1 Statements and predicates

Let M be the set of all men and W the set of all women. Let us consider the following predicates:

$R(m, w)$, $m \in M$, $w \in W$: „Man m and woman w are married.“;

$L_1(m, w)$, $m \in M$, $w \in W$: „Man m loves woman w.“;

$L_2(m, w)$, $m \in M$, $w \in W$: „Woman w loves man m.“.

1. Write the following statments using quantifiers, logical operators and predicates R, L_1, L_2.

a) Every married man loves his wife.

b) Every woman is loved by a man.

c) Every woman has at most one husband.

d) Every man has at most one wife. (Is d) the same statement as in c)?)

e) There exists a married woman.

f) There exists a married man. (Is f) the same statement as in e)?)

g) There exists an unfaithful woman. (A woman is called *unfaithful* if she is married and loves a man which is not her husband.)

2. Translate the following formulas to english.

h) $\exists m \in M \forall w \in W (\neg R(m, w))$;

i) $\exists w \in W \forall m \in M (L_1(m, w) \Rightarrow \neg L_2(m, w))$;

j) $\exists w \in W \forall m \in M (L_2(m, w) \Rightarrow \neg L_1(m, w))$;

k) $\forall w \in W ((\exists m \in M : L_2(m, w)) \Rightarrow (\exists m \in M : L_1(m, w) \ \& \ \neg L_2(m, w))$.

3. Write (in symbols) negations of the following statements. Translate the statements and their negations to English. Decide whether the statements are true or false.

a) $\forall x \in \mathbb{N} \, \exists y \in \mathbb{N} \, x > y$; b) $\forall x \in \mathbb{N} \, \exists y \in \mathbb{N} \, x < y$; c) $\exists y \in \mathbb{N} \, \forall x \in \mathbb{N} \, y \leq x$;

d) $\exists x \in \mathbb{N} \, \forall y \in \mathbb{N} \, x \geq y$; e) $\exists x \in \mathbb{N} \, \forall y \in \mathbb{Z} \, x \leq y$; f) $\forall \varepsilon \in \mathbb{R} \, \exists a \in \mathbb{Q} \, (a - [a] \geq \varepsilon)$;

g) $\forall \varepsilon \in \mathbb{R} \, (\varepsilon < 1 \ \Rightarrow \ \exists a \in \mathbb{Q} : \ a - [a] \geq \varepsilon)$;

4. Draw an example of a function f that satisfy (resp. do not satisfy) the following statement:

a) $\forall x \in \mathbb{R} \, (|x| < 1 \Rightarrow |f(x) - 1| < 3)$; b) $\forall n \in \mathbb{N} \, \exists \delta > 0 \, (|x| < \delta \Rightarrow f(x) > n)$;

c) $\exists a \in \mathbb{R} \, \forall \delta \in \mathbb{R} \, (|x| < \delta \Rightarrow f(x) > \varepsilon)$; d) $\forall n \in \mathbb{N} \, \forall x \in \mathbb{R} \, \forall y \in \mathbb{R} \, |f(x) - f(y)| < \frac{1}{n}$.

5. Write the following statements and predicates using symbols and the predicate $Div(m, n)$, $m, n \in \mathbb{N}$: „m is divisible by n“.

a) „M is the set of all even numbers.“ b) „Each number divisible by 6 is also divisible by 3.“

c) „M is the set of all odd integers larger than 52.“

d) „M and the set of all even integers are disjoint.“

e) „p is a prime.“ f) „P is the set of all prime numbers.“

2 Suprema and infima

Find suprema and infima of the following sets (if they exist) Do there exist maxima and minima?

