Midterm — sample test

Mathematics I, WS2017/18

Compute the limit

$$\lim_{n\to\infty} \frac{3^{n+1}+n^7}{n2^n+3^n} \sqrt[n]{\frac{3^n}{n}-2^n}.$$

Solution. We 'take out the biggest terms' in the numerator and denominator to get

$$\frac{3^{n+1}+n^7}{n2^n+3^n} = \frac{3^n\left(3+\frac{n^7}{3^n}\right)}{3^n\left(\frac{n2^n}{3^n}+1\right)} = \frac{3+\frac{n^7}{3^n}}{\frac{n2^n}{3^n}+1}$$

In the n-th root, we again 'take out the biggest term' and we obtain

$$\sqrt[n]{\frac{3^n}{n}-2^n} = \sqrt[n]{\frac{3^n}{n}\left(1-\frac{n}{3^n}2^n\right)} = \frac{3}{\sqrt[n]{n}}\sqrt[n]{1-n\left(\frac{2}{3}\right)^n}.$$

Together we have

$$\lim_{n\to\infty} \frac{3^n+n^7}{n2^n+3^{n+1}} \sqrt[n]{\frac{3^n}{n}-2^n} = \lim_{n\to\infty} \frac{3+\frac{n^7}{3^n}}{\frac{n2^n}{3^n}+1}\cdot\frac{3}{\sqrt[n]{n}}\sqrt[n]{1-n\left(\frac{2}{3}\right)^n} = \frac{3+0}{0+1}\cdot\frac{3}{1}\cdot 1 = 9.$$

We have used $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ and 'comparison of growths'

$$\lim_{n\to\infty} n\left(\frac{2}{3}\right)^n = \lim_{n\to\infty}\frac{n}{\left(\frac{3}{2}\right)^n} = 0$$

and the sandwich theorem to show that $\lim\limits_{n\to\infty}\sqrt[n]{1-n(2/3)^n}=1$. In particular, we have

$$\frac{1}{2}\le 1-n\left(\frac{2}{3}\right)^n \le 1, \quad \text{and therefore} \quad \sqrt[n]{\frac{1}{2}} \le \sqrt[n]{1-n\left(\frac{2}{3}\right)^n} \le 1.$$

where $\sqrt[n]{1/2}\to 1$.

Grading. 7 points for computations and the result, 3 points for justifications (mainly of the limit of the n-th root).