\lim\limits_{x \to c} f(x) = 0$ and let there be a $\eta > 0$ such that g is bounded on $P(c, \eta)$. Then $\lim\limits_{x \to c} \big(f(x) g(x)\big) = 0$.

The next theorem shows the connection between the limit of a function and the limit of a sequence

Theorem 16 (Heine's theorem)**.** Let $c \in \mathbb{R}^*$, $A \in \mathbb{R}^*$ and for the function f assume that $\lim\limits_{x\to c} f(x) = A$. If the sequence $\{x_n\}$ satisfies $x_n \in D_f$, $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = c$, then it holds that $\lim\limits_{n\to\infty} f(x_n) = A$.

Remark. The name Heine's theorem for the previous claim is not entirely accurate because Heine's theorem is an equivalence and its formulation is more complicated.

It is not hard to formulated Heine's left-hand (or right-hand) theorem for one-sided limits. Similarly one can see the connection between continuity and the limit of sequences. In Chapter **??** we will introduce yet another form of this theorem (Theorem **??**), which we will prove. Using the ideas there one can easily conduct a proof in this setting so we will not prove the theorem above.

Theorem 16 is often useful for proving that a certain limit does not exist.

Example 17. Let us show that the limit $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist.

Solution. Assume that $\lim\limits_{x\to 0} \sin\frac{1}{x} = A \in \mathbb{R}^*$. Let us take the sequence $\{x_n\} = \{\frac{1}{n\pi}\}$. For every $n \in \mathbb{N}$ we have $\sin\frac{1}{x_n} = \sin n\pi = 0$, and therefore $\lim\limits_{n\to\infty} \sin\frac{1}{x_n} = 0$. However $\lim\limits_{n\to\infty} x_n = 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Further let us take the sequence $\{y_n\} = \big\{\frac{1}{\frac{\pi}{2}+2n\pi}\big\}$. We have $\sin\frac{1}{y_n} = \sin\big(\frac{\pi}{2} + 2n\pi\big) = 1$ for every $n \in \mathbb{N}$, and therefore $\lim\limits_{n\to\infty} \sin\frac{1}{y_n} = 1$. Simultaneously however $\lim\limits_{n\to\infty} y_n = 0$ and for all $n \in \mathbb{N}$ it holds that $y_n \neq 0$. By Heine's theorem we must have $\lim\limits_{n\to\infty} \sin\frac{1}{x_n} = \lim\limits_{n\to\infty} \sin\frac{1}{y_n} = A$. On the other hand however $\lim\limits_{n\to\infty} \sin\frac{1}{x_n} \neq \lim\limits_{n\to\infty} \sin\frac{1}{y_n}$, and that is a contradiction. Therefore the limit $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist. ♣

Theorem 8 says how the limit of a function behaves with respect to addition, subtraction, multiplication and division. The following theorem clarifies the relationship that the limit has with composition of functions.

Theorem 18 (limit of a compound function)**.** Let $c, D, A \in \mathbb{R}^*$. Let us have the functions f and g which satisfy $\lim\limits_{x\to c} g(x) = D$ and $\lim\limits_{y\to D} f(y) = A$. Let us also assume that at least one of the following conditions is satisfied:

(P) there exists an $\eta > 0$ such that for all $x \in P(c,\eta)$ it holds that $g(x) \neq D$,

(S) the function f is continuous at the point D.

Then it holds that

$$\lim_{x\to c} (f \circ g)(x) = A.$$

Proof. Let us assume that condition (P) is satisfied. Choose any $\varepsilon > 0$. To this ε there is a $\psi > 0$ such that

$$\forall y \in P(D, \psi)\colon\ f(y) \in B(A, \varepsilon),$$

because $\lim\limits_{y \to D} f(y) = A$. To this ψ one can find a $\delta' > 0$ such that

$$\forall x \in P(c, \delta')\colon\ g(x) \in B(D, \psi),$$

because $\lim\limits_{x \to c} g(x) = D$. Let us put $\delta = \min\{\delta', \eta\}$. For every $x \in P(c, \delta)$ it holds that $g(x) \in B(D, \psi) \setminus \{D\}$, because $g(x) \in P(D, \psi)$. From here $f\big(g(x)\big) \in B(A, \varepsilon)$. Thus the proof is complete for the version with condition (P).

Now let us assume that we have condition (S). Take $\varepsilon > 0$. Then there exists a $\psi > 0$ such that for all $y \in P(D, \psi)$ it holds that $f(y) \in B(A, \varepsilon)$. Because the function f is continuous at the point D, we have $f(D) = A$. Therefore for all $y \in B(D, \psi)$ it holds that $f(y) \in B(A, \varepsilon)$. to any given number ψ we have a $\delta > 0$ such that for all $x \in P(c, \delta)$ it holds that $g(x) \in B(D, \psi)$. Together we get that for $x \in P(c, \delta)$ it holds that $f\big(g(x)\big) \in B(A, \varepsilon)$. And so the proof is completed. ■

Remarks. 1. If neither condition (P) nor (S), then the claim of the theorem may not hold. Consider the case where $f = |\mathrm{sgn}|$, $g = 0$, $c = 0$, $D = 0$, $A = 1$.

2. If the function g is continuous at the point $c \in \mathbb{R}$ and the function f is continuous at the point $g(c)$, then, by the previous theorem, we have that the function $f \circ g$ is continuous at the point c.

The theorem on the limit of compound functions also has its version for one sided limits. The following version is useful.

Theorem 19. Let $c, D, A \in \mathbb{R}^*$. Let the functions f and g satisfy $\lim\limits_{x \to c-} g(x) = D$, $\lim\limits_{y \to D+} f(y) = A$ and let at least one of the following conditions be satisfied:

- (P) there exists an $\eta > 0$ such that for all $x \in P^-(c, \eta)$ it holds that $g(x) > D$,
- (S) the function f is right-continuous at the point D and there exists a number $\eta > 0$ such that for all $x \in P^-(c, \eta)$ it holds that $g(x) \geq D$.

Then it holds that

$$\lim_{x \to c-} (f \circ g)(x) = A.$$

Example 20. It holds that $\lim\limits_{x \to +\infty} f(1/x) = \lim\limits_{y \to 0+} f(y)$, if at least one of the limits exists.

Proof. If we put $g(x) = 1/x$ and use Theorem 19, we get the validity of the expression in the case that the limit $\lim\limits_{y \to 0+} f(y)$ exists. The validity of the relationship under the assumption of the existence of the first limit is proved similarly. ■

Theorem 21 (limit of a monotone function)**.** Let $a, b \in \mathbb{R}^*$, $a < b$. Let the function f be monotone on the interval (a, b). Then there exist $\lim_{x \to a+} f(x)$ and $\lim_{x \to b-} f(x)$, while it holds that:

- If f is non-decreasing on (a, b) then
$$\lim_{x \to a+} f(x) = \inf f\big((a, b)\big) \quad \text{and} \quad \lim_{x \to b-} f(x) = \sup f\big((a, b)\big).$$
- If $f$is non-increasing on (a, b) then
$$\lim_{x \to a+} f(x) = \sup f\big((a, b)\big) \quad \text{and} \quad \lim_{x \to b-} f(x) = \inf f\big((a, b)\big).$$

Proof. Let us prove that $\lim_{x \to a+} f(x) = \inf f\big((a, b)\big)$ holds for a non-decreasing function f bounded from below and for $a \in \mathbb{R}$. The proofs of the other cases are left to the reader. Denote $g = \inf f\big((a, b)\big) \in \mathbb{R}$. Choose the number $\varepsilon > 0$. >From the properties of the infima it follows that there exists a $y \in f\big((a, b)\big)$ such that $y < g + \varepsilon$. >From the definition of the set $f\big((a, b)\big)$ it can be seen that $y = f(x')$ for some $x' \in (a, b)$. Because the function f is non-decreasing we have

$$\forall x \in (a, x')\colon f(x) \leq f(x') < g + \varepsilon.$$

Because g is a lower bound on the set $f\big((a, b)\big)$, we have

$$\forall x \in (a, b)\colon g - \varepsilon < g \leq f(x).$$

Therefore it holds that

$$\forall x \in (a, x')\colon g - \varepsilon < f(x) < g + \varepsilon.$$

Let us put $\delta = x' - a$. Then

$$\forall x \in P^+(a, \delta)\colon f(x) \in (g - \varepsilon, g + \varepsilon).$$

Thus the claim is proven. ■

4.2. Continuous functions on an interval

Definition. Let $J \subset \mathbb{R}$ be a non-degenerate interval (i.e. it contains an infinite number of points). The function $f\colon J \to \mathbb{R}$ is **continuous on the interval** J, if it holds that:

- f is continuous at every inner point of J,
- f is right-continuous at the left end-point of the interval J, if this point belongs to J,
- f is left-continuous at the right end-point of the interval J, if this point belongs to J.

A function continuous on an interval has a number of important properties and we will include the most important of them.

Theorem 22 (Bolzano theorem on intermediate values)**.** Let the function f be continuous on the interval $[a, b]$ and let us assume that $f(a) < f(b)$. then for every $C \in \mathbb{R}$ satisfying $f(a) < C < f(b)$ there exists an $\xi \in (a, b)$ such that $f(\xi) = C$.

Proof. Choose $C \in \big(f(a), f(b)\big)$ and put $M = \{z \in [a, b];\ f(z) < C\}$. The set M is non-empty (because $a \in M$) and bounded from above (the number b is an upper bound on the set M), therefore $\sup M \in \mathbb{R}$. Let us put $\xi = \sup M$. Obviously it holds that $\xi \in [a, b]$. Let us show that $f(\xi) = C$ by eliminating the possibilities that $f(\xi) > C$ and $f(\xi) < C$.

If $f(\xi) > C$, then $\xi > a$ and thanks to the left-continuity of f at ξ we can find $\delta > 0$ such that for all $x \in (\xi - \delta, \xi]$ it holds that $f(x) > C$. This means that $M \subset [a, \xi] \setminus (\xi - \delta, \xi] = [a, \xi - \delta]$, which is in contradiction with the definition ξ.

If $f(\xi) < C$, then $\xi < b$ and thanks to the right-continuity of f at ξ we can find $\delta > 0$ such that for all $x \in [\xi, \xi + \delta)$ it holds that $f(x) < C$. This means that $[\xi, \xi + \delta) \subset M \subset [a, \xi]$, which again is a contradiction. ∎

Remark. The reader can surely formulate the theorem for the case $f(a) > f(b)$. Let us mention that the hypothesis of the theorem do not tell us anything about the number of such points $\xi \in (a, b)$, at which $f(\xi) = C$. Bolzano's theorem about intermediate values claims that there must be at least one such point.

Theorem 23 (image of an interval in a continuous function)**.** Let J be an interval. Let the function $f\colon J \to \mathbb{R}$ be continuous on J. Then $f(J)$ is an interval.

Proof. Let us verify that the set $f(J)$ satisfies the assumption from Lemma 1.11. Choose $y_1, y_2 \in f(J)$ and $z \in \mathbb{R}$, $y_1 < z < y_2$. Then there exist $x_1, x_2 \in J$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. According to Theorem 22 and the following remark f must take the value z at some point, so $z \in f(J)$. According to Lemma 1.11 we have that $f(J)$ is an interval. ∎

Definition. Let $M \subset \mathbb{R}$, $x \in M$ and the function f be defined at least on the set M (i.e. $M \subset D_f$).

- We say that f at x attains its **maximum** (or **minimum**) **on M**, if it holds that

$$\forall y \in M\colon\ f(y) \le f(x) \qquad (\text{or } \forall y \in M\colon\ f(y) \ge f(x)).$$

The point x is then called **the maximal point** (or **minimal point**) of the function f on the set M.

- We say that f has a **local maximum** (or **local minimum**) at the point x if there exists a $\delta > 0$ such that

$$\forall y \in B(x, \delta)\colon\ f(y) \le f(x) \qquad (\text{or } \forall y \in B(x, \delta)\colon\ f(y) \ge f(x)).$$

The point x is called the **point of local maximum** (or **point of local minimum**) of the function f.

• We say that f has a **sharp local maximum** (or **sharp local minimum**) at the point x, if there exists a $\delta > 0$ such that

$$\forall y \in P(x, \delta)\colon f(y) < f(x) \qquad (\text{or } \forall y \in P(x, \delta)\colon f(y) > f(x)).$$

We call the point x the **point of sharp local maximum** (or **sharp local minimum**) of the function f.

• The symbol $\max_M f$ (or $\min_M f$) denotes the largest (or smallest) value attained by the function f over the set M (if such a value exists).

• **The (global) extreme point** is a point of maximum or minimum. **A local extreme point** is a point of local maximum or local minimum.

Remark. Let us notice that if a function f attains a local extreme at the point x then it is defined on some neighborhood of x.

On the picture points x and z are local maximal points of the function f and at the points y and t the function f has a local minimum.

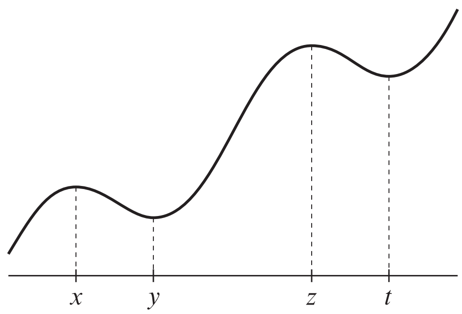


FIGURE 4.

Example 24. 1. The function $f(x) = 1/x$ does not attain an extreme on the interval $(0, 1)$.
2. The function $f\colon [0,1] \to \mathbb{R}$, given by the formula $f(x) = x$ for $x \in (0, 1)$ and $f(0) = f(1) = \frac{1}{2}$, is bounded on the interval $[0, 1]$ but does not attain an extreme.

Theorem 25 (existence of extremal points)**.** Let $a, b \in \mathbb{R}, a < b$, and f be a continuous function on the interval $[a, b]$. Then f attains a maximum and a minimum on $[a, b]$.

Proof. Let us denote $G = \sup f\big([a, b]\big)$. By Lemma 2.21 there exists a sequence $\{y_n\}$ of elements of the set $f\big([a, b]\big)$ such that $\lim y_n = G$. For every $n \in \mathbb{N}$ we find $x_n \in [a,b]$ satisfying $f(x_n) = y_n$. According to Theorem 2.30 we choose a converging subsequence $\{x_{n_k}\}$ from $\{x_n\}$ with the limit x^*. According to Theorem 2.13 the point x^* lies in the interval $[a,b]$. According to the remark after Theorem 16 it holds that

$$\lim_{k\to\infty} f(x_{n_k}) = f(x^*).$$

Because $f(x_{n_k}) = y_{n_k}$, the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ is a subsequence of $\{y_n\}_{n=1}^{\infty}$. According to Theorem 2.7 it holds that

$$f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} y_n = G.$$

Therefore $f(x^*) = G$ and x^* is a maximal point of the function f on the interval $[a,b]$.

For the proof of the existence of a minimal point let us define the function $g\colon [a,b] \to \mathbb{R}$ with the formula $g(x) = -f(x)$. The function g is continuous on $[a,b]$ and therefore it must have a maximum on $[a,b]$ by what we have just proven. Let this be at the point $x_* \in [a,b]$. Then it holds that $g(x) \leq g(x_*)$ whenever $x \in [a,b]$. This means that $f(x) \geq f(x_*)$ for every $x \in [a,b]$, and f attains its minimum on $[a,b]$ at the point x_*. Thus the theorem is proven. ∎

Corollary 26. Let f be a continuous function on the interval $[a,b]$. Then f is bounded on $[a,b]$.

Proof. By the previous theorem the function f attains a maximum and minimum on the interval $[a,b]$ and that at the points $x^*, x_* \in [a,b]$. It therefore holds that $f(x_*) \leq f(x) \leq f(x^*)$ for all $x \in [a,b]$, so the set $f\big([a,b]\big)$ is bounded. ∎

Remark. Finding the extremes of a function on a set is one of the most important mathematical tasks you will meet. Theorem 25 does not tell us how to find these extreme points, but it does give us useful information that at least one such maximum (or minimum) exists (given the hypothesis of the theorem).

In the following we will show some simple criteria how to determine that a function does not have an extreme at a given point. At points where the criteria is not met there may or may not be an extreme. We often refer to these points as "candidates for extreme points". there are generally only a few such points. If we know (for example by using Theorem 25), that our function attains a maximum (or a minimum) on a given set, then the maximum (or minimum) will be that candidate for the extreme point where the function attains the highest (or lowest) value.

At the end of this section we will consider the relationship between continuity and the inverse mapping. A continuous function on the interval J maps this interval onto the interval $f(J)$ (Theorem 23). If f is increasing on J (or decreasing), then f is one-to-one from J onto $f(J)$ and there exists an inverse mapping $f^{-1}\colon f(J) \to J$. This mapping is a function and therefore we will talk about f^{-1} as the **inverse**

function. The following theorem claims that both the type of monotonicity and the continuity of the function are inherited by its inverse function.

Theorem 27 (continuity of the inverse function)**.** Let f be a continuous and increasing (decreasing) function on the interval J. Then the function f^{-1} is continuous and increasing (decreasing) on the interval $f(J)$.

Proof. Without loss of generality we may assume that f is increasing otherwise we would work with the function $-f$. Then by Theorem 23 the function f^{-1} is defined on the interval $f(J)$ and is increasing, which is easy to see. We will now prove the continuity of the function f^{-1} on $f(J)$. Let $y_0 \in f(J)$ not be the right end-point of the interval $f(J)$. We prove the right-continuity of f^{-1} at the point y_0. Let us denote $x_0 = f^{-1}(y_0)$. The point x_0 is not the right end-point of J, because f is increasing on J. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Let us choose $x_1 \in J \cap (x_0, x_0 + \varepsilon)$ and put $\delta = f(x_1) - y_0$. >From here we see that $[y_0, y_0 + \delta] \subset f(J)$. Because it holds that $B^+(y_0, \delta) = [y_0, y_0 + \delta) = \big[f(x_0), f(x_1)\big)$ and the function f^{-1} is increasing on the interval $\big[f(x_0), f(x_1)\big)$, we get for every $y \in B^+(y_0, \delta)$

$$f^{-1}(y) \in [x_0, x_1) \subset B(x_0, \varepsilon) = B(f^{-1}(y_0), \varepsilon).$$

Similarly we can prove the left-continuity of f^{-1} at points in $f(J)$, which are not the left end-point of $f(J)$. >From here the continuity of the functionf^{-1} on $f(J)$ follows. ∎

Remark. If $n \in \mathbb{N}$ is even then the function $x \mapsto x^n$ is a continuous increasing function on $[0, +\infty)$, and therefore by Theorem 27 the function $x \mapsto \sqrt[n]{x}$ is continuous (and increasing) on $[0, +\infty)$. If $n \in \mathbb{N}$ is odd then the function $x \mapsto x^n$ is a continuous increasing function on $\mathbb{R}$, and therefore by Theorem 27 the function $x \mapsto \sqrt[n]{x}$ is continuous (and increasing) on $\mathbb{R}$.