1. a) $M_1 = \{q \in \mathbb{Q} : \ q \in (0,2)\}$, b) $M_2 = \{2^n : \ n \in \mathbb{N}\}$, c) $M_3 = \{2^n : \ n \in \mathbb{Z}, \ n \leq 5\}$

2. $A = \{p/(p+q); \ p \in \mathbb{N}, \ q \in \mathbb{N}\}$,

3. a) $B_1 = \{\sin x; \ x \in \langle 0, 2\pi \rangle\}$, b) $B_2 = \{\sin x; \ x \in (0, 2\pi)\}$, c) $B_3 = \{\sin x; \ x \in (0, \pi)\}$,

4. a) $C_1 = \{n^2 - m^2; \ n \in \mathbb{N}, \ m \in \mathbb{N}\}$, b) $C_2 = \{n^2 - m^2; \ n \in \mathbb{N}, \ m \in \mathbb{N}, n > m\}$,

5. a) $D_1 = \{2^{-n} + 3^{-n}; \ n \in \mathbb{N}\}$, b) $D_2 = \{2^{-n} + 3^{-n}; \ n \in \mathbb{Z}\}$,

6. $E = \{5^{(-1)^j 3^k}; \ j \in \mathbb{Z}, \ k \in \mathbb{Z}\}$.

7. a) $F_1 = \{\cos(n + \frac{1}{n})\pi; \ n \in \mathbb{N}\}$, b) $F_2 = \{\cos(n + \frac{1}{n})\pi; \ n \in \mathbb{N}$ sudé$\}$,

8. Let $A, B \subset \mathbb{R}$ and $S = \sup A, \ s = \inf A, \ T = \sup B, \ t = \inf B$. What can you say about infima and suprema of the following sets?

a) $A \cup B$, b) $A \setminus B$, c) $A \triangle B = (A \setminus B) \cup (B \setminus A)$

d) $A \cup \{1, 2, 3, 4\}$, *e) $A - B = \{a - b \mid a \in A, b \in B\}$, *f) $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$.

9. Let $f, g : \mathbb{R} \to (0, +\infty)$ be two functions and let us denote $a = \sup\{f(x) : \ x \in \mathbb{R}\}$, $b = \sup\{g(x) : \ x \in \mathbb{R}\}$. What can you say about the following suprema?

a) $\sup\{(f(x))^2 : x \in \mathbb{R}\}$, b) $\sup\{f(x)g(x) : \ x \in \mathbb{R}\}$, c) $\sup\{f(x) - g(x) : \ x \in \mathbb{R}\}$

3 Limits of sequences

1. Compute the limits

a) $\lim\limits_{n\to\infty} \frac{2n^2+n-3}{n^3-1}$ b) $\lim\limits_{n\to\infty} \frac{2n^3+6n}{n^3-7n+7}$ c) $\lim\limits_{n\to\infty} \frac{2n^5+3n-2}{n^5-3n^3+1}$ d) $\lim\limits_{n\to\infty} \frac{2n^3+\sqrt{n}+1}{3n^2+n^4}$

2. Compute the following limits using growth orders

a) $\lim\limits_{n\to\infty} \frac{2^n+3^n}{2^n+n!}$ b) $\lim\limits_{n\to\infty} \frac{n+n^2+2^{-n}}{1+3^{-n}+4^n}$ c) $\lim\limits_{n\to\infty} \frac{\sqrt{n}+\sin n}{\sqrt[3]{n^2}+1}$ d) $\lim\limits_{n\to\infty} \frac{n^2\sqrt{n}+n\sin n}{n^3+2n+\sin n}$

e) $\lim\limits_{n\to\infty} \frac{n^3 2^n}{3^n+2^n+2n^2}$ f) $\lim\limits_{n\to\infty} \frac{4^{n+3}+2^{2n-1}}{4^{n-1}+n^2 3^{n-1}}$ g) $\lim\limits_{n\to\infty} \frac{2^n+(n-2)!}{3^n+n!}$ h) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n!}+2^n}{\sqrt{n!}+2^n}$

3. Compute the following limits

a) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+5}-\sqrt{n^2+1}\right)$ b) $\lim\limits_{n\to\infty} \left(\sqrt[3]{n+11}-\sqrt[3]{n}\right)$ c) $\lim\limits_{n\to\infty} (-1)^n\sqrt{n}(\sqrt{n^4+n}-n^2)$

4. Compute with justification

a) $\lim\limits_{n\to\infty} \frac{n+1}{2^n}$ b) $\lim\limits_{n\to\infty} \frac{2^n}{n!}$ c) $\lim\limits_{n\to\infty} \frac{n2^{3n-1}}{(n-1)!}$ d) $\lim\limits_{n\to\infty} \frac{n+\sqrt{n+2}}{n^2+2^{n+1}}$

