Midterm 2

for Mathematics I, WS2017/18

Compute the limit

$$\lim_{n\to\infty} \frac{(2n-1)!}{7^n + (2n+1)!}\left(\sqrt{2n^4+1} - \sqrt{n+1}\right).$$

If you compare orders of growth, justify the corresponding limit.

Solution. We 'take out the biggest term' in the denominator and get

$$\frac{(2n-1)!}{7^n + (2n+1)!} = \frac{(2n-1)!}{(2n+1)!\left(\frac{7^n}{(2n+1)!}+1\right)} = \frac{1}{2n(2n+1)\left(\frac{7^n}{(2n+1)!}+1\right)} = \frac{1}{2n^2\left(2+\frac{1}{n}\right)\left(\frac{7^n}{(2n+1)!}+1\right)}.$$

We can see that $\sqrt{2n^4+1}$ is much larger than $\sqrt{n+1}$ so we 'take out the biggest term' to get

$$\sqrt{2n^4+1} - \sqrt{n+1} = n^2\left(\sqrt{2+\frac{1}{n^4}} - \sqrt{\frac{1}{n^3}+\frac{1}{n^4}}\right).$$

Now, we have

$$\lim_{n\to\infty} \frac{(2n-1)!}{7^n + (2n+1)!}\left(\sqrt{2n^4+1} - \sqrt{n+1}\right) = \lim_{n\to\infty} \frac{n^2\left(\sqrt{2+\frac{1}{n^4}} - \sqrt{\frac{1}{n^3}+\frac{1}{n^4}}\right)}{2n^2\left(2+\frac{1}{n}\right)\left(\frac{7^n}{(2n+1)!}+1\right)} = \frac{\sqrt{2}-\sqrt{0}}{2(2-0)(0+1)} = \frac{\sqrt{2}}{4}.$$

We used $a_n \to A \;\Rightarrow\; \sqrt{a_n} \to \sqrt{A}$ and $\lim \frac{7^n}{(2n+1)!} = 0$ (growth orders). To justify the last limit we compute

$$\lim_{n\to\infty} \frac{\frac{7^{n+1}}{(2(n+1)+2)!}}{\frac{7^n}{(2n+1)!}} = \lim_{n\to\infty} \frac{7}{(2n+3)(2n+2)} = 0$$

and use Lemma 14.

Grading. Bracket 3 pts (3 pts for taking out the biggest term or 1 pt for difference of square roots trick and 2 pts for taking out the biggest term), fraction 6 pts (2 pts — taking out the factorial, 2 pts — canceling factorials and taking out the n, 2 pts — justifying the growth orders), result 1pt. Further: each secondary school error -0.5 pts, partial limit -2 pts, further partial limits -1 pt