4.3. Elementary functions

This term generally covers those functions which we typically meet most often in practice while calculating – a kind of basic repertoire of the mathematician. it is exactly this “commonness” which makes it worthwhile making clear sense of these elementary functions – define them well, give the properties, which we will consider basic, and learn the relevant techniques for calculations.

It is somewhat up to us, which functions we consider to be elementary. In accordance to most users of mathematics we will take them to be polynomials, the logarithm, the exponential function (which we will also use to define the general power a^b for $a > 0$, $b \in \mathbb{R}$), the trigonometric functions and the cyclometric functions. Let us start with the logarithm function.

Theorem 28 (the logarithm)**.** There exists exactly one function (we denote it as $\log$ and call it the **natural logarithm**), which has these properties:

(i) $D_{\log} = (0, +\infty)$,
(ii) $\log$ is increasing on $(0, +\infty)$,
(iii) $\forall x, y \in (0, +\infty)\colon\ \log xy = \log x + \log y$,
(iv) $\lim\limits_{x \to 1} \frac{\log x}{x-1} = 1$.

Remark. We will leave the proof of this theorem until we have the knowledge necessary to allow us to conduct a simple proof (see for example chapter **??**).

In the theorem the function $\log$ is characterized using certain properties, which we will take to be basic. The theorem states that these properties are not mutually exclusive (that such a function does exist) and that they determine the defined object uniquely (that there can only be one such function).

Let us now find further useful properties which are important to help us calculate with the logarithm.

Properties of the function logarithm.

- $\log 1 = 0$
  It holds that $\log 1 = \log(1 \cdot 1) = \log 1 + \log 1 = 2 \cdot \log 1$, and from here we see that $\log 1 = 0$.

- $\forall x \in (0, +\infty)\colon\ \log(1/x) = -\log x$
  Let us write $0 = \log 1 = \log\left(x \cdot \frac{1}{x}\right) = \log x + \log \frac{1}{x}$, from whence we can see that the desired relationship follows.

- $\forall n \in \mathbb{Z}\ \forall x \in (0, +\infty)\colon\ \log x^n = n \log x$
  Try to prove this relationship yourself using the previous claim and mathematical induction.

- $\lim\limits_{x\to+\infty} \log x = +\infty,\ \lim\limits_{x\to 0+} \log x = -\infty$
  Because $\log$ is an increasing function on the interval $(0, +\infty)$, the existence of both limits is guaranteed by Theorem 21 about the limit of monotone functions. In order to prove that both limits are infinite it suffices to prove that the function $\log$ is not bounded on $(0, +\infty)$ from above or bellow. to do this it suffices to notice that the sequence $\{\log 2^n\}$ is not bounded from above and the sequence $\{\log 2^{-n}\}$ is not bounded from below.

- The function $\log$ is continuous on its domain.
  It holds that

$$\lim_{x\to 1} \log x = \lim_{x\to 1} \left(\frac{\log x}{x-1} \cdot (x-1) \right) = 1 \cdot 0 = 0 = \log 1,$$

and so it is proven that the function $\log$ is continuous at the point 1. Let us now show the continuity at any point $c \in (0, +\infty)$. For $x \in (0, +\infty)$ we have

$$\log x = \log\left(c \cdot \frac{x}{c}\right) = \log c + \log \frac{x}{c}.$$

On the basis of the claim about the continuity of the function $\log$ at the point 1 which we have already proven and using Theorem 18 on the limit of compound functions we have $\lim\limits_{x\to c} \log \frac{x}{c} = 0$. From here we get $\lim\limits_{x\to c} \log x = \log c + 0 = \log c$, and so the continuity at the point c is proven.

- $R_{\log} = \mathbb{R}$
  The range of the logarithm is an interval (Theorem 23), which is not bounded from above or below. Therefore $R_{\log} = \mathbb{R}$.

The logarithm is one-to-one and maps $(0, +\infty)$ onto $\mathbb{R}$, therefore there must be exactly one number $e \in (0, +\infty)$ satisfying $\log e = 1$. The number e, which has great importance and meaning in mathematics, is irrational and approximately equal to 2.71828.

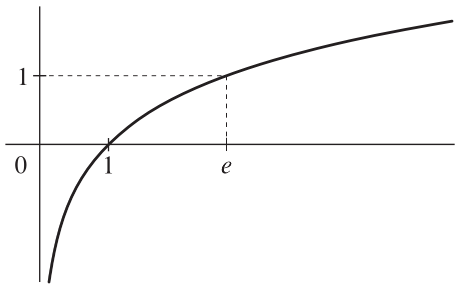


FIGURE 5. The graph of the logarithm function

In the following theorem we define the sine function in a similar way to the logarithm function in Theorem 28.

Theorem 29 (the function $\sin$ and the number π)**.** There exists exactly one real number (we will call it π) and exactly one function **sine** (we will denote it as $\sin$), which has the following properties:

(i) $D_{\sin} = \mathbb{R}$,
(ii) $\sin$ is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$,
(iii) $\sin 0 = 0$,
(iv) $\forall x, y \in \mathbb{R}$:

$$\sin(x+y) = \sin x \cdot \sin\left(\frac{\pi}{2} - y\right) + \sin\left(\frac{\pi}{2} - x\right) \cdot \sin y, \tag{6}$$

(v) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

The proof of this theorem is not easy and so we will not do it. The number we have just introduced π is the well-known number used to calculate the circumference of a circle. It is an irrational number and is approximately equal to $3{,}14159$.

Some other functions are defined on the basis of the sine function – the functions cosine tangents and cotangents. We call this collection of functions **trigonometric functions**.

Definition. The function **cosine**, which we will denote as $\cos$, is defined by the formula $\cos x = \sin(\frac{\pi}{2} - x)$ for $x \in \mathbb{R}$.

Notice that property (iv) can be rewritten using the cosine as follows: for all $x, y \in \mathbb{R}$ it holds that

$$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y, \tag{7}$$

which is a well known summation rule. Before we define the functions tangents and cotangents, let us show how one can deduce further properties of the function sine and cosine from the basic ones.

Properties of the function sine and cosine.

(SC1) The function $\cos$ is decreasing on the interval $[0, \pi]$.

The claim follows from the definition of the cosine and property (ii).

(SC2) $\cos \frac{\pi}{2} = 0$

It follows from (iii).

(SC3) $\cos 0 = \sin \frac{\pi}{2} = 1$

If in the expression (6) we choose $x = \frac{\pi}{2}$ and $y = 0$, we get, by using (iii), the following equality $\sin \frac{\pi}{2} = \sin^2(\frac{\pi}{2})$. >From here, from (iii) and from the strict monotonicity of sine on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ it immediately follows that $\sin \frac{\pi}{2} = 1$.

(SC4) $\sin \pi = 0$

It follows from (7) upon putting $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ and using (SC2).

(SC5) $\cos \pi = \sin(-\frac{\pi}{2}) = -1$

We put $x = \pi$ and $y = -\frac{\pi}{2}$ into (6). Using (SC3) and (SC4) we get $1 = \sin^2(-\frac{\pi}{2})$. Properties (ii) and (iii) then give the required relationship.

(SC6) $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

The equality $\sin \frac{\pi}{4} = \cos \frac{\pi}{4}$ follows from the definition of the function $\cos$. We put $x = y = \frac{\pi}{4}$ into (6) and using (SC3) we get $1 = 2\sin^2(\frac{\pi}{4})$. Properties (ii) and (iii) then give the required relationship.

(SC7) $\forall x \in \mathbb{R}\colon\ \sin(x + \pi) = -\sin x$

It suffices in (7) to put $y = \pi$ and use (SC4) and (SC5).

(SC8) The functions $\sin$ and $\cos$ are 2π-periodic.

According to (SC7) we get $\sin(x + 2\pi) = -\sin(x + \pi) = \sin x$.

(SC9) The function $\cos$ is even.

In (7) it suffices to put $x = \frac{\pi}{2}$, the for any $y \in \mathbb{R}$ we get by using (SC3) and (SC2) $\cos(-y) = \sin\left(\frac{\pi}{2} + y\right) = \cos y$.

(SC10) The function $\sin$ is odd.

By using (SC9) and then (SC7) we get for $x \in \mathbb{R}$

$$\sin(-x) = \cos\left(\frac{\pi}{2} + x\right) = \cos\left(-\frac{\pi}{2} - x\right) = \sin(\pi + x) = -\sin x.$$

(SC11) $\forall x \in \mathbb{R}\colon\ \sin^2 x + \cos^2 x = 1$

It suffices to substitute $y = \frac{\pi}{2} - x$ into (7) and use (SC3).

(SC12) $\forall x \in \mathbb{R}\colon\ |\sin x| \le 1,\ |\cos x| \le 1$

Follows from (SC11).

(SC13) $\forall x, y \in \mathbb{R}\colon\ \sin x - \sin y = 2\sin\Big(\dfrac{x-y}{2}\Big)\cdot\cos\Big(\dfrac{x+y}{2}\Big).$

For $a, b \in \mathbb{R}$ it holds that (by using (SC9) and (SC10))

$$\sin(a-b) = \sin a\cos(-b) + \cos a\sin(-b) = \sin a\cos b - \cos a\sin b. \qquad (8)$$

If we subtract this equation from $\sin(a+b) = \sin a\cos b + \cos a\sin b$, we get $\sin(a+b) - \sin(a-b) = 2\cos a\sin b$. For a given $x, y \in \mathbb{R}$ it suffices to put $a = (x+y)/2$ and $b = (x-y)/2$ and we get the required relationship.

(SC14) The function $\sin$ is continuous on $\mathbb{R}$.

According to (iii) we have $\sin 0 = 0$. Further by (v)

$$\lim_{x\to 0}\sin x = \lim_{x\to 0}\left(x\cdot\frac{\sin x}{x}\right) = 0\cdot 1 = 0.$$

The function $\sin$ is therefore continuous at the point 0.

Choose $x_0 \in \mathbb{R}$. by using (SC13) we get

$$\begin{aligned}\lim_{x\to x_0}\sin x &= \lim_{x\to x_0}\big(\sin x_0 + (\sin x - \sin x_0)\big) = \\ &= \sin x_0 + \lim_{x\to x_0}\left(2\sin\Big(\frac{x-x_0}{2}\Big)\cdot\cos\Big(\frac{x+x_0}{2}\Big)\right) = \\ &= \sin x_0 + 0 = \sin x_0.\end{aligned}$$

During the calculation of the last limit we used the fact that $\lim\limits_{x\to x_0}\sin\left(\frac{x-x_0}{2}\right) = 0$ (Theorem 18 and the continuity of sine at 0), the property (SC12) and Theorem 15. The function $\sin$ is therefore continuous at every point of its domain.

(SC15) The function $\cos$ is continuous on $\mathbb{R}$.

This follows from (SC14) using Theorem 18 on the limit of compound functions.

(SC16) The sine function is equal to zero at exactly the points $\{k\pi;\ k \in \mathbb{Z}\}$. The function cosine is equal to zero at exactly the points $\{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\}$.

The properties (iii), (SC4) and (SC8) give $\sin k\pi = 0$ for $k \in \mathbb{Z}$. By (ii) and (iii) the sine function is positive on the interval $(0, \frac{\pi}{2}]$. On the interval $[\frac{\pi}{2}, \pi)$ it is positive by the property (SC9) and the definition of the cosine. >From here according to property (SC7) we get, on the interval $(\pi, 2\pi)$, that sine is negative. The function $\sin$ is therefore non-zero on the set $(0,\pi)\cup(\pi,2\pi)$. If $x \in \mathbb{R}\setminus\{k\pi;\ k\in\mathbb{Z}\}$, then there exists an $l \in \mathbb{Z}$ such that $x \in \big(2l\pi, (2l+1)\pi\big)\cup\big((2l+1)\pi, (2l+2)\pi\big)$, and it holds that $\sin x = \sin(x - 2l\pi) \neq 0$.

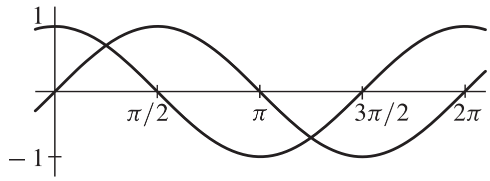


FIGURE 6. The graphs of the functions sine and cosine

Using the functions sine and cosine which we have just defined we will introduce two more trigonometric functions.

Definition. We define:

- the function **tangent** (and we denote it as tg) by the formula $\mathrm{tg}\, x = \frac{\sin x}{\cos x}$ for every real x, for which the quotient is defined i.e. $D_{\mathrm{tg}} = \mathbb{R} \setminus \{(2k+1)\frac{\pi}{2};\ k \in \mathbb{Z}\}$ (compare with (SC16));
- the function **cotangent** (cotg) by the formula $\mathrm{cotg}\, x = \frac{\cos x}{\sin x}$ for every real x, for which the quotient is defined, i.e. $D_{\mathrm{cotg}} = \mathbb{R} \setminus \{k\pi;\ k \in \mathbb{Z}\}$ (compare with (SC16)).

Now we will deduce the basic properties of the functions that we have just defined.

Properties of the tangent function.

- $\mathrm{tg}\, \frac{\pi}{4} = 1$

  This follows from (SC6).

- The function tg is continuous at every point of its domain.

  This follows from (SC14) and (SC15) using Theorem 8.

- The function tg is odd.

  This follows from (SC10) and (SC9).

- The function tg is periodic with the period π.

If $x \in D_{\mathrm{tg}}$, then also $x + \pi \in D_{\mathrm{tg}}$ and $x - \pi \in D_{\mathrm{tg}}$. For $x \in D_{\mathrm{tg}}$ by (SC7), (SC10) and (SC9) we have:

$$\begin{aligned}\mathrm{tg}(x+\pi) &= \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{\sin(-\frac{\pi}{2} - x)} = \frac{-\sin x}{-\sin(\frac{\pi}{2} + x)} = \\ &= \frac{\sin x}{\sin(\frac{\pi}{2} + x)} = \frac{\sin x}{\cos(-x)} = \frac{\sin x}{\cos x} = \mathrm{tg}\, x.\end{aligned}$$

- The function tg is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

  Let $-\frac{\pi}{2} < y < x < \frac{\pi}{2}$. Then $x - y \in (0, \pi)$. The function $\sin$ is positive on the interval $(0, \pi)$ (see the proof of (SC16)). We use the relationship (8) and get

$$0 < \sin(x - y) = \sin x \cos y - \cos x \sin y. \tag{9}$$

The function $\cos$ is positive on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (because sine is positive on the interval $(0, \pi)$), and therefore by dividing (9) by the expression $\cos x \cos y$ we get

$$0 < \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y},$$

from here the desired claim follows.

- $\lim\limits_{x \to \pi/2-} \operatorname{tg} x = +\infty$

  The claim follows from the remark after Theorem 9.

- $\lim\limits_{x \to -\pi/2+} \operatorname{tg} x = -\infty$

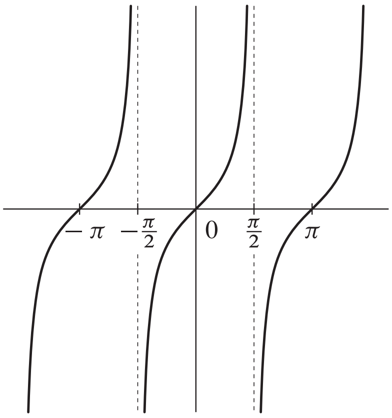
  The claim follows from the remark after Theorem 9, however one must also use the Theorem on the arithmetic of limits (Theorem 8).

- $R_{\operatorname{tg}} = \mathbb{R}$

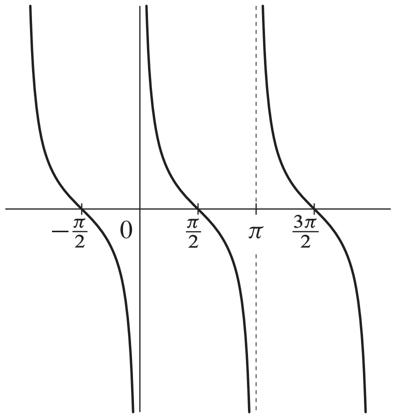
  The function tg is continuous on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and maps this interval onto an interval (Theorem 23), which, by the previous two statements is neither bounded from above no below. Therefore it must be $\operatorname{tg}\big((-\frac{\pi}{2}, \frac{\pi}{2})\big) = \mathbb{R}$, Thus the claim is proved.

Properties of the cotangent. During the proof of the following properties we can use the same approach as was used for the tangent function and therefore we do not include the reasoning.

- The function cotg is continuous at every point of its domain.
- The function cotg is odd.
- The function cotg is periodic with the period π.
- The function cotg is decreasing on the interval $(0, \pi)$.
- $\lim\limits_{x \to 0+} \operatorname{cotg} x = +\infty$
- $\lim\limits_{x \to \pi-} \operatorname{cotg} x = -\infty$
- $\operatorname{cotg} \frac{\pi}{4} = 1$
- $R_{\operatorname{cotg}} = \mathbb{R}$



OBR. 7. The graph of the tangent function



OBR. 8. The graph of the cotangent function

We will define a further elementary function as the inverse of the functions $\log$ and the inverse of the trigonometric functions which we have defined above

Definition. The **exponential function** is the inverse function to the function $\log$. We will use the symbol $\exp$ to denote it.

Properties of the exponential function.

- $D_{\exp} = \mathbb{R}$
- $R_{\exp} = (0, +\infty)$
- The function $\exp$ is increasing on $\mathbb{R}$.
- The function $\exp$ is continuous on $\mathbb{R}$.
- $\exp 0 = 1$

All five of these claims follow from the properties of the logarithmic function and from Theorem 27 on inverse functions.