5. Compute with justification

a) $\lim\limits_{n\to\infty} \sqrt[n]{n+(-1)^n}$ b) $\lim\limits_{n\to\infty} \sqrt[n]{n^3+3n^2+1}$ c) $\lim\limits_{n\to\infty} (-1)^n(\sqrt{n^2+n}-\sqrt{n^2+1})$

d) $\lim\limits_{n\to\infty} \sqrt[n]{2^n+3^n+5^n}$ e) $\lim\limits_{n\to\infty} \frac{n+[\sqrt[3]{n}]^3}{n-[\sqrt{n+9}]}$ ([...] being the integer part)

6. Compute the limits (with justification)

a) $\lim\limits_{n\to\infty} \left(\sqrt[3]{\sqrt{n^7}+\sqrt[3]{n^7}}-\sqrt[3]{\sqrt{n^7}-\sqrt[3]{n^7}}\right)$ b) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^9+4n^2+2}-n^3}{n^6-\sqrt{n^{12}-2n^2}}$

c) $\lim\limits_{n\to\infty} \frac{(n!)^2\left(\sqrt{n^2+\sin(1/n)}-\sqrt{n^2-\sin(1/n)}\right)}{(4^n+(n-1)!\sin n+n!)(n-2)!}$ d) $\lim\limits_{n\to\infty} \frac{\sqrt{2n-1}+3\sqrt{n}}{n^2+2^{n+1}}\left(\sqrt{3^n+2^n}-\sqrt{3^n}\right)$

4 Mappings

Consider the following mappings (P is the set of all primes).

$$
\begin{aligned}
f &: \mathbb{R} \setminus \{0\} \to \mathbb{R}, & f(x) &= \tfrac{1}{x} \\
g_1 &: \mathbb{R} \to \mathbb{R}, & g_1(x) &= \sin(3x+1) \\
g_2 &: (-1,1) \to [-1,1], & g_2(x) &= \sin(3x+1) \\
h_1 &: P \times P \to \mathbb{N}, & h_1(p,q) &= p \cdot q \\
h_2 &: \mathbb{N} \times \mathbb{N} \to \mathbb{N}, & h_2(p,q) &= p \cdot q \\
i &: \mathbb{N} \to \{0,1,\dots,9\}, & i(n) &= \text{last digit of } n
\end{aligned}
$$

1. Decide whether the above mappings are onto (resp. one-to-one, bijection) or not.

2. Find ranges of the above mappings.

3. Compute

a) $f((0,2))$ b) $g_1((0,1))$ c) $h_2(\mathbb{N} \times \{2\})$ d) $i(P)$

4. Compute

a) $f_{-1}((-1,2))$ b) $(g_1)_{-1}((-1,1))$ c) $(g_2)_{-1}([-1,1])$ d) $(h_1)_{-1}(P)$
e) $(h_2)_{-1}(P)$ f) $i_{-1}(\{2\})$ g) $f_{-1}([1,+\infty))$ g) $(h_2)_{-1}([\frac{1}{2},1])$

5 Limits of functions

1. Compute the limits.

a) $\lim\limits_{x\to+\infty} \frac{(x+5)^4-(x+3)^4}{x^3+2x+2}$ b) $\lim\limits_{x\to3} \frac{x^2-5x+6}{x^2-8x+15}$ c) $\lim\limits_{x\to0} \frac{(1+x)(1+2x)(1+3x)-1}{x}$

d) $\lim\limits_{x\to1} \frac{x+x^2+\cdots+x^8-8}{x-1}$ e) $\lim\limits_{x\to1} \frac{x^8-1}{x^{12}-1}$ f) $\lim\limits_{x\to0} \frac{(1+7x)^8-(1+8x)^7}{x^2}$

2. Compute the limits.

a) $\lim\limits_{x\to+\infty} \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{2x+3}} \cdot x$ b) $\lim\limits_{x\to7} \frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$ c) $\lim\limits_{x\to0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$

d) $\lim\limits_{x\to0} \frac{\sqrt[11]{1+x}-1}{x}$ e) $\lim\limits_{x\to0} \frac{\sqrt[7]{1+ax}-\sqrt[8]{1+bx}}{x}$, $a, b > 0$ f) $\lim\limits_{x\to-\infty} (x^2+3x)\frac{\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{2}{x}}}{\sqrt{x^2+x}}$

