Midterm 1

for Mathematics I, WS2017/18

Compute the limit

$$\lim_{n\to\infty} \frac{\sqrt{n^3+3n^2\sqrt{n}}-\sqrt{n^3+n}}{n+\sqrt[n]{n^3+2^n}}.$$

Solution. Let us first simplify the numerator (we use $a^2-b^2=(a-b)(a+b)$)

$$\sqrt{n^3+3n^2\sqrt{n}}-\sqrt{n^3+n} = \frac{n^3+3n^2\sqrt{n}-(n^3+n)}{\sqrt{n^3+3n^2\sqrt{n}}+\sqrt{n^3+n}} = \frac{3n^2\sqrt{n}-n}{\sqrt{n^3+3n^2\sqrt{n}}+\sqrt{n^3+n}}.$$

We 'take out the biggest term' in the numerator and denominator

$$= \frac{n^{5/2}\left(3-\frac{1}{n^{3/2}}\right)}{n^{3/2}\left(\sqrt{1+\frac{3}{\sqrt{n}}}+\sqrt{1+\frac{1}{n^2}}\right)} = \frac{n\left(3-\frac{1}{n^{3/2}}\right)}{\sqrt{1+\frac{3}{\sqrt{n}}}+\sqrt{1+\frac{1}{n^2}}}.$$

To simplify the denominator of the original limit, we can see that $\lim_{n\to\infty}\sqrt[n]{n^3+2^n}=2$ (to be justified below). So, 'taking out the biggest term' of the denominator we get

$$n\left(1+\frac{1}{n}\sqrt[n]{n^3+2^n}\right).$$

Together, we have

$$\lim_{n\to\infty} \frac{\sqrt{n^3+3n^2\sqrt{n}}-\sqrt{n^3+n}}{n+\sqrt[n]{n^3+2^n}} = \lim_{n\to\infty} \frac{3-\frac{1}{n^{3/2}}}{\left(1+\frac{1}{n}\sqrt[n]{n^3+2^n}\right)\left(\sqrt{1+\frac{3}{\sqrt{n}}}+\sqrt{1+\frac{1}{n^2}}\right)} = \frac{3-0}{(1+0)(1+1)} = \frac{3}{2}.$$

We justify $\lim_{n\to\infty}\sqrt[n]{n^3+2^n}=2$. Since $\lim_{n\to\infty}\frac{n^3}{2^n}=0$ (orders of growth), we have $n^3\le 2^n$ for large n, and therefore

$$2=\sqrt[n]{2^n}\le\sqrt[n]{2^n+n^3}\le\sqrt[n]{2\cdot 2^n}=2\sqrt[n]{2}\le 2\sqrt[n]{n}.$$

Since $\lim_{n\to\infty}2\sqrt[n]{n}=2$, by sandwich theorem we have $\lim_{n\to\infty}\sqrt[n]{n^3+2^n}=2$.
Grading. Numerator 4 pts (2 pts — difference of square roots trick, 2 pts — taking out the biggest term), denominator 5 pts (1 pt — lim $\sqrt[n]{\dots}$, 2 pts — taking out the biggest term, 2 pts — sandwich theorem).
Further: each secondary school error -0.5 pts, partial limit -2 pts, further partial limits -1 pt