- $\forall x, y \in \mathbb{R}\colon\ \exp(x+y) = \exp x \cdot \exp y$

We write $\log(\exp x \cdot \exp y) = \log(\exp x) + \log(\exp y) = x + y$. The expression $\log(\exp x \cdot \exp y)$ is therefore equal to $x + y$, and therefore $\exp\big(\log(\exp x \cdot \exp y)\big) = \exp(x+y)$. If we take into consideration that the compound mapping $\exp \circ \log$ is the identity we immediately get the required equality $\exp x \cdot \exp y = \exp(x+y)$.

- $\forall x \in \mathbb{R}\colon\ \exp(-x) = 1/\exp x$

This relationship easily follows from the following equality:

$$1 = \exp 0 = \exp(x - x) = \exp(x)\exp(-x), \quad x \in \mathbb{R}.$$

- $\forall n \in \mathbb{Z}\, \forall x \in \mathbb{R}\colon\ \exp(nx) = (\exp x)^n$

  This can be proved by mathematical induction and using the previous properties.

- $\lim\limits_{x\to-\infty} \exp x = 0,\ \lim\limits_{x\to+\infty} \exp x = +\infty$

  The claim follows from the fact that $R_{\exp} = (0, +\infty)$ and from Theorem 21 on the limit of monotone functions.

- $\lim\limits_{x\to 0} \dfrac{\exp(x) - 1}{x} = 1$

  The function $\exp$ is one-to-one and therefore $\exp x \neq 1$ for every $x \in \mathbb{R} \setminus \{0\}$. By using the version of Theorem 18 with condition (P) and the properties of the logarithm function we get

$$\lim_{x\to 0} \frac{\log(\exp x)}{\exp(x) - 1} = 1.$$

>From here, after some algebraic operations, we get

$$\lim_{x\to 0} \frac{x}{\exp(x) - 1} = 1.$$

The relationship we wish to prove now follows from Theorem 8 on the arithmetic of limits.

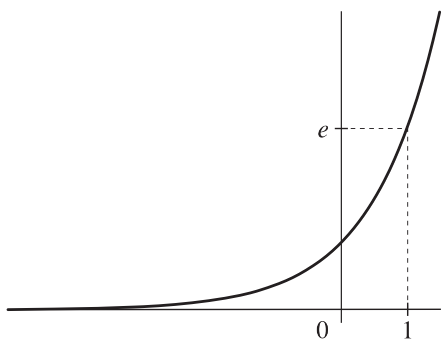


FIGURE 9. The graph of the exponential function

Example 30. For every $r \in \mathbb{Q}$ and $x \in \mathbb{R}$ it holds that $\exp(rx) = (\exp x)^r$.

Proof. Let $r = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Then from the basic properties of the exponential function it follows that

$$\big(\exp(rx)\big)^q = \left(\exp\Big(\frac{p}{q}x\Big)\right)^q = \exp\Big(q\frac{p}{q}x\Big) = \exp(px) = (\exp x)^p,$$

and therefore

$$\exp(rx) = \sqrt[q]{(\exp x)^p} = (\exp x)^{\frac{p}{q}} = (\exp x)^r.$$

■

Definition. Let $a, b \in \mathbb{R}$, $a > 0$. The **general power** a^b is defined by the formula

$$a^b = \exp(b \cdot \log a).$$

This definition extends the already existing one a^b for all real exponents b if the base a is greater than 0. Until now we had only defined the expression a^b, $a > 0$, for b rational. According to Example 30 the new definition gives the same result as the original and therefore is an extension of it. Instead of $\exp x$ we often write e^x, which is correct by the original definition of the power.

The expression a^b is defined in the following cases:

- $a \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$ arbitrary,
- $a \in \mathbb{R}$ arbitrary and $b \in \mathbb{N}$,
- $a \in \mathbb{R}$, $a \neq 0$, and $b \in \mathbb{Z}$, $b < 0$.

Later in this text we will use the convention that the expression $f(x)^{g(x)}$ will be understood as $\exp\bigl(g(x) \log f(x)\bigr)$, if g is a non-constant function.

On the basis of the properties of the functions $\log$ and $\exp$ it can be proven that all the usual rules for calculation hold.

Definition. Let $a \in \mathbb{R}$, $a > 0$, $a \neq 1$. Let us now define the function called the **base a logarithm** by the formula

$$\log_a x = \frac{\log x}{\log a} \qquad \text{for } x \in (0, +\infty).$$

Remark. The properties of the function $\log_a$ easily follow from the properties of the function $\log$. The function $\log$ is equal to the function $\log_e$. The base a logarithm of x is the exponent, such that the base a to that exponent gives the number x.

Example 31. Let $a \in \mathbb{R}$, $a > 0$. The function $x \mapsto x^a$ is continuous on $(0, +\infty)$ and it holds that $\lim\limits_{x \to 0+} x^a = 0$, $\lim\limits_{x \to +\infty} x^a = +\infty$.

Proof. According to our definition it holds that $x^a = \exp(a \log x)$. Our function is therefore the composition of two continuous functions and therefore is continuous on the interval $(0, +\infty)$.

Further it holds that $\lim\limits_{x \to 0+} (a \log x) = -\infty$, because $a > 0$, and $\lim\limits_{y \to -\infty} \exp y = 0$. According to the appropriate form of Theorem 19 (P) we get $\lim\limits_{x \to 0+} x^a = 0$. Similarly it holds that $\lim\limits_{x \to +\infty} a \log x = +\infty$ a $\lim\limits_{y \to +\infty} \exp y = +\infty$. >From here we see $\lim\limits_{x \to +\infty} x^a = +\infty$. ■

Example 32. Let $n \in \mathbb{N}$. It holds that $\lim_{x\to+\infty} \sqrt[n]{x} = +\infty$. If n is odd then also $\lim_{x\to-\infty} \sqrt[n]{x} = -\infty$.

Proof. The first claim follows from the previous example. If n is odd then the function $x \mapsto \sqrt[n]{x}$ is odd and the second claim follows from the previous by using Theorem 8. ■

Trigonometric functions are not one-to-one on their entire domains and so none of them have an inverse function. It is however useful to define the so called cyclometric functions as the inverse to a certain restriction of the trigonometric functions.

The function $\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ is continuous and increasing on its domain. According to Theorem 23 it maps the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ onto the interval $[-1,1]$. Therefore, by Theorem 27, it has an inverse function $\left(\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}\right)^{-1}$ which is defined on the interval $[-1,1]$. This function is continuous and increasing on the interval $[-1,1]$ and maps it onto the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$. The function $\left(\sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}\right)^{-1}$ is called **arcsine** and therefore we will use the notation $\arcsin$. For a given $y \in [-1,1]$ the equation $y = \sin x$ has infinitely many solutions. The value $\arcsin y$ is the unique solution of this equation that lies in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$.

Similarly if we restrict the function $\cos$ onto the interval $[0,\pi]$, then this function is decreasing and continuous on its domain $[0,\pi]$, which it maps onto the interval $[-1,1]$. There exists therefore an inverse function to the function $\left(\cos|_{[0,\pi]}\right)^{-1} : [-1,1] \to [0,\pi]$, which is decreasing and continuous on $[-1,1]$. We call it **arcosine** and we will denote it as $\arccos$.

Properties of arcsine and arcosine.

- The function $\arcsin$ is odd.

- The following equalities follow from the known properties of the functions $\sin$ and $\cos$.

$$\begin{aligned}
\arcsin(-1) &= -\frac{\pi}{2}, & \arccos(-1) &= \pi,\\
\arcsin 1 &= \frac{\pi}{2}, & \arccos 1 &= 0,\\
\arcsin 0 &= 0, & \arccos 0 &= \frac{\pi}{2},\\
\arcsin \frac{\sqrt{2}}{2} &= \frac{\pi}{4}, & \arccos \frac{\sqrt{2}}{2} &= \frac{\pi}{4}.
\end{aligned}$$

- $\forall x \in [-1,1]\colon\ \arcsin x + \arccos x = \frac{\pi}{2}$

Let us take any $x \in [-1,1]$ and denote $y = \arcsin x$. Then it holds that $x = \sin y = \cos(\frac{\pi}{2} - y)$. Because $y \in [-\pi/2, \pi/2]$, we have $\frac{\pi}{2} - y \in [0,\pi]$, and so $\arccos x = \frac{\pi}{2} - y = \frac{\pi}{2} - \arcsin x$, from here we have our desired equality.

- $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = 1$
  The function $\arcsin$ is one-to-one and therefore $\arcsin x \neq 0$ for all $x \in [-1,1] \setminus \{0\}$. Using Theorem 18 (P) and the properties of the sine function we get

$$\lim_{x\to 0} \frac{\sin(\arcsin x)}{\arcsin x} = 1.$$

>From here, after some algebraic operations, it follows that

$$\lim_{x\to 0} \frac{x}{\arcsin x} = 1.$$

The desired relationship follows from Theorem 8 on the arithmetic of limits.

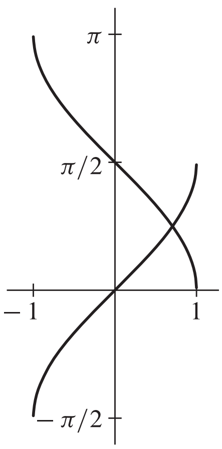


FIGURE 10. The graphs of the functions arcsine and arcosine

The function **arctangent** is defined as the inverse of the function tg restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. We denote it as arctg.

Properties of the arctangent function.

- $D_{\mathrm{arctg}} = \mathbb{R}$
- $R_{\mathrm{arctg}} = (-\frac{\pi}{2}, \frac{\pi}{2})$
- arctg is continuous increasing and odd on $\mathbb{R}$
- $\lim\limits_{x\to+\infty} \mathrm{arctg}\, x = \frac{\pi}{2}$, $\lim\limits_{x\to-\infty} \mathrm{arctg}\, x = -\frac{\pi}{2}$
- $\mathrm{arctg}\, 0 = 0$, $\mathrm{arctg}\, 1 = \frac{\pi}{4}$, $\mathrm{arctg}(-1) = -\frac{\pi}{4}$

- $\lim\limits_{x\to 0} \frac{\operatorname{arctg} x}{x} = 1$

Similarly we define the function **arcotangent** ($\operatorname{arccotg}$) as the inverse of the restriction of the function cotg to the interval $(0, \pi)$.

Properties of the arcotangent function.

- $D_{\operatorname{arccotg}} = \mathbb{R}$
- $R_{\operatorname{arccotg}} = (0, \pi)$
- $\operatorname{arccotg}$ is a continuous and decreasing function on $\mathbb{R}$
- $\lim\limits_{x\to+\infty} \operatorname{arccotg} x = 0, \ \lim\limits_{x\to-\infty} \operatorname{arccotg} x = \pi$
- $\operatorname{arccotg} 0 = \frac{\pi}{2}$, $\operatorname{arccotg} 1 = \frac{\pi}{4}$
- $\forall x \in \mathbb{R}\colon\ \operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}$

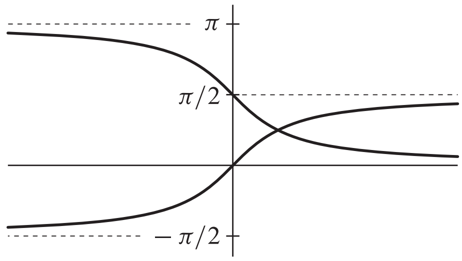


FIGURE 11. The graphs of the arctangent and arcotangent functions

The collection of functions $\arcsin$, $\arccos$, arctg and $\operatorname{arccotg}$ is called the **cyclometric functions**.

Example 33. Calculate $\lim\limits_{x\to 0} \dfrac{\cos x + 1}{\cos x - 1}$.

Solution. The limit of the numerator at the point 0 is 2, the limit of the denominator at this point is 0. On an appropriate deleted neighborhood of the point 0 the inequality $\cos x < 1$ holds, and therefore $\cos x - 1 < 0$. According to Theorem 9 this allows us to easily deduce that the limit is equal to $-\infty$. ♣

Example 34. Calculate $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x^2}$.

Solution. Let us put $f(y) = \frac{\sin y}{y}$ and $g(x) = x^2$. Then $\lim\limits_{x\to 0} g(x) = 0$ and $\lim\limits_{y\to 0} f(y) = 1$. Also put $g(x) \neq 0$ for $x \neq 0$, therefore the condition (P) of Theorem 18 is satisfied. Thus we have $\lim\limits_{x\to 0} \frac{\sin(x^2)}{x^2} = 1$. ♣

Example 35. Calculate $\lim\limits_{x\to-\infty} x\Big(\sqrt{x^2+9} - \sqrt{x^2-9}\Big)$.

Solution. As with the corresponding limits of sequences we will use the equality $a^2 - b^2 = (a-b)(a+b)$ in order to eliminate the difference of two roots. This difference is unpleasant since it is of the form "$+\infty - (+\infty)$", and therefore one cannot use the relevant claim of Theorem 8. For $x < -3$ it holds that

$$x\Big(\sqrt{x^2+9} - \sqrt{x^2-9}\Big) = x\Big(\sqrt{x^2+9} - \sqrt{x^2-9}\Big)\cdot\frac{\sqrt{x^2+9}+\sqrt{x^2-9}}{\sqrt{x^2+9}+\sqrt{x^2-9}} =$$
$$= \frac{x(x^2+9-x^2+9)}{\sqrt{x^2+9}+\sqrt{x^2-9}} = \frac{18x}{\sqrt{x^2+9}+\sqrt{x^2-9}}\cdot\frac{x^{-1}}{x^{-1}} =$$
$$= \frac{18}{\frac{\sqrt{x^2+9}}{x}+\frac{\sqrt{x^2-9}}{x}} = \frac{18}{-\sqrt{1+\frac{9}{x^2}} - \sqrt{1-\frac{9}{x^2}}}.$$

But beware, for $x < 0$ we have $\sqrt{x^2} = |x| = -x$. The limit of the expressions under the roots is equal to 1. Because the square root is continuous at 1 by the remark on page 67, we can use Theorem 18 (S). The limit is therefore equal to -9 by Theorem 8. ♣

Example 36. Calculate $\lim\limits_{x\to+\infty} \sin\left(\pi\cdot\frac{4\sqrt{x}-3\sqrt[3]{x}}{2\sqrt[4]{x^2+1}}\right)$.

Solution. We will approach this, again, by using Theorem 18 on the limit of compound functions. Firstly we calculate

$$\lim_{x\to+\infty} \pi\cdot\frac{4\sqrt{x}-3\sqrt[3]{x}}{2\sqrt[4]{x^2+1}} = \lim_{x\to+\infty} \pi\cdot\frac{4-3\frac{1}{\sqrt[6]{x}}}{2\sqrt[4]{1+\frac{1}{x^2}}} = 2\pi.$$

We used the continuity of the root and Theorem 8 together with Example 32. The sine function is continuous at 2π and therefore

$$\lim_{x\to+\infty} \sin\left(\pi\cdot\frac{4\sqrt{x}-3\sqrt[3]{x}}{2\sqrt[4]{x^2+1}}\right) = \sin 2\pi = 0.$$

♣

Example 37. Calculate $\lim\limits_{x\to+\infty} \log\left(\sqrt{x^2+4}-x\right)$.

Solution. Let us firstly calculate the limit of the inner function $\sqrt{x^2+4}-x$:

$$\lim_{x\to+\infty}\left(\sqrt{x^2+4}-x\right)\cdot\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x} = \lim_{x\to+\infty}\frac{4}{\sqrt{x^2+4}+x} = 0.$$

Let us realize that for all $x > 0$ we have $\frac{4}{\sqrt{x^2+4+x}} > 0$ and $\lim\limits_{y\to 0+}\log y = -\infty$. >From here by using Theorem 19 with condition (P) we deduce the result of the task is $-\infty$. ♣

Example 38. Calculate $\lim\limits_{x\to 0} \dfrac{4^{3x}-1}{\log(x+1)}$.

Solution. Firstly let us rewrite the limit as follows:

$$\lim_{x\to 0} \frac{\exp(3x\log 4)-1}{\log(x+1)}.$$

We know that $\lim\limits_{x\to 0} \frac{\exp(x)-1}{x} = 1$, and from the relationship $\lim\limits_{x\to 1} \frac{\log x}{x-1} = 1$ we can easily deduce, by using Theorem 18, that $\lim\limits_{x\to 0} \frac{\log(x+1)}{x} = 1$. Combining these results we get

$$\begin{aligned}\lim_{x\to 0} \frac{4^{3x}-1}{\log(x+1)} &= \lim_{x\to 0} \frac{\exp(3x\log 4)-1}{3x\log 4} \cdot \frac{3x\log 4}{x} \cdot \frac{x}{\log(x+1)} = \\ &= 1\cdot 3\log 4\cdot 1 = 3\log 4.\end{aligned}$$

♣

When calculating limits of the type $\lim\limits_{x\to c} f(x)^{g(x)}$ it tends to be useful to rewrite the expression $f(x)^{g(x)}$ using the definition of the general power as $\exp\bigl(g(x)\log f(x)\bigr)$. In order to calculate the limit $\lim\limits_{x\to c} \exp\bigl(g(x)\log f(x)\bigr)$ let us use Theorem 18, with the outer function being $y \mapsto \exp(y)$ and the inner function being $x \mapsto g(x)\log f(x)$. Let us now notice that if $\lim\limits_{x\to c} g(x)\log f(x)$ is a real number then we can use Theorem 18 with the condition (S). If $\lim\limits_{x\to c} g(x)\log f(x)$ is infinite then we use Theorem 18 with condition (P), which is satisfied automatically in this case, because the real value $g(x)\log f(x)$ is always different from $\pm\infty$.

Example 39. Calculate $\lim\limits_{x\to+\infty} \left(\dfrac{x+4}{x+2}\right)^{4x-1}$.