3. Compute the limits (easy applications of compound function limit).

a) $\lim\limits_{x\to0} \frac{\sin 5x-\sin 3x}{\sin x}$ b) $\lim\limits_{x\to7} \frac{\sin(\sqrt{x+2}-3)}{x-7}$ c) $\lim\limits_{x\to0} \frac{\sin(x^2+2x+1)}{x-1}$

d) $\lim\limits_{x\to2} \frac{e^{3x-6}-1}{x^2-4}$ e) $\lim\limits_{x\to+\infty} x^5 \log\left(1+\frac{1}{x^5}\right)$ f) $\lim\limits_{x\to0} \frac{1-\cos x^2}{x^4}$

4. Compute the limits (advanced applications of compound function limit).

a) $\lim\limits_{x\to0} \frac{\log\cos x}{x^2}$ b) $\lim\limits_{x\to0} \frac{1+\sin x-\cos x}{1-\sin x-\cos x}$ c) $\lim\limits_{x\to0} \frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$

d) $\lim\limits_{x\to0} \frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$ e) $\lim\limits_{x\to+\infty} x\log\left(\frac{x^2+1}{x^2-3x}\right)$ f) $\lim\limits_{x\to0} \frac{e^{x\cos x}-1}{x^2}$

5. Compute the limits (further tricks to learn).

a) $\lim\limits_{x\to\frac{\pi}{3}} \frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}$ b) $\lim\limits_{x\to\frac{\pi}{4}} \frac{\sin x-\cos x}{x-\frac{\pi}{4}}$ c) $\lim\limits_{x\to3} \frac{e^{x^2-7}-e^{5-x}}{x-3}$

d) $\lim\limits_{x\to\infty} \left(\frac{x+2}{2x+1}\right)^{x^2}$ e) $\lim\limits_{x\to0} (1+x^2)^{\operatorname{cotg}^2 x}$ f) $\lim\limits_{x\to0} \left(\frac{1+\operatorname{tg} x}{1+\sin x}\right)^{\frac{1}{\sin^2 x}}$

g) $\lim\limits_{x\to\frac{\pi}{4}} \operatorname{tg} x^{\operatorname{tg} 2x}$ h) $\lim\limits_{x\to0} \left(\frac{1+x\cdot2^x}{1+x\cdot3^x}\right)^{\frac{1}{x^2}}$ i) $\lim\limits_{x\to\infty} \left(\cos\sqrt{x+1}-\cos\sqrt{x-1}\right)$

g) $\lim\limits_{x\to\infty} \frac{\log\left(x^2-x+1\right)}{\log(x^{10}+x+1)}$ h) $\lim\limits_{x\to+\infty} e^{-x^2+x}\log\left(\frac{x+3}{x^2-1}\right)$ i) $\lim\limits_{x\to+\infty} \log(1+2^x)\cdot\log\left(1+\frac{3}{x}\right)$

6 Derivatives

1. Find the domain and compute the derivative of the following functions

a) $\cos(2x+1)$ b) $\log(x^2+x+1)$ c) $\log(xe^x)$
d) $(x^2+51x+119)^{87}$ e) $(x+15)^3(x-17)^{10}x^9$ f) $\frac{e^{x^2+1}\cdot\cos x}{(x+1)^2\cdot\log(x^2+1)}$
g) x^x h) $(1+\sin x)^{\cos x}$ i) $\sin\left(\cos\left((x^3+17x^2-56x+1)^{18}\right)\right)$
j) $(\operatorname{arctg} x)^{\arcsin x}$ k) $\operatorname{arctg} e^x - \log\sqrt{\frac{e^{2x}}{e^{2x}+1}}$ l) $\operatorname{arctg}\sqrt{x^2-1} - \frac{\log x}{\sqrt{x^2-1}}$

