Sample Question

Mathematics 1, WS 2017/18

1. (8 points) Write definition of an *onto* mapping. Give positive and negative examples.

2. (12 points) Formulate Theorem 31 (Bolzano intermediate value theorem). Explain the theorem using a picture.

3. (8 points) Formulate Theorem 23 ("Division by positive zero" for functions) and prove it for A finite.

4. (12 points) Compute the limit of the constant sequence $a_n = -1$ using definition of a limit. Show that $b_n = (-1)^n$ does not have a limit. Which theorems do you need?

ANSWERS.

1. (8 points) Write definition of an *onto* mapping. Give positive and negative examples.
Answer: The mapping $f : X \to Y$ is onto, if for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
For example, $f : \mathrm{R} \to \mathrm{R}$, $f(x) = x^2$ is not onto, $f : \mathrm{R} \to [0, +\infty)$, $f(x) = x^2$ is onto. Graphs of these functions and explanation.
Additional questions: Draw a graph of a function which maps $[0, 1]$ onto $[1, 3]$. Is the sine function onto?
Answer: Picture of a straight segment connecting points $[0, 1]$ and $[1, 3]$. Sine is onto $[-1, 1]$ but it is not onto R.

2. (12 points) Formulate Theorem 31 (Bolzano intermediate value theorem). Explain the theorem using a picture.
Answer: If $I \subset \mathrm{R}$ is an interval, $f : I \to \mathrm{R}$ is continuous and $a, b \in I$ with $f(a) < f(b)$, then for every $c \in (f(a), f(b))$ there exists $\xi \in I$ with $f(\xi) = c$.
A picture of a continuous functions with a, b, c and ξ and explanation.
Additional question: Show that functions which are not continuous do not have the property.
Answer: Picture of function sgn with explanation for what a, b, c we cannot find ξ with the desired property.

3. (8 points) Formulate Theorem 23 ("Division by positive zero" for functions) and prove it for A finite.
Answer: statement and proof of the theorem ...

4. (12 points) Compute the limit of the constant sequence $a_n = -1$ using definition of a limit. Show that $b_n = (-1)^n$ does not have a limit. Which theorems do you need?
Answer: We show that the limit is -1. We take $\varepsilon > 0$ arbitrary and find n_0 such that for all $n \geq n_0$ it holds that $|a_n - (-1)| < \varepsilon$. Since $a_n = -1$ for all n, we have $|a_n - (-1)| = 0 < \varepsilon$ for all n, so we can take e.g. $n_0 = 1$.
Similarly, we can show that $\lim 1 = 1$. We can take two constant subsequences of $(-1)^n$, in particular $b_{2n} = 1$, $b_{2n+1} = -1$ with different limits 1 and -1. This contradicts the Theorem "If a sequence has a limit, then every subsequence has the same limit."