Solution. We rewrite this limit using the exponential function as

$$\lim_{x\to+\infty} \exp\left((4x-1)\log\frac{x+4}{x+2}\right).$$

Let us firstly calculate the limit of the exponent and then use the Theorem 18 on the limit of compound functions:

$$\begin{aligned}\lim_{x\to+\infty} (4x-1)\log\frac{x+4}{x+2} &= \lim_{x\to+\infty} \left((4x-1)\cdot\frac{\log\frac{x+4}{x+2}}{\frac{x+4}{x+2}-1}\cdot\left(\frac{x+4}{x+2}-1\right)\right) = \\ &= \lim_{x\to+\infty} \left(\frac{\log\frac{x+4}{x+2}}{\frac{x+4}{x+2}-1}\cdot\frac{2(4x-1)}{x+2}\right) = 1\cdot 8 = 8,\end{aligned}$$

and therefore

$$\lim_{x\to+\infty}\left(\frac{x+4}{x+2}\right)^{4x-1} = \lim_{x\to+\infty}\exp\left((4x-1)\log\frac{x+4}{x+2}\right) = e^8. \qquad \clubsuit$$

4.4. Derivatives

Definition. Let f be a real function and $a \in \mathbb{R}$. Then the

- **derivative of the function f at the point a** is the number

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

  if this limit exists,
- **right-hand derivative of the function** f at the point a is the number

$$f'_+(a) = \lim_{h\to 0+}\frac{f(a+h)-f(a)}{h},$$

  if this limit exists,
- **the left-hand derivative of the function** f at the point a is the number

$$f'_-(a) = \lim_{h\to 0-}\frac{f(a+h)-f(a)}{h},$$

  if this limit exists.

Remark. During the calculation of the derivative of the function f at the point $a \in \mathbb{R}$ the following cases may occur:

$$\text{the derivative at the point } a \begin{cases} \text{does not exist,} \\ \text{exists and is } \begin{cases} \textbf{real}\text{, i.e. is equal to a real number,} \\ \textbf{infinite}\text{, i.e. is equal to } +\infty \text{ or } -\infty. \end{cases} \end{cases}$$

Further remarks. 1. According to the theorem on the limit of compound functions it holds that

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

if at least one side of the equation is defined. We will often use this formulation.

2. The existence of $f'(a)$ (either real or infinite) implies that there exists a neighborhood of the point a, on which the function f is defined.

3. Notice that the derivative of the function f at the point $a \in \mathbb{R}$ exists if and only if the right and left-hand derivatives exist at a and are equal.

Geometrically, the concept of the derivative can be understood as follows.

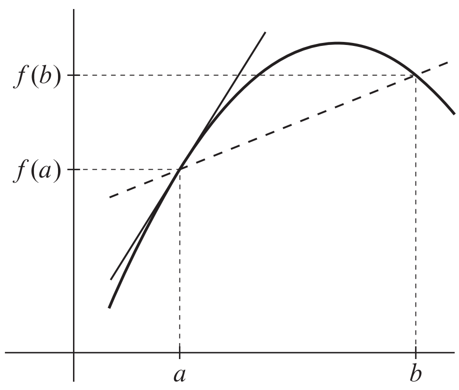


FIGURE 12.

The quotient $\frac{f(b)-f(a)}{b-a}$ is the gradient of the secant of the graph of the function, i.e. the line that intersects the points $[a, f(a)]$ a $[b, f(b)]$. If the point b approaches the point a, then the line approaches the line which intersects the point $[a, f(a)]$ with the gradient $f'(a)$ (if $f'(a)$ exists and is real). this line is uniquely determined because the uniqueness of the derivative follows from Theorem 2 on the uniqueness of the limit. We will call it the **tangent to the graph of the function** f at the point $[a, f(a)]$. The equation of the tangent to the graph of the function f at the point $[a, f(a)]$ is

$$y = f(a) + f'(a) \cdot (x - a).$$

Example 40. 1. Let $c \in \mathbb{R}$ and $f(x) = c$ on a certain neighborhood of the point $x_0 \in \mathbb{R}$. Then $f'(x_0) = 0$.

2. Let $n \in \mathbb{N}$, $f(x) = x^n$ and $x_0 \in \mathbb{R}$. By simple computation we get

$$f'(x_0) = \lim_{x \to x_0} \frac{x^n - {x_0}^n}{x - x_0} =$$
$$= \lim_{x \to x_0} \frac{(x - x_0) \cdot (x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-1})}{x - x_0} = n \cdot x_0^{n-1}.$$

3. We can easily calculate

$$\mathrm{sgn}'_+(0) = \lim_{x \to 0+} \frac{1-0}{x} = +\infty$$

and similarly $\mathrm{sgn}'_-(0) = +\infty$.

The function sgn has one-sided derivatives at the point 0 and the values of the derivatives coincide. >From here it follows that the derivative of the function sgn at the point 0 exists and is equal to $+\infty$.

Example 41. Let us calculate the derivative of the function $f(x) = |x|$.

Solution. The domain D_f is the entire $\mathbb{R}$. On the interval $(-\infty, 0)$ we have $f(x) = -x$, and therefore $f'(x) = -1$ at all points of this interval. On the interval $(0, +\infty)$ we have $f(x) = x$, and therefore $f'(x) = 1$ at every point of this interval. At the point 0 the derivative of the function $|x|$ does not exist – this is because the (double-sided) limit $\lim\limits_{x\to 0} \frac{|x|-0}{x-0}$ does not exist. The one-sided limits do exist however and therefore it holds that $f'_+(0) = \lim\limits_{x\to 0+} |x|/x = 1$ a $f'_-(0) = \lim\limits_{x\to 0-} |x|/x = -1$. We have that $f'(x) = \operatorname{sgn} x$ pro $x \in \mathbb{R} \setminus \{0\}$. At the point $x = 0$ the one-sided derivatives are $f'_+(0) = 1$ and $f'_-(0) = -1$. ♣

Theorem 42 (derivatives and continuity)**.** Let the function f have a real derivative at the point $x_0 \in \mathbb{R}$. Then the function f at the point x_0 is continuous.

Proof. Using Theorem 8 on the arithmetic of the limit we calculate:

$$\lim_{x\to x_0} \big(f(x) - f(x_0)\big) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x\to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0,$$

that is $\lim\limits_{x\to x_0} f(x) = f(x_0)$. ■

Remark. The opposite implication in the previous theorem does not generally hold, i.e. the continuity of a function at a point does not imply the existence of the derivative at that point. For example the function $x \mapsto |x|$ is continuous at the point 0, but does not have a derivative at that point (see Example 41).

An important word in the formulation of Theorem 42 is the word "real" derivative – the existence of an infinite derivative does not guarantee that the function is continuous at that point. An example of this is the function sgn, whose derivative at the point 0 is $+\infty$ but it is not continuous at the point 0.

Our aim now is to learn to calculate the derivative of any function which occurs from elementary functions using arithmetic operations and composition, and that at all points that the derivative exists. In order to do this on the one hand we will need sum, subtraction, multiplication and quotient rules for the derivative, but on the other we will also need a chain rule for the derivation of compound functions and a rule for the derivation of inverse functions. Further we need to be able to differentiate the elementary functions themselves.

Theorem 43 (arithmetic of the derivative)**.** Let $\alpha \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and $f'(x_0)$, $g'(x_0)$ be real. Then it holds that:

(i) $(f + g)'(x_0) = f'(x_0) + g'(x_0)$,
(ii) $(\alpha \cdot f)'(x_0) = \alpha \cdot f'(x_0)$,
(iii) $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$,

(iv) if $g(x_0) \neq 0$ then

$$(f/g)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)}.$$

Proof. We will restrict ourselves to proving the last claim. The function g has a real derivative at the point x_0 and so it is continuous at x_0. According to the remark after Theorem 13 g is non-zero on a certain neighborhood $P(x_0, \Delta)$. On this neighborhood we will rewrite the quotient in the limit which we are to calculate as follows.

$$\begin{aligned} \frac{(f/g)(x) - (f/g)(x_0)}{x - x_0} &= \\ &= \frac{f(x) \cdot g(x_0) - f(x_0) \cdot g(x)}{x - x_0} \cdot \frac{1}{g(x)g(x_0)} = \\ &= \frac{\frac{f(x)-f(x_0)}{x-x_0} \cdot g(x_0) - f(x_0) \cdot \frac{g(x)-g(x_0)}{x-x_0}}{g(x)g(x_0)}. \end{aligned} \tag{10}$$

Because g is continuous at the point x_0 and because $g(x_0) \neq 0$, the limit of the last expression in (10) at the point x_0 equals

$$\frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)}.$$

>From the equation (10) we now get the required relationship. ∎

Later we will also need this stronger version of claim (i) of the previous theorem.

Theorem 44. Let $f'(x_0)$ and $g'(x_0)$ exist and the expression $f'(x_0) + g'(x_0)$ is defined. Then $(f+g)'(x_0)$ exists and it holds that $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Proof. By the theorem on the arithmetic of the limit (Theorem 8) it holds that

$$\begin{aligned} (f+g)'(x_0) &= \lim_{x \to x_0} \frac{\big(f(x) + g(x)\big) - \big(f(x_0) + g(x_0)\big)}{x - x_0} = \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0). \end{aligned}$$

∎

Theorem 45 (the chain rule or derivative of compound functions)**.** Let the function f have a real derivative at the point $y_0 \in \mathbb{R}$, let the function g have a real derivative at the point $x_0 \in \mathbb{R}$ and $y_0 = g(x_0)$. Then

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Proof. Let us put

$$F(y) = \begin{cases} \frac{f(y)-f(y_0)}{y-y_0} & \text{for } y \in D_f \setminus \{y_0\}, \\ f'(y_0) & \text{for } y = y_0. \end{cases}$$

The function F is continuous at the point y_0, because

$$\lim_{y \to y_0} F(y) = \lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0} = f'(y_0) = F(y_0).$$

The function g has a real derivative at the point x_0 so it is continuous at this point by Theorem 42. According to Theorem 18 on the limit of compound functions with the condition (S) we get $\lim\limits_{x \to x_0} F\big(g(x)\big) = F\big(g(x_0)\big) = F(y_0) = f'(y_0)$. Now we can write

$$\frac{f\big(g(x)\big) - f\big(g(x_0)\big)}{x - x_0} = \frac{f\big(g(x)\big) - f(y_0)}{x - x_0} = F\big(g(x)\big) \cdot \frac{g(x) - g(x_0)}{x - x_0} \tag{11}$$

on some neighborhood $P(x_0, \Delta)$. This equation can be checked by considering the cases $g(x) \neq g(x_0)$, and $g(x) = g(x_0)$ separately. Because the limit of the right-hand side in (11) for $x \to x_0$ is equal to $f'(y_0) \cdot g'(x_0)$, this is also the limit of the left-hand side in (11). This limit is exactly the derivative $(f \circ g)'(x_0)$ by definition. ■

The last result of this section concerns the derivative of inverse functions.

Theorem 46 (derivative of the inverse function)**.** Let the function f defined on the interval (a, b) be continuous and strictly monotone and have a real, non-zero derivative at the point $x_0 \in (a, b)$. The the function f^{-1} is differentiable at $y_0 = f(x_0)$ and the following equality holds

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\big(f^{-1}(y_0)\big)}.$$

Proof. Let us define the tentative function h as:

$$h(x) = \frac{x - x_0}{f(x) - f(x_0)}, \quad x \in (a, b) \setminus \{x_0\}.$$

The function f has a real derivative at x_0 which is non-zero and therefore it holds that $\lim\limits_{x \to x_0} h(x) = 1/f'(x_0)$. According to Theorem 27 the function f^{-1} is continuous and strictly monotone on the interval $f\big((a, b)\big)$. The function f is strictly monotone and therefore the point y_0 is an interior point of the interval $f\big((a, b)\big)$. Therefore it holds that

$$\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0.$$

By using Theorem 18 with condition (P) we get

$$\lim_{y\to y_0}(h\circ f^{-1})(y)=\frac{1}{f'(x_0)}=\frac{1}{f'\big(f^{-1}(y_0)\big)}.$$

With respect to the fact that for $y\in f\big((a,b)\big)\setminus\{y_0\}$ it holds that

$$(h\circ f^{-1})(y)=\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0},$$

we get

$$(f^{-1})'(y_0)=\frac{1}{f'\big(f^{-1}(y_0)\big)}.$$

■

Derivative of elementary functions.

- $(x^n)'=nx^{n-1}$ for $x\in\mathbb{R}$, $n\in\mathbb{N}$
  See Example 40.
- $(x^n)'=nx^{n-1}$ for $x\in\mathbb{R}\setminus\{0\}$, $n\in\mathbb{Z}$, $n<0$
  According to Theorem 43 on the derivative of the quotient (and using the fact that we already know the derivative of x^{-n}) we get

$$(x^n)'=\left(\frac{1}{x^{-n}}\right)'=\frac{0-1\cdot(-n)x^{-n-1}}{(x^{-n})^2}=nx^{n-1}.$$

- $\log' x=1/x$ for $x\in(0,+\infty)$
  We will calculate this derivative from the definition. We get

$$\log' x=\lim_{h\to0}\frac{\log(x+h)-\log x}{h}=\frac{1}{x}\cdot\lim_{h\to0}\frac{\log(1+\frac{h}{x})}{(1+\frac{h}{x})-1}=\frac{1}{x}.$$

We used the relationship $\lim\limits_{y\to1}\frac{\log y}{y-1}=1$ and Theorem 18 on the limit of compound functions.

- $\exp' x=\exp x$ pro $x\in\mathbb{R}$
  According to Theorem 46 on the derivative of the inverse function we can write

$$\exp' x=\frac{1}{\log'(\exp x)}=\exp x.$$

- $\sin' x=\cos x$ for $x\in\mathbb{R}$
  We use the properties of the function $\sin$ and $\cos$, Theorem 18 on the limit of compound functions and the equality

$$\lim_{x\to0}\frac{\sin x}{x}=1.$$

Let us calculate:

$$\sin' x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{2}{h} \cdot \sin\frac{h}{2} \cdot \cos\Big(x + \frac{h}{2}\Big) = \cos x.$$

- $\cos' x = -\sin x$ pro $x \in \mathbb{R}$

  If we use the previous result and Theorem 45 on the derivative of compound functions we get

$$\cos' x = \big(\sin(\tfrac{\pi}{2} - x)\big)' = -\cos(\tfrac{\pi}{2} - x) = -\sin x.$$

- $\operatorname{tg}' x = 1/\cos^2 x$ for $x \in D_{\operatorname{tg}}$

  We calculate using Theorem 43 (derivative of the quotient)

$$\operatorname{tg}' x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

- $\operatorname{cotg}' x = -1/\sin^2 x$ for $x \in D_{\operatorname{cotg}}$

  We get the claim in the same way as in the previous case.

- $\arcsin' x = 1/\sqrt{1-x^2}$ for $x \in (-1,1)$

  We differentiate the function $\arcsin$ at all points of the open interval $(-1,1)$ using Theorem 46 on the derivative of the inverse function. If $x \in (-1,1)$, then

$$\arcsin' x = \frac{1}{\cos(\arcsin x)}.$$

This expression can be simplified if we consider that

$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)} = \sqrt{1-x^2}.$$

The sign at the root is positive because $\cos(\arcsin x) > 0$.

- $\arccos' x = -1/\sqrt{1-x^2}$ for $x \in (-1,1)$

  The formula follows from the previous claim and the relationship $\arcsin x + \arccos x = \pi/2$, which holds for $x \in [-1,1]$ (see page 78).

- $\operatorname{arctg}' x = 1/(1+x^2)$ for $x \in \mathbb{R}$

  It holds that

$$\operatorname{arctg}' x = \frac{1}{\frac{1}{\cos^2(\operatorname{arctg} x)}} = \cos^2(\operatorname{arctg} x).$$

By using the relationship $\cos^2 y = 1/(1 + \operatorname{tg}^2 y)$, which holds for all $y \in D_{\operatorname{tg}}$, we get the equation we wanted.

- $\operatorname{arccotg}' x = -1/(1+x^2)$ for $x \in \mathbb{R}$

  The equation follows from the previous claim and from the relationship $\operatorname{arctg} x + \operatorname{arccotg} x = \pi/2$, which holds for $x \in \mathbb{R}$ (see page 80).

- Let $a \in \mathbb{R}$, then $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$.
  The given function can be differentiated as follows
  $$(x^a)' = \big(\exp(a \cdot \log x)\big)' = \exp(a \cdot \log x) \cdot \frac{a}{x} = a \cdot x^{a-1}.$$
- Let $b \in \mathbb{R}$, $b > 0$, then $(b^x)' = b^x \log b$ for $x \in \mathbb{R}$.
  Here it holds that
  $$(b^x)' = \big(\exp(x \cdot \log b)\big)' = \exp(x \cdot \log b) \cdot \log b = b^x \cdot \log b.$$

Example 47. Compute the derivative of the function $f(x) = \dfrac{1}{4} \log \dfrac{x^2-1}{x^2+1}$ everywhere it exists.

Solution. Firstly we determine D_f. By solving the inequality $(x^2-1)/(x^2+1) > 0$ we get $D_f = (-\infty, -1) \cup (1, +\infty)$.

While computing the derivative of the function f we will use Theorem 45, the chain rule, the equation $\log' y = 1/y$ and Theorem 43 (derivative of the quotient). Firstly we find all x, for which the hypothesis of these claims are satisfied. The function $x \mapsto \frac{x^2-1}{x^2+1}$ can be differentiated at any point $x \in \mathbb{R}$. The equation $\log' y = 1/y$ holds for $y \in (0, +\infty)$, so the chain rule can be used for $x \in (-\infty, -1) \cup (1, +\infty)$. We get
$$f'(x) = \frac{1}{4} \frac{1}{\frac{x^2-1}{x^2+1}} \cdot \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2}$$
for $x \in (-\infty, -1) \cup (1, +\infty)$. After alterations we have
$$f'(x) = \frac{x}{x^4-1}, \quad x \in (-\infty, -1) \cup (1, +\infty). \tag{12}$$
We have calculated the derivative at all points of the domain.

If at some point of the domain of function f the hypothesis of the theorems we used were not satisfied then this would not necessarily mean that the derivative does not exist at that point. We would have to use different approaches to investigate the derivatives at those points (see Example 49).

Notice also that the expression on the right-hand side of (12) is actually defined for all x z $\mathbb{R}$ except for 1 and -1. this does not mean that the function f is differentiable at all points of the set $\mathbb{R} \setminus \{-1, 1\}$. Our function cannot be differentiated at the points of the set $[-1, 1]$, because the elements do not lie in D_f. ♣

Remark. In the following examples and exercises we will not particularly emphasize that the derivative should be calculated "everywhere that it exists". This will be taken as automatically meant similarly as the investigation of the existence of single-sided derivatives at the points of D_f, where the double-sided derivatives do not exist.