2. Find domains, intervals of monotonicity, and local extrema of the following functions.

a) $x^3+12x-10$ b) xe^x c) $\log(x^2+x+1)$
d) $f(x)=\frac{2x}{1-x^2}$ e) $f(x)=x\sqrt{1-x^2}$ f) $f(x)=\sqrt[3]{x^3-x^2-x+1}$
g) $\sin x+\cos^2 x$ h) $\operatorname{arctg}\frac{x+1}{x-1}$ i) e^{x^2-3x+7}

3. Draw graph of f in a neighborhood of x_0 using one-sided limits of f and f'. Compute also $f'_\pm(x_0)$ and decide whether $f'(x_0)$ exists. Square brackets $[\dots]$ in e) denote the integer part.

a) $f(x)=\frac{\sin x}{x^2+e^x}$, $x_0=0$, b) $f(x)=\sqrt[3]{x^2-1}$, $x_0=1$, c) $f(x)=\begin{cases} e^{\sin x} & x>0 \\ \log(x+e^{x+1}) & x\le 0 \end{cases}$, $x_0=0$
d) $f(x)=|ex^2-e^x|$, $x_0=1$, e) $f(x)=x[\sin x]$, $x_0=\pi$, f) $f(x)=\max\{x^2,2x\}$, $x_0=2$,
g) $f(x)=\begin{cases} x+3 & x\ge -1 \\ x-1 & x<-1 \end{cases}$, $x_0=-1$, h) $f(x)=\begin{cases} \sin x & x>0 \\ \sin(x^2) & x\le 0 \end{cases}$, $x_0=0$
i) $f(x)=e^{|x^2-5x+6|}$, $x_0=3$, j) $f(x)=|x|\sin x$, $x_0=0$

7 l'Hospital rule

1. Compute the following limits using the l'Hospital rule

a) $\lim\limits_{x\to\pi} \frac{\sin x}{1+\cos x}$ b) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin 6x+\pi}{\cos 7x}$ c) $\lim\limits_{x\to-\pi} \frac{\pi\sin(\frac{x}{2})+x}{(x+\pi)\sin x}$

d) $\lim\limits_{x\to 1} \frac{\ln x-x+1}{x^2}$ e) $\lim\limits_{x\to 0} \frac{\sin x-x}{x^3}$ f) $\lim\limits_{x\to 0} \frac{e^x-1-\sin x}{x^2}$

2. Compute the following limits. Combine the l'Hospital rule with classical methods.

a) $\lim\limits_{x\to\pi} \frac{\sin(x^6)}{(1+\cos x)^3}$ b) $\lim\limits_{x\to 0} \frac{\sin\left(\frac{x^4}{(1+x^3)^4}\right)}{(1-\sin(x+\frac{\pi}{2}))^3}$ c) $\lim\limits_{x\to\pi} \frac{(x-\pi)(e^{x-\pi}-\sin(\frac{x}{2}))}{(1+\cos x)\sin x}$

d) $\lim\limits_{x\to 0} \frac{x\sin x+\ln x-x+1}{x^2}$ e) $\lim\limits_{x\to 0} \frac{\sin x-x\sqrt{1+x}}{x\sin(x^2)}$ f) $\lim\limits_{x\to 0} \frac{\sin x-x\cos x}{\sqrt{1+x^3}-\sqrt{1-x^3}}$

8 Investigation of functions

1. Investigate the following functions. It means: determine the domain, compute the first and second derivatives, limits in the endpoints (, and asymptotes), investigate monotonicity, convexity and local extrema and finaly draw graph of the function.

a) $f(x) = \frac{2x}{1-x^2}$ b) $f(x) = x\sqrt{1-x^2}$ c) $f(x) = \sqrt[3]{x^3 - x^2 - x + 1}$
d) $f(x) = \sqrt[3]{x^2} - \frac{2}{\sqrt{x}}$ e) $f(x) = \operatorname{arctg} \frac{x+1}{x-1}$ f) $f(x) = \frac{x^4+8}{x^3+1}$
g) $f(x) = 1 - x + \sqrt{\frac{x^3}{3+x}}$ h) $f(x) = \sin x + \cos^2 x$ i) $f(x) = \exp(-x^2 + 3x - 7)$

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