Example 48. Compute the derivative of the function $f(x) = \sqrt[n]{x}$ for $n \in \mathbb{N}$ odd, $n > 1$.

Solution. The domain D_f is the entire $\mathbb{R}$. On the interval $(0, +\infty)$ we have $f(x) = x^{\frac{1}{n}}$, and therefore $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1} = \frac{1}{n}\sqrt[n]{x^{1-n}}$ at every point in this interval. On the interval $(-\infty, 0)$ we have $f(x) = -\sqrt[n]{-x}$, and by the previous calculations we have

$$f'(x) = -\left(\frac{1}{n}\sqrt[n]{(-x)^{1-n}}\right) \cdot (-1) = \frac{1}{n}\sqrt[n]{(-x)^{1-n}} = \frac{1}{n}\sqrt[n]{x^{1-n}}$$

at every point in the interval $(-\infty, 0)$. In the final equation we have used the fact that $n-1$ is even. Together we have $f'(x) = \frac{1}{n}\sqrt[n]{x^{1-n}}$ for $x \in \mathbb{R} \setminus \{0\}$.

At the point 0 we will compute the derivative from the definition. It is

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[n]{x} - \sqrt[n]{0}}{x - 0} = \lim_{x \to 0} \frac{\sqrt[n]{x}}{x} = \lim_{x \to 0} \sqrt[n]{\frac{1}{x^{n-1}}} = +\infty,$$

where we have again used the fact that $n-1$ is even. ♣

Example 49. Compute the derivative of the function $f(x) = (\sin x)^{|\cos x|}$.

Solution. Firstly we will determine the domain. Let us rewrite the function using the exponential form (this form is also the most suitable for calculating the derivative):

$$f(x) = \exp\bigl(|\cos x| \log(\sin x)\bigr).$$

It can be seen that the domain of the function f is determined by the condition $\sin x > 0$, therefore

$$D_f = \bigcup_{k \in \mathbb{Z}} \bigl(2k\pi, (2k+1)\pi\bigr).$$

Firstly we will make use of the chain rule. If $\cos x \neq 0$, then the assumptions of the theorem are satisfied (see Example 41). Then for every $x \in D_f \setminus \{\pi/2 + 2k\pi;\ k \in \mathbb{Z}\}$ it holds that

$$f'(x) = \exp\bigl(|\cos x| \log(\sin x)\bigr) \cdot \\ \cdot \left(\operatorname{sgn}(\cos x)(-\sin x)\log(\sin x) + |\cos x| \frac{\cos x}{\sin x}\right),$$

which, after manipulation, is equal to

$$f'(x) = (\sin x)^{|\cos x|} \operatorname{sgn}(\cos x) \left(\frac{\cos^2 x}{\sin x} - \sin x \log(\sin x)\right).$$

We will use the definition to compute the derivative at the point $\pi/2+2k\pi$, $k\in\mathbb{Z}$. We get

$$\begin{aligned}
&\lim_{x\to\pi/2+2k\pi}\frac{\exp\big(|\cos x|\log(\sin x)\big)-1}{x-\pi/2-2k\pi}=\\
&=\lim_{x\to\pi/2+2k\pi}\frac{\exp\big(|\cos x|\log(\sin x)\big)-1}{|\cos x|\log(\sin x)}\cdot\frac{|\cos x|\log(\sin x)}{x-\pi/2-2k\pi}=\\
&=\lim_{x\to\pi/2+2k\pi}\frac{\exp\big(|\cos x|\log(\sin x)\big)-1}{|\cos x|\log(\sin x)}\cdot\frac{\log(\sin x)}{\sin x-1}\cdot\frac{\sin x-1}{x-\pi/2-2k\pi}|\cos x|=\\
&=1\cdot 1\cdot\sin'(\pi/2+2k\pi)\cdot|\cos(\pi/2+2k\pi)|=\\
&=\cos(\pi/2+2k\pi)\cdot|\cos(\pi/2+2k\pi)|=0.
\end{aligned}$$

We used Theorem 18 twice here both times with the condition (P). Firstly we computed the limit

$$\lim_{x\to\pi/2+2k\pi}\frac{\exp\big(|\cos x|\log(\sin x)\big)-1}{|\cos x|\log(\sin x)}.$$

Here the inner function is $x\mapsto|\cos x|\log(\sin x)$, which is non-zero on $P(\pi/2+2k\pi,\pi/2)$ ($k\in\mathbb{Z}$) and

$$\lim_{x\to\pi/2+2k\pi}|\cos x|\log(\sin x)=0,\quad k\in\mathbb{Z}.$$

The outer function is $y\mapsto\frac{e^y-1}{y}$, and its limit at the point 0 is 1. Further we computed the limit

$$\lim_{x\to\pi/2+2k\pi}\frac{\log(\sin x)}{\sin x-1}.$$

Here conduct the reasoning of the use of Theorem 18 yourself.

Because $\operatorname{sgn}\big(\cos(\pi/2+2k\pi)\big)=0$, it holds that

$$f'(x)=(\sin x)^{|\cos x|}\operatorname{sgn}(\cos x)\left(\frac{\cos^2 x}{\sin x}-\sin x\log(\sin x)\right),\quad x\in D_f.$$

♣

We have already mentioned the importance of the task of finding extremes of a function. The following theorem is useful when solving that type of task.

Theorem 50 (necessary condition for a local extreme)**.** Let $x_0\in\mathbb{R}$ be a local-extreme point of the function f. If $f'(x_0)$ exists then $f'(x_0)=0$.

Proof. Let $f'(x_0)$ exist and be non-zero. If $f'(x_0)>0$, then by Theorem 13 there exists a $\delta>0$ such that for every $x\in P(x_0,\delta)$ we have

$$\frac{f(x)-f(x_0)}{x-x_0}>0.$$

Here, if $x \in (x_0, x_0 + \delta)$, then $f(x) > f(x_0)$. If $x \in (x_0 - \delta, x_0)$, we get $f(x) < f(x_0)$. This means that at the point x_0 f does not have a local extreme. The case $f'(x_0) < 0$, can similarly be brought to a contradiction. So if $f'(x_0)$ exists it must be 0. ∎

Remark. Let $f\colon [a, b] \to \mathbb{R}$ and let f attain its maximum over $[a, b]$ at the point $x_0 \in (a, b)$. Then x_0 is a point of local maximum of the function f. This is obvious because it holds that $f(x) \leq f(x_0)$ for all x from the interval $[a, b]$, and therefore holds for all $x \in (x_0 - \Delta, x_0 + \Delta)$, where Δ is an arbitrary positive number such that $a < x_0 - \Delta$ and $x_0 + \Delta < b$.

>From here we see that the candidates for the maximum and minimum points of f on the interval $[a, b]$ are

- the end points a and b,
- the points $x \in (a, b)$, where the derivative of f is zero (Theorem 50),
- the points $x \in (a, b)$, where $f'(x)$ does not exist (Theorem 50).

If we know that $f\colon [a, b] \to \mathbb{R}$ attains its extreme on $[a, b]$, which is the case, for example, for continuous functions f on $[a, b]$, then the previous analysis gives us an idea of how to find the extremes. Put

$$M = \{a, b\} \cup \{x \in (a, b);\ f'(x) = 0\} \cup \{x \in (a, b);\ f'(x) \text{ does not exist}\}.$$

According to the above, all extremes of the function f on $[a, b]$ lie in the set M. Then it suffices to compare the functional values of $f(z)$, where $z \in M$. The maximum is at the point where the value of $f(z)$ is greatest and the minimum is where the value $f(z)$ is smallest.

Notice that it suffices to find the set $\tilde{M}$, which contains M, and compare functional values of f at the points of the set $\tilde{M}$. Sometimes it is easier to consider and compare more points than to find the set M exactly. For example at some points it is easier to calculate the functional value than to determine the existence of the derivative.

Example 51. Find the extremes of the function $f(x) = \sqrt[3]{(x^2 - 3x)^2}$ on the closed interval $[-1, 4]$.

Solution. The function f is continuous since it is the composition of continuous functions. We have to find the extremes of a continuous function on a closed interval. In this case we know that the given function on the given set attains both its maximum and its minimum (Theorem 25). Candidate points are those where the derivative is 0, also points where $f'(x)$ does not exist and finally the end points of the interval.

Let us calculate the derivative of the function f. According to Example 48 it holds that

$$f'(x) = \frac{2}{3}\sqrt[3]{((x^2 - 3x)^2)^{-2}}(x^2 - 3x)(2x - 3)$$

for all $x \in (-\infty, 0) \cup (0, 3) \cup (3, +\infty)$. We compare the functional at the points -1, 4 (end-points of the interval $[-1, 4]$), at the point $3/2$ ($f'(3/2) = 0$) an the points 0, 3 (where investigating the derivative is more complicated than evaluating the function). The functional values at these points are

$$f(-1) = 2\sqrt[3]{2},\ f(0) = 0,\ f(3/2) = \frac{3}{2}\sqrt[3]{3/2},\ f(3) = 0,\ f(4) = 2\sqrt[3]{2}.$$

Comparing the functional values we see that the function f attains its minimum over $[-1, 4]$ at the points 0 and 3 so we have $\min_{[-1,4]} f = 0$; the maximum is attained at the points -1 and 4 and it holds that $\max_{[-1,4]} f = 2\sqrt[3]{2}$. ♣

Example 52. Determine the extremes of the function $f(x) = 2\sqrt{1-x^2} + \operatorname{arctg} \dfrac{x}{\sqrt{1-x^2}}$.

Solution. Firstly let us determine the domain:

$$D_f = \{x \in \mathbb{R};\ 1 - x^2 > 0\} = (-1, 1).$$

The function is continuous on its domain. Let us compute the limit at the end-points of the domain: $\lim\limits_{x \to -1+} f(x) = -\pi/2$, $\lim\limits_{x \to 1-} f(x) = \pi/2$. Let us now define the tentative function $\bar{f}\colon [-1, 1] \to \mathbb{R}$ as:

$$\bar{f}(x) = \begin{cases} f(x) & \text{for } x \in (-1, 1), \\ \pi/2 & \text{for } x = 1, \\ -\pi/2 & \text{for } x = -1. \end{cases}$$

The function $\bar{f}$ is continuous of the interval $[-1, 1]$. We say that we have *assigned continuous values* to the function f at the points -1 and 1. >From Theorem 25 it follows that the function $\bar{f}$ attains a maximum and minimum over $[-1, 1]$. Let us show that it holds that $\sup_{(-1,1)} f = \max_{[-1,1]} \bar{f}$ and $\inf_{(-1,1)} f = \min_{[-1,1]} \bar{f}$.

Let us assume that the function f attains its maximum at the point $x_0 \in [-1, 1]$. The inequality $\sup_{(-1,1)} f \leq \bar{f}(x_0)$ is obvious. Further there exists a sequence $\{x_n\}$ of points in $(-1, 1)$, which converge to x_0. By Heine's theorem it holds that $\bar{f}(x_0) = \lim \bar{f}(x_n) = \lim f(x_n)$, and therefore $\sup_{(-1,1)} f = \bar{f}(x_0)$ according to Lemma 2.21. The relationship for the infimum can be explained similarly.

Let us therefore investigate the extremes of the function $\bar{f}$ on $[-1, 1]$. Let us first compute the derivative of the function $\bar{f}$:

$$\bar{f}'(x) = f'(x) = \frac{1 - 2x}{\sqrt{1 - x^2}}, \quad x \in (-1, 1).$$

The derivative is equal to 0 only for $x = 1/2$. Let us then compare the functional values of the function $\bar{f}$ at the points -1, $1/2$, 1:

$$\bar{f}(-1) = -\pi/2, \qquad \bar{f}(1/2) = \sqrt{3} + \pi/6, \qquad \bar{f}(1) = \pi/2.$$

We have $\max_{[-1,1]} \bar{f} = \bar{f}(1/2)$ and $\min_{[-1,1]} \bar{f} = \bar{f}(-1)$. If we realize the mutual relationship between the functions $\bar{f}$ and f, then we get $\max_{(-1,1)} f = f(1/2) = \sqrt{3} + \pi/6$. The function $\bar{f}$ is continuous on $[-1,1]$ and therefore $\inf\{f(x);\ x \in (-1,1)\} = \bar{f}(-1)$. >From here it follows that the function f does not attain a minimum over the interval $(-1,1)$. ♣

4.5. Deeper theorems about the derivative of a function

Theorem 53 (Rolle's theorem)**.** Let $a, b \in \mathbb{R}$, $a < b$, and the function f have the following properties:

(i) it is continuous on the interval $[a,b]$,
(ii) has (real or infinite) derivative at every point of the interval (a,b),
(iii) it holds that $f(a) = f(b)$.

Then there exists an $\xi \in (a,b)$ such that $f'(\xi) = 0$.

Proof. According to Theorem 25 the function f attains a maximum and a minimum over $[a,b]$. Firstly let us assume that the maximum and minimum points lie in the set $\{a,b\}$. Then from (iii) it immediately follows that the function f is constant on $[a,b]$. But constant functions have zero derivative at every point of the interval (a,b), so in this case the point ξ can be taken arbitrarily in the interval (a,b).

Another possible situation is that at least one of the points where the function attains its extreme lies in the interval (a,b). Let us denote this point as ξ. We already know that ξ is a point of local extreme for f, and we also know (hypothesis (ii)), that $f'(\xi)$ exists. According to Theorem 50 it is necessary that $f'(\xi) = 0$ and the claim of the theorem holds in this case as well. Because there are no other possible cases, the theorem has been proven. ∎

Remark. Geometrically Theorem 53 can be interpreted that under assumptions (i)–(iii) the graph of the function f contains a point $[\xi, f(\xi)]$, in which the tangent to the graph of the function f is parallel with the x-axis.

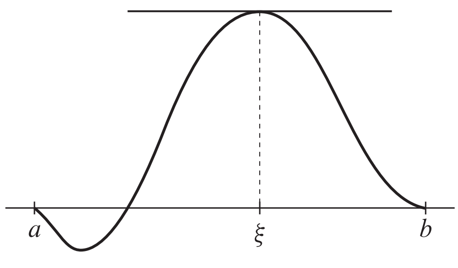


FIGURE 13.

The following theorem which is a corollary of Theorem 53, has a wide range of applications. It is the so called Lagrange mean value theorem.

Theorem 54 (Lagrange theorem)**.** Let $a, b \in \mathbb{R}$, $a < b$, let the function f be continuous on the interval $[a, b]$ and have (real or infinite) derivative at every point of the interval (a, b). Then there exists an $\xi \in (a, b)$ satisfying

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof. The function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

satisfies the assumptions of Rolle's theorem on the interval $[a, b]$. Conditions (i) and (iii) are easy to check, condition (ii) follows from Theorem 44. There exists an $\xi \in (a, b)$ such that $F'(\xi) = 0$. Because by Theorem 44 it holds that

$$F'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a},$$

we get the required equality from this immediately. ∎

Remark. Under the assumptions of Theorem 54 we can express the growth of the function f over the interval $[a, b]$, which is equal to $f(b) - f(a)$, as:

$$f(b) - f(a) = f'(\xi) \cdot (b - a),$$

that is as a the product of the increase in the variable x and the derivative at the point ξ, about whom we know only that it belongs in (a, b).

Notice that neither Theorem 53 nor Theorem 54 say anything about the number of points ξ with the given property. They only say that one such point must exist.

Geometrically Theorem 54 can be interpreted as saying that under the given assumptions the graph of the function f contains the point $[\xi, f(\xi)]$, where the tangent to the graph of f is parallel to the line connecting the points $[a, f(a)]$ and $[b, f(b)]$.

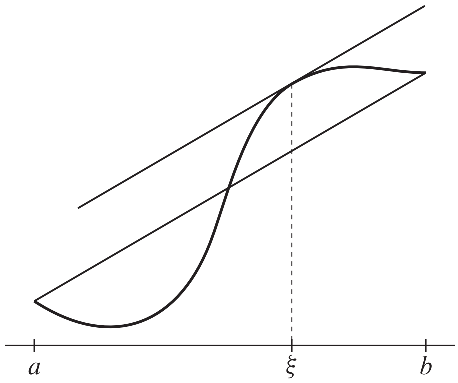


FIGURE 14.

One of the nice applications of the Lagrange theorem is an elegant proof of the theorem about the relationship between the monotonicity of a function and the sign of the derivative.

Theorem 55 (sign of the derivative and monotonicity)**.** Let $J \subset \mathbb{R}$ be a non-degenerate interval. Let f be continuous on J and in every interior point of J (we denote the set of interior points of J as $\operatorname{Int} J$) has a (real or infinite) derivative.

- If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.
- If $f'(x) < 0$ for all $x \in \operatorname{Int} J$, then f is decreasing on J.
- If $f'(x) \geq 0$ for all $x \in \operatorname{Int} J$, then f is non-decreasing on J.
- If $f'(x) \leq 0$ for all $x \in \operatorname{Int} J$, then f is non-increasing on J.

Proof. Let us prove the first claim of the theorem. Chose to points x_1, x_2 such that $x_1, x_2 \in J$, $x_1 < x_2$. On the interval $[x_1, x_2]$ the function f is continuous and has a derivative at the points of the interval (x_1, x_2). The function f then satisfies the assumptions for the Lagrange theorem (Theorem 54) on the interval $[x_1, x_2]$. Therefore there exists a $\xi \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi).$$

According to the hypothesis, however, $f'(\xi) > 0$, and because the denominator in the quotient in the given inequality is positive then we must have $f(x_2) - f(x_1) > 0$.

We have proved that

$$\forall x_1, x_2 \in J, x_1 < x_2 \colon f(x_1) < f(x_2),$$

or f is increasing on the interval J.

The other claims can be proven similarly. ■

Remark. >From the previous theorem it follows that a continuous function having zero derivative at every interior point of the interval J is constant on J; this is because it is both non-increasing and non-decreasing on J.

The following theorem is useful for calculating single-sided derivatives.

Theorem 56. Let the function f be right-continuous at the point $a \in \mathbb{R}$ and let $\lim\limits_{x \to a+} f'(x)$ exist. Then also $f'_+(a)$ exists and we have

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

Proof. Denote $A = \lim\limits_{x \to a+} f'(x)$. Choose any $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Then, from that, we find a $\delta \in \mathbb{R}$, $\delta > 0$, such that $f'(x) \in B(A, \varepsilon)$ for every $x \in (a, a + \delta)$. The function f has a real derivative at every point of the interval $(a, a + \delta)$.

Now let us take any point x from the interval $(a, a + \delta)$. The function f is continuous on the interval $[a, x]$ since it is differentiable at all points of the interval $(a, x]$ and its right-continuity at the point a. On (a, x) f is differentiable. According to the Lagrange theorem there is a point $\xi \in (a, x)$, such that

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}.$$

Because $\xi \in (a, a + \delta)$, we have $\frac{f(x)-f(a)}{x-a} = f'(\xi) \in B(A, \varepsilon)$. According to the definition of right-continuity therefore $f'_+(a) = A$. ■

Remark. The corresponding theorem holds for left-continuity.

Example 57. Using Theorem 56 on the calculation of the single-sided derivative we can easily find that the the relevant single-sided derivatives of $\arcsin$ at the points 1 and -1 are equal to $+\infty$.

Example 58. Calculate the derivative of the function $f(x) = \sqrt{\operatorname{arctg}(\log^2 x)}$ everywhere it exists.

Solution. The domain of the function f is $D_f = (0, +\infty)$. It holds that $(\sqrt{z})' = \frac{1}{2\sqrt{z}}$ for $z \in (0, +\infty)$. Firstly let us calculate the derivative at the point where the condition $\operatorname{arctg}(\log^2 x) \neq 0$ is satisfied, i.e. at the points of $D_f \setminus \{1\}$. At those points we can compute in a standard way using the chain rule, Theorem 45. For $x \in D_f \setminus \{1\}$ we have

$$f'(x) = \frac{\log x}{x\sqrt{\operatorname{arctg}(\log^2 x)}(1 + \log^4 x)}.$$

Because the function f is also defined at the point 1, we ask whether the derivative exists, or at least a single-sided derivative. We have two possibilities how we can make the computation: use the definition or Theorem 56 on the computation of single-sided derivatives. Using the definition is a universal approach but sometimes can be difficult to calculate the limit in question. In order to use the theorem we have to:

(i) verify the one-sided continuity of the function,
(ii) verify the existence of the one-sided limit of the function f' and compute it.

The function we are investigating is continuous on all of D_f, therefore also at the point 1. Further

$$\begin{aligned}\lim_{x\to1+} f'(x) &= \lim_{x\to1+} \frac{\log x}{x\sqrt{\operatorname{arctg}(\log^2 x)}(1+\log^4 x)} = \\ &= \lim_{x\to1+} \frac{1}{x}\cdot\sqrt{\frac{\log^2 x}{\operatorname{arctg}(\log^2 x)}}\cdot\frac{1}{1+\log^4 x} = 1,\\ \lim_{x\to1-} f'(x) &= \lim_{x\to1-} -\frac{1}{x}\cdot\sqrt{\frac{\log^2 x}{\operatorname{arctg}(\log^2 x)}}\cdot\frac{1}{1+\log^4 x} = -1.\end{aligned}$$

We used $\lim\limits_{x\to0} \frac{\operatorname{arctg} x}{x} = 1$. According to Theorem 56 on the computation of single-sided derivatives we have

$$\begin{aligned}f'_+(1) &= \lim_{x\to1+} f'(x) = 1,\\ f'_-(1) &= \lim_{x\to1-} f'(x) = -1.\end{aligned}$$

The left and right derivatives of the function f at the point 1 do not agree, hence $f'(1)$ does not exist. ♣

Let us give also the so called l'Hopital's rule, which we will not prove. It allows us to compute limits in situations which we will describe as "$\frac{0}{0}$ type" and "typ $\frac{\infty}{\infty}$ type", that is in those cases where the claim of the quotient rule in Theorem 8 cannot be used. We include l'Hoptial's rule here because its proof rests in a certain generalization of Theorem 54.

Theorem 59 (l'Hospital's rule)**.** Let the functions $f$and g have real derivatives on a certain deleted neighborhood of the point $a \in \mathbb{R}^*$.

(i) If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ and there exists $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$, then also $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

(ii) If $\lim_{x\to a} |g(x)| = +\infty$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then also $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 60. Compute $\lim_{x\to+\infty} (\pi - 2\operatorname{arctg} x)\log x$.

Solution. The product rule cannot be directly applied because the limit of the expression in the brackets is 0 and the limit of the other factor is $+\infty$. This a so called "indeterminate expression of the type $0 \cdot (+\infty)$".

In order to use l'Hospital's rule we have to modify the expression somewhat. We will calculate

$$\lim_{x\to+\infty} \frac{\pi - 2\operatorname{arctg} x}{\frac{1}{\log x}}, \tag{13}$$

which is an "indeterminate expression of the type $\frac{0}{0}$".

We differentiate the numerator and the denominator in the quotient (13) and calculate

$$\lim_{x\to+\infty} \frac{-\frac{2}{1+x^2}}{-\frac{1}{\log^2 x}\cdot\frac{1}{x}} = \lim_{x\to+\infty} \frac{2x\log^2 x}{1+x^2}.$$

Now we have got an "indeterminate expression of the type $\frac{+\infty}{+\infty}$" and again we try to use l'Hospital's rule. We will use it several times yet (at every step check that the hypothesis for its use are satisfied). We get:

$$\lim_{x\to+\infty} \frac{2\log^2 x + 4\log x}{2x}, \tag{14}$$

$$\lim_{x\to+\infty} \frac{4\log x \cdot \frac{1}{x} + \frac{4}{x}}{2} = \lim_{x\to+\infty} \frac{2\log x + 2}{x}, \tag{15}$$

$$\lim_{x\to+\infty} \frac{2}{x}.$$

With the final usage of l'Hospital's rule we got to a function who's limit can easily be calculated and it is equal to 0. By l'Hospital's rule it follows that all the limits in (15), (14) and (13) are equal to 0. Therefore in the end we get

$$\lim_{x\to+\infty} (\pi - 2\operatorname{arctg} x)\log x = 0.$$

Let us remark that the original limit could also have been modified to take the form

$$\lim_{x\to+\infty} \frac{\log x}{\frac{1}{\pi - 2\operatorname{arctg} x}},$$

then we would have calculated a limit of the type "$\frac{+\infty}{+\infty}$".

♣

Example 61. Compute $\lim\limits_{x\to 0} \dfrac{x^2 \sin \frac{1}{x}}{\sin x}$.

Solution. Because $\lim\limits_{x\to 0} x^2 \sin \frac{1}{x} = 0$ a $\lim\limits_{x\to 0} \sin x = 0$, we will try to use l'Hospital's rule to compute

$$\lim_{x\to 0} \frac{2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot (-\frac{1}{x^2})}{\cos x} = \lim_{x\to 0} \frac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{\cos x}.$$

This limit does not exist however. This does not mean that the original limit does not exist, only that one cannot use l'Hospital's rule because the limit of the quotient of derivatives does not exist.

Let us rewrite our function as:

$$\frac{x^2 \sin \frac{1}{x}}{\sin x} = \frac{x}{\sin x} \cdot x \sin \frac{1}{x}.$$

Now we easily see (by using $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, $\lim\limits_{x\to 0} x \sin \frac{1}{x} = 0$ and the product rule), that

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0.$$

♣

Example 62. Let $a, \beta, \gamma \in (0, +\infty)$. Then

$$\lim_{x\to +\infty} \frac{\log^\beta x}{x^\gamma} = 0, \qquad \lim_{x\to +\infty} \frac{x^\gamma}{a^x} = 0.$$

Solution. Let us firstly compute the limit $\lim\limits_{x\to +\infty} \frac{\log x}{x^\gamma}$ using l'Hospital's rule:

$$\lim_{x\to +\infty} \frac{\log x}{x^\gamma} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{\gamma x^{\gamma-1}} = \lim_{x\to +\infty} \frac{1}{\gamma x^\gamma} = 0.$$

By using this result we get

$$\lim_{x\to +\infty} \frac{\log^\beta x}{x^\gamma} = \lim_{x\to +\infty} \left(\frac{\log x}{x^{\gamma/\beta}} \right)^\beta = 0,$$

We calculated the final limit using the theorem on composite functions, and the fact that the function $y \mapsto y^\beta$ is right continuous at the point 0 (see Example 31).

The second limit can be computed similarly. ♣

Example 63. Compute $\lim\limits_{y\to 0+} y \log y$.

Solution. According to Example 20 it suffices to compute the limit $\lim\limits_{x\to +\infty} \frac{-\log x}{x}$, which is equal to 0 by Example 62. ♣

4.6. Convex and concave functions

In the following section we will accustom ourselves with the following terms, which are useful in analyzing a function.

Definition. Let I be an interval and $f\colon I \to \mathbb{R}$. We say that the function f is

- **convex** on the interval I, if
$$\forall x_1, x_2 \in I\ \forall\lambda \in [0,1]\colon f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2),$$
- **strictly convex** on the interval I, if
$$\forall x_1, x_2 \in I, x_1 \neq x_2\ \forall\lambda \in (0,1)\colon f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2),$$
- **concave** on the interval I, if
$$\forall x_1, x_2 \in I\ \forall\lambda \in [0,1]\colon f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda) f(x_2),$$
- **strictly concave** on the interval I, if
$$\forall x_1, x_2 \in I, x_1 \neq x_2\ \forall\lambda \in (0,1)\colon f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda) f(x_2).$$

Remark. The following claim can be seen to hold immediately from the definition. The function f is concave on I, if and only if the function $-f$ is convex on I. The function f is strictly concave on I, if and only if the function $-f$ is strictly convex on I.

Remark. The geometrical implication of the inequalities in the definition of a convex function are the following. The function f is convex on I is such that whenever we take the line that intersects a pair of points $[x_1, f(x_1)]$, $[x_2, f(x_2)]$, $x_1, x_2 \in I$, $x_1 < x_2$, then all the points of the set $\{[x, f(x)] \in \mathbb{R}^2;\ x \in [x_1, x_2]\}$ are under the line or on the line.

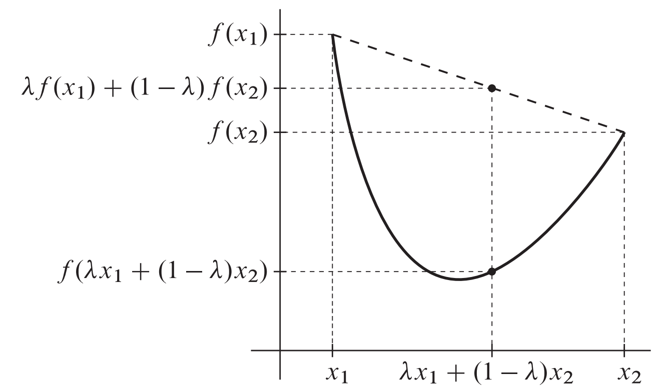


FIGURE 15.

In the following lemma we will describe convexity somewhat differently than in the previous definition.

Lemma 64. The function f is convex on$I$, if and only if, for every three points $x_1, x, x_2 \in I$, $x_1 < x < x_2$, it holds that

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}. \tag{16}$$

The following picture illustrated the claim of the lemma because the gradient of a line passing through the points $[x_1, f(x_1)]$ and $[x, f(x)]$, respectively the points $[x, f(x)]$ and $[x_2, f(x_2)]$, is equal to $\frac{f(x)-f(x_1)}{x-x_1}$, respectively $\frac{f(x_2)-f(x)}{x_2-x}$.

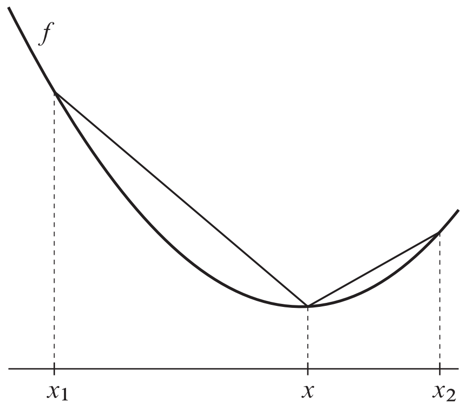


FIGURE 16.

Proof. Let us assume that firstly f is convex on I. We take the points $x_1, x, x_2 \in I$ such that $x_1 < x < x_2$. If we put

$$\lambda = \frac{x_2 - x}{x_2 - x_1},$$

then we have $\lambda \in (0, 1)$ a $x = \lambda x_1 + (1 - \lambda)x_2$. Therefore we can write

$$\begin{aligned} f(x) = f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) = \\ = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2). \end{aligned}$$

Upon making the necessary modifications we get (16).

Now let us prove the opposite implication. Let us take $x_1, x_2 \in I$ and $\lambda \in [0, 1]$. If $x_1 = x_2$ or $\lambda = 0$ or $\lambda = 1$, then the inequality in the definition of a convex function surely holds. Without loss of generality we may assume that $x_1 < x_2$

and $\lambda \in (0,1)$. Let us put $x = \lambda x_1 + (1-\lambda)x_2$. Then we have $x_1 < x < x_2$ and according to the hypothesis it holds that

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}.$$

Now we substitute the expression $\lambda x_1 + (1-\lambda)x_2$ instead of x and upon modifications we get the inequality $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$. ■

Definition. Let the function f have a real derivative on a neighborhood of $a \in \mathbb{R}$. **The second derivative** of the function f at the point a is

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h},$$

if this limit exists.

Let $n \in \mathbb{N}$ and the function f have a real n-th derivative on a neighborhood of the point $a \in \mathbb{R}$ (we will denote it as $f^{(n)}$). Then the $\boldsymbol{(n+1)}$**-th derivative** of the function f at the point a is

$$f^{(n+1)}(a) = \lim_{h \to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h},$$

if the limit exists.

In this context we take the first derivative as defined above.

Remark. Let us notice the inductive nature of the previous definition. Firstly we define the concept of the first derivative and then, assuming that the n-th derivative has already been defined , we define the $(n+1)$-th derivative.

The relationship between convexity and the sign of the second derivative is the subject of the following theorem.

Theorem 65 (second derivative and convexity)**.** Let the function f have non-negative second derivative at all points of the open interval I. Then f is convex on I.

Proof. It follows from the hypothesis and from Theorem 42 that the function f' is continuous on I. Using the non-negativity of f'' in Theorem 55, we get that the function f' is non-decreasing on I.

Now take $x_1 < x < x_2$ three points from the interval I. By the Lagrange theorem, applied to the function f on the intervals $[x_1, x]$, $[x, x_2]$, there exist two numbers ξ and η such that $x_1 < \xi < x < \eta < x_2$ and

$$f'(\xi) = \frac{f(x)-f(x_1)}{x-x_1}, \quad f'(\eta) = \frac{f(x_2)-f(x)}{x_2-x}.$$

Because $\xi < \eta$ and f' is non-decreasing we get

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}.$$

>From here using Lemma 64 we prove our claim. ■

Remark. Similarly it can be proven that if the function f has a positive real second derivative on I (respectively non-positive or negative), then f is strictly convex (respectively concave or strictly concave) on I.

Example 66. The function $\log$ has a negative second derivative at all points of $x \in (0, +\infty)$ because $\log'' x = -\frac{1}{x^2}$. According to the previous remark, the function $\log$ is $(0, +\infty)$ strictly concave. Similarly the function $\exp$ is strictly convex on its domain.

Let us have the function f, which has a real derivative at the point $x_0 \in \mathbb{R}$. Let us denote

$$T_{x_0} = \{[x, y] \in \mathbb{R}^2;\ x \in \mathbb{R},\ y = f(x_0) + f'(x_0)(x - x_0)\}.$$

As we know, the set T_{x_0} is the tangent to the graph of the function f at the point $[x_0, f(x_0)]$.

Definition. Let the function f have a real derivative at the point $x_0 \in \mathbb{R}$ and $x \in D_f$. Let us say that

- the point $[x, f(x)]$ **lies under the tangent** T_{x_0}, if

$$f(x) < f(x_0) + f'(x_0)(x - x_0),$$

- the point $[x, f(x)]$ **lies above the tangent** T_{x_0}, if

$$f(x) > f(x_0) + f'(x_0)(x - x_0).$$

>From the point of view of the analysis of functions the points, where the function “goes from being under the tangent to above it” or the opposite are of particular interest. Let us describe the situation exactly.

Definition. Let the function f have a real derivative at the point $x_0 \in \mathbb{R}$. Let us say that x_0 is an **inflection point** for the function f, if there exists a $\Delta > 0$ such that

- $\forall x \in (x_0 - \Delta, x_0)\colon [x, f(x)]$ lies under the tangent T_{x_0},
- $\forall x \in (x_0, x_0 + \Delta)\colon [x, f(x)]$ lies above the tangent T_{x_0},

or

- $\forall x \in (x_0 - \Delta, x_0)\colon [x, f(x)]$ lies above the tangent T_{x_0},
- $\forall x \in (x_0, x_0 + \Delta)\colon [x, f(x)]$ lies under the tangent T_{x_0}.

The following picture illustrates the concept that we have just defined of an inflection point.

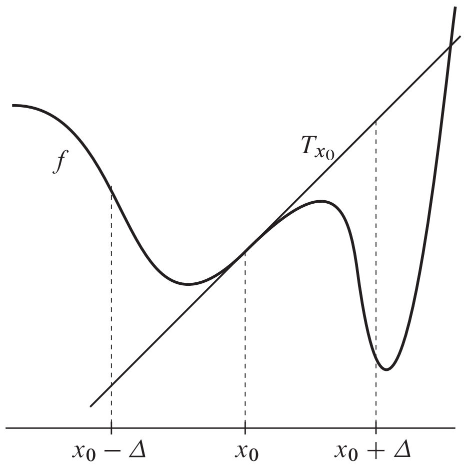


FIGURE 17.

Theorem 67 (necessary condition for inflection)**.** Let $x_0 \in \mathbb{R}$ be an inflection point of the function f. Then $f''(x_0) = 0$, or $f''(x_0)$ does not exist.

Proof. Let us assume firstly that $f''(x_0) > 0$. Then from the definition of the second derivative at the point x_0 it follows that there exists a $\Delta > 0$ such that $f'(x)$ is real for all $x \in (x_0 - \Delta, x_0 + \Delta)$ and it holds that

- $\forall x \in (x_0 - \Delta, x_0)\colon f'(x) < f'(x_0)$,
- $\forall x \in (x_0, x_0 + \Delta)\colon f'(x_0) < f'(x)$.

We get the continuity of f from the existence of the real derivative on $(x_0 - \Delta, x_0 + \Delta)$.

Let $x \in (x_0 - \Delta, x_0)$. The function f is continuous on $[x, x_0]$ and is differentiable on (x, x_0). Therefore the assumptions of the Lagrange theorem are satisfied and there exists a point $\xi_1 \in (x, x_0)$ such that $f(x_0) - f(x) = f'(\xi_1)(x_0 - x)$. It holds that $f'(\xi_1) < f'(x_0)$, and therefore $f(x_0) - f(x) < f'(x_0)(x_0 - x)$. We have proven that for any x in the interval $(x_0 - \Delta, x_0)$ we have $f(x) > f(x_0) + f'(x_0)(x - x_0)$, because the point $[x, f(x)]$ lies above the tangent T_{x_0}.

Let $x \in (x_0, x_0 + \Delta)$. According to the Lagrange theorem there exists a point $\xi_2 \in (x_0, x)$ such that $f(x) - f(x_0) = f'(\xi_2)(x - x_0)$. Because, however $f'(\xi_2) > f'(x_0)$, we have $f(x) - f(x_0) > f'(x_0)(x - x_0)$. In this case it holds, for any x from the interval $(x_0, x_0 + \Delta)$, that $f(x) > f(x_0) + f'(x_0)(x - x_0)$, because the point $[x, f(x)]$ lies above the tangent T_{x_0}.

This means that the point x_0 is not an inflection point of the function f. Similarly we can prove that the assumption $f''(x_0) < 0$ implies that x_0 is not an inflection point of the function f. From here we retrieve the claim. ∎

Remark. The fact that $f''(x_0)$ is zero is only a necessary (not a sufficient) condition for x_0 to be an inflection point of f. The function $f\colon x \mapsto x^4$, proves this at the point 0 where its second derivative is equal 0, but every point $[x, x^4], x \neq 0$, lies above the tangent $T_0 = \{[x,y] \in \mathbb{R}^2;\ y = 0\}$ to the graph of the function f at the point $[0,0]$.

Theorem 68 (sufficient condition for inflection)**.** Let the function f have a continuous first derivative on the interval (a,b) and $x_0 \in (a,b)$. Let it hold that:

$$\forall x \in (a, x_0)\colon f''(x) > 0 \quad \text{and} \quad \forall x \in (x_0, b)\colon f''(x) < 0.$$

Then x_0 is an inflection point of the function f.

Proof. According to Theorem 55 the function f' is increasing on the interval $(a, x_0]$ and decreasing on the interval $[x_0, b)$. Therefore we have

- $\forall x \in (a, x_0)\colon f'(x) < f'(x_0)$,
- $\forall x \in (x_0, b)\colon f'(x) < f'(x_0)$.

Take $x \in (x_0, b)$. The assumptions of the Lagrange theorem (Theorem 54) are satisfied by f on the interval $[x_0, x]$. Therefore there exists a $\xi \in (x_0, x)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(\xi) < f'(x_0).$$

>From here $f(x) < f(x_0) + f'(x_0)(x - x_0)$, because $[x, f(x)]$ lies under the tangent T_{x_0}.

For $x \in (a, x_0)$ we reason similarly. Therefore there exists a number $\eta \in (x, x_0)$ such that

$$\frac{f(x_0) - f(x)}{x_0 - x} = f'(\eta) < f'(x_0).$$

>From here we see that $f(x) > f(x_0) + f'(x_0)(x - x_0)$, because $[x, f(x)]$ lies above the tangent T_{x_0}. This proves that x_0 is an inflection point of the function f.

■

Remark. The claim of the theorem holds also, of course, in the case where both of the inequalities in the previous theorem are interchanged.

At the end of this section we will use the concavity and monotonicity of $\log$ to prove an inequality between the arithmetic and geometric means. Firstly let us realize that the following property is the result of convexity.

Lemma 69. Let the function f be convex on the interval I. Then it holds that

$$\begin{aligned} &\forall m \in \mathbb{N}\, \forall x_1, \dots, x_m \in I\, \forall \lambda_1 \geq 0, \dots, \lambda_m \geq 0,\ \lambda_1 + \cdots + \lambda_m = 1: \\ &\qquad\qquad f(\lambda_1 x_1 + \cdots + \lambda_m x_m) \leq \lambda_1 f(x_1) + \cdots + \lambda_m f(x_m). \end{aligned} \tag{17}$$

Proof. Let us use mathematical induction. For $m = 1$ the claim is obvious.

Let the claim (17) hold for $m = n$. We will prove that it is also true for $m = n + 1$. Let us have $x_1, \dots, x_n, x_{n+1} \in I$ and the non-negative numbers $\lambda_1, \dots, \lambda_n, \lambda_{n+1}$, whose sum is equal to 1. If all numbers $\lambda_1, \dots, \lambda_n$ are null then $\lambda_{n+1} = 1$ and the inequality in (17) is satisfied.

If at least one of the numbers $\lambda_1, \dots, \lambda_n$ is non-zero then the sum $\lambda_1 + \cdots + \lambda_n$ is a positive number. Let us denote the value of this sum as μ, and also denote

$$y = \frac{\lambda_1}{\mu} x_1 + \cdots + \frac{\lambda_n}{\mu} x_n.$$

The point y lies in I, because $\min\{x_1, \dots, x_n\} \leq y \leq \max\{x_1, \dots, x_n\}$, and also $\mu + \lambda_{n+1} = 1$. >From the convexity of the function f it immediately follows that

$$f(\mu y + \lambda_{n+1} x_{n+1}) \leq \mu f(y) + \lambda_{n+1} f(x_{n+1}). \tag{18}$$

Now we will use the induction hypothesis in order to estimate $f(y)$. We get

$$f(y) = f\left(\frac{\lambda_1}{\mu} x_1 + \cdots + \frac{\lambda_n}{\mu} x_n\right) \leq \frac{\lambda_1}{\mu} f(x_1) + \cdots + \frac{\lambda_n}{\mu} f(x_n). \tag{19}$$

>From the inequalities (18) and (19) we immediately get the inequality that we want to prove (17). ∎

Example 70. Let $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathbb{R}$ be non-negative numbers. Then it holds that

$$\frac{x_1 + x_2 + \cdots + x_m}{m} \geq \sqrt[m]{x_1 x_2 \cdots x_m},$$

which is a well known inequality between arithmetic and geometric means.

Proof. If at least on of the numbers $x_1, \dots, x_m$ is equal to zero then the inequality is obvious. Then let $x_1, \dots, x_m$ be positive numbers. Let us put $\lambda_i = \frac{1}{m}$, $i = 1, \dots, m$. Because the function $-\log$ is convex on the interval $(0, +\infty)$, we have, by the previous lemma, that

$$\begin{aligned} -\log\left(\frac{x_1}{m} + \frac{x_2}{m} + \cdots + \frac{x_m}{m}\right) &\leq -\frac{1}{m}\left(\log x_1 + \log x_2 + \cdots + \log x_m\right) = \\ &= -\frac{1}{m}\log(x_1 x_2 \cdots x_m) = -\log \sqrt[m]{x_1 x_2 \cdots x_m}. \end{aligned}$$

Because $-\log$ is decreasing on $(0, +\infty)$ the required inequality follows immediately. ∎

4.7. Investigating a function

The material we have covered until now will often help us get an idea about what a given function "does". Let us also mention the useful concept of asymptotes of a function which sometimes enable us to sketch the graph of a function more precisely.

Definition. The line, which is the graph of the affine function $x \mapsto kx+q$, is called the **asymptote** of the function f **at** $+\infty$, respectively **at** $-\infty$, if

$$\lim_{x\to+\infty}\bigl(f(x) - (kx+q)\bigr) = 0, \quad \text{respectively} \quad \lim_{x\to-\infty}\bigl(f(x) - (kx+q)\bigr) = 0.$$

Determining the asymptotes is usually done using the following theorem.

Theorem 71. The function f has the asymptote at $+\infty$ described by the affine function $x \mapsto kx + q$ if and only if

$$\lim_{x\to+\infty}\frac{f(x)}{x} = k \in \mathbb{R} \quad \text{and} \quad \lim_{x\to+\infty}\bigl(f(x) - kx\bigr) = q \in \mathbb{R}.$$

Proof. Let us assume that the asymptote of the function f at $+\infty$ is given by the affine function $x \mapsto kx + q$. Then it hods that

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \left(\frac{f(x) - kx - q}{x} + \frac{kx + q}{x} \right) = k,$$
$$\lim_{x\to+\infty} (f(x) - kx) = \lim_{x\to+\infty} (f(x) - kx - q + q) = q$$

according to Theorem 8. The opposite implication follows immediately from the relationship for q. ∎

Remarks. 1. Similar conditions hold for the existence of the asymptotes at $-\infty$.

2. If $\lim\limits_{x\to+\infty} f(x) = A \in \mathbb{R}$, then obviously the asymptote of the function f v $+\infty$ is a line parallel to the x-axis having the equation $y = A$.

Investigating a function.

1. Determine the domain and discuss the continuity of the function.
2. Determine the symmetry of the function: odd, even, periodic.
3. Compute the limits at the "edges of the domain".
4. Investigate the first derivative, determine intervals of monotonicity and find global and local extremes.
5. Investigate the second derivative and determine the intervals where the function f is convex or concave. Determine inflection points.
6. Determine the asymptotes of the function if the exist.
7. Sketch the graph of the function.

Example 72. Investigate the function $f(x) = \sqrt[3]{(x^4 - 1)^2}$.

Solution. 1. In our case $D_f = \mathbb{R}$ and the function is continuous on its entire domain.

2. The function is even, which will aid us in sketching its graph.

3. At the "end" points of the domain we compute

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$

4. We compute the first derivative of the function:

$$f'(x) = \frac{8}{3} \sqrt[3]{((x^4 - 1)^2)^{-2}} (x^4 - 1)x^3 = \frac{8}{3} \frac{x^3}{\sqrt[3]{x^4 - 1}}, \quad x \in \mathbb{R} \setminus \{-1, 1\}.$$

Using Theorem 56 on the computation of single-sided derivatives we get $f'_+(1) = +\infty$ and $f'_-(1) = -\infty$. The function f is even and therefore for all $x \in D_f$ we have the equation $f'_+(x) = -f'_-(-x)$, if at least one of the derivatives exists. >From here $f'_+(-1) = +\infty$ and $f'_-(-1) = -\infty$. The derivative of the function is 0 only for $x = 0$. For $x \in (0, 1)$ we have $f'(x) < 0$, and therefore the function f is decreasing here; for $x \in (1, +\infty)$ we have $f'(x) > 0$ and the function f is increasing on this interval. >From the symmetry of f we get that f is decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$. >From here it also follows that f has a local maximum at the point 0 and a local minimum at the points 1 and -1.

The function f attains it minimum over D_f at the points ± 1 (the value is equal to 0), and f does not have a maximum on D_f because it is not bounded from above.

5. Now let us investigate the convexity and concavity of the function using the second derivative:

$$f''(x) = \frac{8}{9} \cdot \frac{x^2(5x^4 - 9)}{\sqrt[3]{(x^4 - 1)^4}}, \quad x \in \mathbb{R} \setminus \{-1, 1\}.$$

Therefore $f''(x) = 0$ if and only if $x = 0$ or $x = \sqrt{3/\sqrt{5}}$ or $x = -\sqrt{3/\sqrt{5}}$. For $x \in (-1, 1)$ we have $f''(x) \leq 0$ and the function is concave on this interval; for $x \in \left(1, \sqrt{3/\sqrt{5}}\right)$ we have $f''(x) < 0$ and the function is strictly concave on this interval; similarly we get that f is strictly convex on the interval $\left(\sqrt{3/\sqrt{5}}, +\infty\right)$. The points $\pm\sqrt{3/\sqrt{5}}$ are inflection points of the function f.

6. It holds that

$$\lim_{x \to +\infty} \frac{f(x)}{x} = +\infty \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{f(x)}{x} = -\infty,$$

therefore the function f does not have an asymptote at $+\infty$ nor at $-\infty$.

7. Here is the graph of the function f.

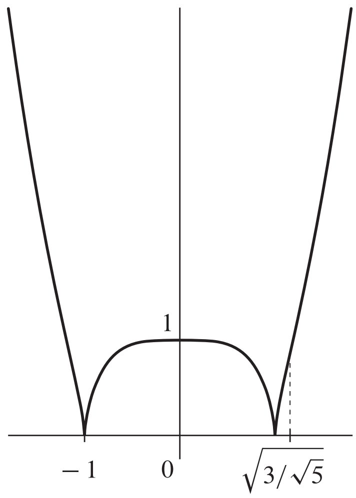


FIGURE 18.

♣

Example 73. Investigate the function $f(x) = |x| \exp(-|x-1|)$.

Solution. The domain of the function f is the entire $\mathbb{R}$. The function is continuous on $\mathbb{R}$; it is not even, odd or periodic. Using Example 62 we can easily compute the following limit:

$$\lim_{x\to+\infty} |x| \exp(-|x-1|) = \lim_{x\to+\infty} \frac{x}{\exp(x-1)} = \\ = \lim_{x\to+\infty} \frac{ex}{\exp x} = 0.$$

Similarly $\lim\limits_{x\to-\infty} |x| \exp(-|x-1|) = 0$.

For all $x \in \mathbb{R} \setminus \{0, 1\}$ we calculate the derivative

$$f'(x) = \operatorname{sgn} x \cdot \exp(-|x-1|) - |x| \exp(-|x-1|) \operatorname{sgn}(x-1) = \\ = \exp(-|x-1|) \left(\operatorname{sgn} x - |x| \operatorname{sgn}(x-1)\right).$$

According to Theorem 56 on the computation of single-sided derivatives we get $f'_-(0) = -1/e$, $f'_+(0) = 1/e$ also $f'_-(1) = 2$, $f'_+(1) = 0$. The derivative f' is equal to 0 only for $x = -1$.

For $x \in (-\infty, -1)$ we have $f'(x) > 0$ and the function f is increasing on this interval. For $x \in (-1, 0)$ we have $f'(x) < 0$ and the function is decreasing on this interval. Similarly we find that the function f is increasing on the interval $(0, 1)$ and decreasing on the interval $(1, +\infty)$.

>From here we see that the function f has local maximums at the points -1 and 1 and f has a local minimum at the point 0. By computing the values of f at the local extreme points and by comparing with the limits at the points $+\infty$ and $-\infty$ we finally get, that the function f attains its minimum over $\mathbb{R}$ at the point 0 and its maximum at the point 1.

Now let us calculate the second derivative:

$$f''(x) = \begin{cases} -(x+2)\exp(x-1) & \text{for } x \in (-\infty, 0), \\ (x+2)\exp(x-1) & \text{for } x \in (0,1), \\ (x-2)\exp(-x+1) & \text{for } x \in (1, +\infty). \end{cases}$$

The second derivative is zero at the points $x = -2$ a $x = 2$. >From the sign of the second derivative we see that the function f is convex on each of the intervals $(-\infty, -2)$, $(0,1)$ and $(2, +\infty)$ and concave on each of the intervals $(-2, 0)$ and $(1, 2)$. The points 2 and -2 are inflection points. The point 0 is not an inflection point because we do not have a real derivative $f'(0)$.

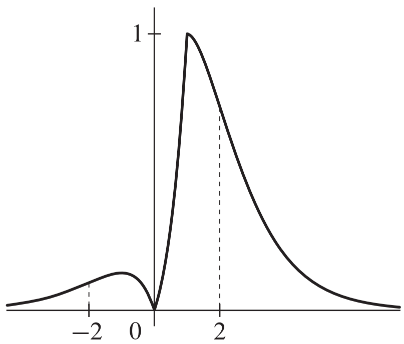


FIGURE 19.

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We will proceed more quickly in the following example. We leave it to the reader to justify the computations considering each claim carefully.

Example 74. Investigate the function $f(x) = |x-2| - 2 \operatorname{arctg} x$.

Solution. The function is continuous on its domain $D_f = \mathbb{R}$, it is not even, odd nor periodic, $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to -\infty} f(x) = +\infty$. It holds that:

$$f'(x) = \operatorname{sgn}(x-2) - \frac{2}{1+x^2}, \quad x \in \mathbb{R} \setminus \{2\}.$$

According to Theorem 56 on the computation of single-sided derivatives we get $f'_+(2) = 3/5$, $f'_-(2) = -7/5$. At no point of the domain therefore does the function f have derivative equal to 0.

>From the sign of the first derivative we get that the function f is decreasing on the interval $(-\infty, 2)$ and increasing on the interval $(2, +\infty)$. At the point 2 the function f attains its minimum over $\mathbb{R}$.

For the second derivative it holds that

$$f''(x) = \frac{4x}{(1+x^2)^2}, \quad x \in \mathbb{R} \setminus \{2\}.$$

>From here it follows that f is concave on $(-\infty, 0)$ and convex on each of the intervals $(0, 2)$ and $(2, +\infty)$. At the point 0 the function has an inflection point.

Calculating the asymptote at $+\infty$:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = 1,$$
$$\lim_{x\to+\infty} (f(x) - 1 \cdot x) = -2 - \pi.$$

The asymptote at $+\infty$ therefore exists and it is the line given by the equation $y = x - 2 - \pi$. Similarly the asymptote at $-\infty$ is the line given by the equation $y = -x + 2 + \pi$.

We finish the investigation by supplementing with the computation $f(0) = 2$, $f'(0) = -3$, $f(2) = -2 \operatorname{arctg} 2$ and the graph of the function and its asymptotes.

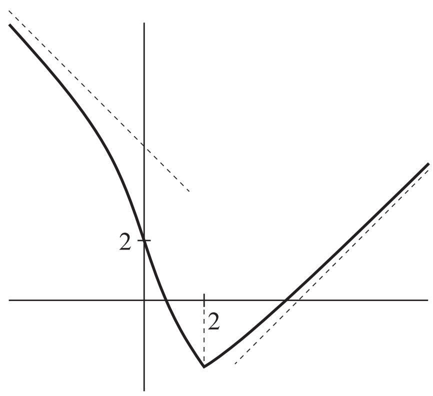

FIGURE 20.

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4.8. Exercises

Compute the following limits if they exist.

1. $\lim\limits_{x\to 9} \dfrac{2x - 6\sqrt{x}}{x^2 - 8x - 9}$

2. $\lim\limits_{x\to +\infty} x\left(\sqrt{x^2+1} - x\right)$

3. $\lim\limits_{x\to 1} \dfrac{2x^3 - 5x^2 + x + 2}{x^2 - 1}$

4. $\lim\limits_{x\to 3} \dfrac{x+1}{(x-3)^2}$

5. $\lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{\sin^2 x}$

6. $\lim\limits_{x\to 0} \dfrac{\sin x + 1}{\sin x}$

7. $\lim\limits_{x\to \pi/2} \dfrac{\operatorname{tg} x}{x - \pi/2}$

8. $\lim\limits_{x\to 16} \sqrt{\dfrac{4 - \sqrt{x}}{64 - \sqrt{x^3}}}$

9. $\lim\limits_{x\to 0} \exp\left(\dfrac{\sqrt[3]{1-x^2} - 1}{5x^2}\right)$

10. $\lim\limits_{x\to -\infty} \dfrac{\log(1 + e^x)}{x}$

11. $\lim\limits_{x\to \infty} \sqrt{\dfrac{\cos x + 2}{x^2 + x}}$

12. $\lim\limits_{x\to 0} \exp\left(\dfrac{\operatorname{cotg} x}{\log(1-x)}\right)$

13. $\lim\limits_{x\to +\infty} x(2^{1/x} - 1)$

14. $\lim\limits_{x\to 0} \dfrac{\log(1+x^2)}{\log(1-x^2)}$

15. $\lim\limits_{x\to 3} \dfrac{\arcsin(x-3)}{x^2 - 3x}$

16. $\lim\limits_{x\to -\infty} \dfrac{\operatorname{arccotg} x}{x}$

17. $\lim\limits_{x\to 2} \left(\operatorname{arctg}\left(\dfrac{1}{2-x}\right)\right)^2$

18. $\lim\limits_{x\to +\infty} \left(\dfrac{3x+2}{2x+3}\right)^{2x-1}$

19. $\lim\limits_{x\to 0} (1 + 4x)^{1/3x}$

20. $\lim\limits_{x\to +\infty} \left(\dfrac{x^2+3}{x^2+7}\right)^x$

21. $\lim\limits_{x\to 0} (\cos 3x)^{1/x^2}$

22. $\lim\limits_{x\to 0} (\cos x)^{\operatorname{cotg}^2 x}$

23. $\lim\limits_{x\to 0} \left(\dfrac{2^x + 8^x}{2}\right)^{1/x}$

24. $\lim\limits_{x\to \pi/2} \left(\operatorname{tg} x + \dfrac{1}{x - \pi/2}\right)$

25. $\lim\limits_{x\to 0} \dfrac{\exp(-1/x^2)}{x}$

26. $\lim\limits_{x\to 0} (\operatorname{tg}^2 x)^{\sin^2 x}$

In the following tasks you are to compute the derivative of the function f.

27. $\left|\dfrac{x-1}{1-2x}\right|$

28. $\dfrac{(\log x)^x}{x^{\log x}}$

29. $\sqrt[3]{(1-\exp(1-x^2))^2}$

30. $\sqrt{1-e^{-x^2}}$

31. $\arcsin\left|\dfrac{5x+2}{3x-6}\right|$

32. $\sqrt{\sin x\cos x}$

In the following tasks you are to determine the extremes of the function f on D_f.

33. $x\sqrt{2-x^2}$

34. $\sin^3 x+\cos^3 x$

35. $\arccos\left(\dfrac{-x^2-x+2}{4}\right)$

36. $\left|\dfrac{x}{1+x^2}\right|$

In the following tasks investigate the given function f.

37. $|x-1|\exp\left(-\dfrac{1}{(x-1)^2}\right)$

38. $\dfrac{x^4-1}{x^3+1}$

39. $\arccos\left|\dfrac{1-x}{1-2x}\right|$

40. $(x+2)\exp(1/x)$

41. $\dfrac{\cos x}{2+\sin x}$

42. $\arcsin\left(\dfrac{2x}{1+x^2}\right)-2\operatorname{arctg} x$

43. $f_n(x)=e^x(x+1)^n$, where $n\in\mathbb{N}$

Solutions

1. $1/10$ **2.** $1/2$ **3.** $-3/2$ **4.** $+\infty$ **5.** 0 **6.** neexistuje **7.** $-\infty$
8. $1/(4\sqrt{3})$ **9.** $\exp(-1/15)$ **10.** 0 **11.** 0 **12.** 0 **13.** $\log 2$
14. -1 **15.** $1/3$ **16.** 0 **17.** $\pi^2/4$ **18.** $+\infty$ **19.** $\exp(4/3)$ **20.** 1
21. $\exp(-9/2)$ **22.** $\exp(-1/2)$ **23.** 4 **24.** 0 **25.** 0 **26.** 1

27. $D_f=(-\infty,1/2)\cup(1/2,+\infty)$;

$$f'(x)=\operatorname{sgn}\left(\frac{x-1}{1-2x}\right)\cdot\frac{-1}{(1-2x)^2},\quad x\in D_f\setminus\{1\};$$

$f'_+(1)=1$, $f'_-(1)=-1$

28. $D_f = (1, +\infty)$;

$$f'(x) = \frac{(\log x)^x}{x^{\log x}} \left(\log(\log x) + \frac{1}{\log x} - \frac{2}{x} \log x \right), \quad x \in D_f = (1, +\infty)$$

29. $D_f = \mathbb{R}$;

$$f'(x) = \frac{4}{3} \left(\left(1 - \exp(1 - x^2)\right)^2 \right)^{-2/3} \left(1 - \exp(1 - x^2)\right) \exp(1 - x^2) x,$$

$x \in D_f \setminus \{-1, 1\}$; $f'_+(1) = +\infty$, $f'_-(1) = -\infty$, $f'_+(-1) = +\infty$, $f'_-(-1) = -\infty$

30. $D_f = \mathbb{R}$;

$$f(x) = \frac{xe^{-x^2}}{\sqrt{1 - e^{-x^2}}}, \quad x \in \mathbb{R} \setminus \{0\};$$

$f'_+(0) = 1$, $f'_-(0) = -1$

31. $D_f = [-4, 1/2]$;

$$f'(x) = \frac{-6 \operatorname{sgn}(5x + 2)}{(x - 2)\sqrt{2(x + 4)(1 - 2x)}}, \quad x \in D_f \setminus \{-4, -2/5, 1/2\};$$

$f'_+(-4) = -\infty$, $f'_-(-2/5) = -25/36$, $f'_+(-2/5) = 25/36$, $f'_-(1/2) = +\infty$

32. $D_f = \bigcup_{k \in \mathbb{Z}} [k\pi, \pi/2 + k\pi]$;

$$f'(x) = \frac{\cos^2 x - \sin^2 x}{2\sqrt{\sin x \cos x}}, \quad x \in \bigcup_{k \in \mathbb{Z}} (k\pi, \pi/2 + k\pi);$$

$f'_+(2k\pi) = +\infty$, $f'_-(\pi/2 + 2k\pi) = -\infty$,
$f'_+(\pi + 2k\pi) = +\infty$, $f'_-(3\pi/2 + 2k\pi) = -\infty$, $k \in \mathbb{Z}$

33. $D_f = [-\sqrt{2}, \sqrt{2}]$; $\min\limits_{[-\sqrt{2}, \sqrt{2}]} f = f(-1) = -1$, $\max\limits_{[-\sqrt{2}, \sqrt{2}]} f = f(1) = 1$

34. This function attains its maximum at every point $x = 2k\pi$ and $x = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$. Its value is equal to 1. The minimum of the function is the value -1, which is attained at the points $x = \pi + 2k\pi$ and $3\pi/2 + 2k\pi$, $k \in \mathbb{Z}$. Further at the points $\pi/4 + 2k\pi$, $k \in \mathbb{Z}$, the function attains a local minimum, and a local maximum at the points$5\pi/4 + 2k\pi$, $k \in \mathbb{Z}$.

35. $D_f = [-3, 2]$; $\min\limits_{[-3,2]} f = f(-1/2)$, $\max\limits_{[-3,2]} f = f(-3) = f(2) = \pi$

36. $D_f = \mathbb{R}$; $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to -\infty} f(x) = 0$, $\min\limits_{\mathbb{R}} f = f(0) = 0$, $\max\limits_{\mathbb{R}} f = f(1) = f(-1) = 1/2$

37. $D_f = \mathbb{R} \setminus \{1\}$; $\lim\limits_{x \to \pm\infty} f(x) = +\infty$, $\lim\limits_{x \to 1} f(x) = 0$; therefore we can assign a continuous value to the function f at the point 1 with its limit (put $f(1) = 0$); this extended function (we will continue to call it f) then has $D_f = \mathbb{R}$ and is continuous on the entire $\mathbb{R}$;[2]

$$f'(x) = \exp\left(-\frac{1}{(x-1)^2}\right)\operatorname{sgn}(x-1)\left(1+\frac{2}{(x-1)^2}\right), \quad x \in \mathbb{R}\setminus\{1\};$$

the derivative of the function f at 1 can easily be calculated using Theorem 56 on single-sided derivatives; it is $\lim_{x\to 1\pm} f'(x) = 0$, and so $f'(1) = 0$; the function f is decreasing on $(-\infty, 1)$ and increasing on $(1, +\infty)$, at the point 1 it attains its minimum over D_f;

$$f''(x) = 2\exp\left(-\frac{1}{(x-1)^2}\right)\operatorname{sgn}(x-1)\frac{-x^2+2x+1}{(x-1)^5}, \quad x \in \mathbb{R}\setminus\{1\};$$

$f''(0) = 0$, f is concave on $(-\infty, 1-\sqrt{2})$, convex on $(1-\sqrt{2}, 1+\sqrt{2})$ and concave on $(1+\sqrt{2}, +\infty)$; the points $1\pm\sqrt{2}$ are inflection points of f; the asymptote at $-\infty$ is the line given by the equation $y = -x+1$, at $+\infty$ it is the line $y = x-1$

38. $D_f = \mathbb{R}\setminus\{-1\}$; $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-1} f(x) = -4/3$; therefore the function can be continuously defined as $-4/3$ at the point -1 and then it is continuous on the entire $\mathbb{R}$;

$$f'(x) = \frac{x^2(x^2-2x+3)}{(x^2-x+1)^2}, \quad x \in \mathbb{R};$$

f is increasing on D_f;

$$f''(x) = \frac{6x(1-x)}{(x^2-x+1)^3}, \quad x \in \mathbb{R};$$

f is concave on $(-\infty, 0)$, convex on $(0, 1)$, concave on $(1, +\infty)$; inflection points: 0, 1; $f(0) = -1$, $f'(0) = 0$, $f(1) = 0$, $f'(1) = 2$; the common asymptote of f at $\pm\infty$ is the line given by the equation $y = x$

39. $D_f = (-\infty, 0] \cup [2/3, +\infty)$; $\lim_{x\to\pm\infty} f(x) = \pi/3$; $f(0) = f(2/3) = 0$;

$$f'(x) = -\frac{\operatorname{sgn}(1-x)}{(1-2x)\sqrt{x(3x-2)}}, \quad x \in D_f \setminus \{0, 2/3, 1\};$$

$f'_-(0) = -\infty$, $f'_+(2/3) = +\infty$, $f'_-(1) = 1$, $f'_+(1) = -1$; f is decreasing on $(-\infty, 0)$ and on $(1, +\infty)$; f is increasing on $(2/3, 1)$, f attains its maximum over

[2] Assigning the continuous value at 1 may seem unnatural. Our goal is to sketch the graph of the function f a well as possible and so this approach is very useful because it allows us to better investigate the behavior of the original function close to the point 1.

D_f at the point 1 ($\max_{D_f} f = \pi/2$), its minimum over D_f at the points 0 and $2/3$ ($\min_{D_f} f = 0$);

$$f''(x) = \frac{(-12x^2 + 9x - 1)\operatorname{sgn}(1 - x)}{x(1 - 2x)^2(3x - 2)\sqrt{x(3x - 2)}}, \quad x \in D_f \setminus \{0, 2/3, 1\};$$

f is concave on $(-\infty, 0)$, f is concave on $(2/3, 1)$, f is convex on $(1, +\infty)$; the asymptote of the function at the points $\pm\infty$ is the line given by the equation $y = \pi/3$

40. $D_f = (-\infty, 0) \cup (0, +\infty)$;
$\lim\limits_{x\to-\infty} f(x) = -\infty$, $\lim\limits_{x\to 0-} f(x) = 0$, $\lim\limits_{x\to 0+} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = +\infty$;

$$f'(x) = \frac{(x + 1)(x - 2)}{x^2} \exp(1/x), \quad x \in D_f;$$

f is increasing on $(-\infty, -1)$, decreasing on $(-1, 0)$, decreasing on $(0, 2)$ and increasing on $(2, +\infty)$; the function f has a local maximum at the point -1, a local minimum at the point 2, on D_f the function does not attain its maximum or minimum;

$$f''(x) = \frac{5x + 2}{x^4} \exp(1/x), \quad x \in D_f;$$

f is concave on $(-\infty, -2/5)$, convex on $(-2/5, 0)$ and on $(0, +\infty)$, the point $-2/5$ is an inflection point of the function f; the asymptote at $\pm\infty$ is the line given by the equation $y = x + 3$; the function f can be defined at 0 by its left-hand limit, i.e. by putting $f(0) = 0$, and thus defined the function is now defined on the entire $\mathbb{R}$ and at the point 0 it is continuous from the left, then $f'_-(0) = \lim\limits_{x\to 0-} f'(x) = 0$

41. $D_f = \mathbb{R}$; the function f is continuous on the entire D_f;

$$f'(x) = -\frac{1 + 2\sin x}{(2 + \sin x)^2}, \quad x \in \mathbb{R};$$

the function f is decreasing on $(-\pi, -5\pi/6)$, increasing on $(-5\pi/6, -\pi/6)$, decreasing on $(-\pi/6, \pi)$; the function f attains a local maximum at the point $-\pi/6$ (and further at each of the points $-\pi/6 + 2k\pi, k \in \mathbb{Z}$), a local minimum at the point $-5\pi/6$ (and at each of the points $-5\pi/6 + 2k\pi, k \in \mathbb{Z}$); at all local minimum points the function f attains its minimum over $\mathbb{R}$, at all points of local maximum the function attains its greatest value over the entire $\mathbb{R}$;

$$f''(x) = \frac{2\cos x(\sin x - 1)}{(2 + \sin x)^3}, \quad x \in \mathbb{R};$$

the function f is convex on $(-\pi, -\pi/2)$, concave on $(-\pi/2, \pi/2)$, convex on $(\pi/2, \pi)$, the points $-\pi/2$ and $\pi/2$ are inflection points

42. $D_f = \mathbb{R}$; $\lim\limits_{x\to-\infty} f(x) = \pi$, $\lim\limits_{x\to+\infty} f(x) = -\pi$;

$$f'(x) = \frac{2(1-x^2)}{|1-x^2|\,(1+x^2)} - \frac{2}{1+x^2}, \quad x \in \mathbb{R} \setminus \{1, -1\};$$

$f'_-(-1) = -2$, $f'_+(-1) = 0$, $f'_-(1) = 0$, $f'_+(1) = -2$; the function f is decreasing on $(-\infty, -1)$, constant on $(-1, 1)$ ($f(x) = 0$ for $x \in (-1, 1)$), decreasing on $(1, +\infty)$, and so f is non-increasing on D_f, f does not attain a maximum or minimum on D_f;

$$f''(x) = \frac{8x}{(1+x^2)^2}, \quad x \in (-\infty, -1) \cup (1, +\infty);$$

the function f is concave on$(-\infty, 1)$, and convex on $(1, +\infty)$; the asymptote at $-\infty$ is the line given by the equation $y = \pi$, the asymptote at $+\infty$ is the line given by the equation $y = -\pi$

43. We have to investigate the function f_n, whose formula depends on the parameter $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ it holds that $D_{f_n} = \mathbb{R}$, $\lim\limits_{x\to-\infty} f_n(x) = 0$, $\lim\limits_{x\to+\infty} f_n(x) = +\infty$, $f_n(-1) = 0$ a $f_n(0) = 1$. Further

$$f_n'(x) = e^x (x+1)^{n-1}(x+1+n), \quad n \in \mathbb{N},\ x \in \mathbb{R}.$$

We easily see that it is necessary to discern three different cases, which are $n = 1$, n odd and larger than 1 and finally n even.

1. For $n = 1$ the function f_1 is decreasing on $(-\infty, -2)$, increasing on $(-2, +\infty)$; f_1 and attains its minimum on D_{f_1} at the point -2;

$$f_1''(x) = e^x(x+3), \quad x \in D_{f_1};$$

f_1 is concave on $(-\infty, -3)$, convex on $(-3, +\infty)$, and -3 is an inflection point.

2. For odd $n \in \mathbb{N}$, $n > 1$, we have f_n decreasing on $(-\infty, -n-1)$, increasing on $(-n-1, +\infty)$; and attains a minimum over D_{f_n} at the point $-(n+1)$;

$$f_n''(x) = e^x(x+1)^{n-2}(x+n+1+\sqrt{n})(x+n+1-\sqrt{n}), \quad x \in D_{f_n};$$

f_n is concave on $(-\infty, -n-1-\sqrt{n})$, convex on $(-n-1-\sqrt{n}, -n-1+\sqrt{n})$, concave on $(-n-1+\sqrt{n}, -1)$, convex on $(-1, +\infty)$, the points $-n-1-\sqrt{n}$, $-n-1+\sqrt{n}$, -1 are inflection points.

3. For even $n \in \mathbb{N}$ the function f_n is increasing on $(-\infty, -n-1)$, decreasing on $(-n-1, -1)$ and increasing on $(-1, +\infty)$; f_n it has a local maximum at the point $-(n+1)$, and its minimum over D_{f_n} is at the point -1; the second derivative has the same form as before; f_n is convex on $(-\infty, -n-1-\sqrt{n})$, concave on $(-n-1-\sqrt{n}, -n-1+\sqrt{n})$, convex on $(-n-1+\sqrt{n}, +\infty)$, and $-n-1-\sqrt{n}$, $-n-1+\sqrt{n}$ are inflection points.

In all cases the asymptote of the function f_n at $-\infty$ is the x-axis. The asymptote at $+\infty$ doesn't exist.

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