HOMEWORK 8

due date: November 29, 2017

a) Find a suitable trigonometric identity and compute the limit

$$\lim_{x\to 3} \frac{2\cos(\frac{\pi}{4}x)+\sqrt{2}}{x-3}.$$

b) Compute the limit

$$\lim_{x\to +\infty} \frac{\log(2^x+3^x+x^7)}{\log(2^x+\cos(x^2))}.$$

I both a) and b), if you use the compound function theorem (Theorem 25), write what is the inner function g, the outer function f and write the relevant limits from Theorem 25 (what is c, A, B). Voluntarily: Verify condition (I).

Solution. a) Write the numerator as

$$2\left(\cos(\frac{\pi}{4}x)+\frac{\sqrt{2}}{2}\right) = 2\left(\cos(\frac{\pi}{4}x)+\cos(\frac{\pi}{4})\right).$$

Use the formula $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ and then $\cos y = \sin(\frac{\pi}{2}-y)$ to obtain

$$4\cos(\frac{\pi}{8}(x+1))\cos(\frac{\pi}{8}(x-1)) = 4\sin(\frac{\pi}{2}-\frac{\pi}{8}(x+1))\cos(\frac{\pi}{8}(x-1)) = 4\sin(\frac{\pi}{8}(3-x))\cos(\frac{\pi}{8}(x-1)).$$

Now, we can write

$$\lim_{x\to 3}\frac{2\cos(\frac{\pi}{4}x)+\sqrt{2}}{x-3} = \lim_{x\to 3} 4\cos(\frac{\pi}{8}(x-1))\lim_{x\to 3}\frac{\sin(\frac{\pi}{8}(3-x))}{x-3} = 4\cos\frac{\pi}{4}\cdot\lim_{x\to 3}(-\frac{\pi}{8})\frac{\sin(\frac{\pi}{8}(3-x))}{\frac{\pi}{8}(3-x)}$$

$$= -\frac{\pi\sqrt{2}}{4}\lim_{x\to 3}\frac{\sin(\frac{\pi}{8}(3-x))}{\frac{\pi}{8}(3-x)} = -\frac{\pi\sqrt{2}}{4}.$$

Here we used CFT with $f(y)=\frac{\sin y}{y}\to 1$, $g(x)=\frac{\pi}{8}(3-x)\to 0$, $c=3$, $A=0$, $B=1$. Condition (I) is satisfied since $\frac{\pi}{8}(3-x)=0$ only for $x=3$.
b) Take out the biggest term in both logarithms and obtain

$$\frac{\log(2^x+3^x+x^7)}{\log(2^x+\cos(x^2))} = \frac{\log(3^x(\frac{2^x}{3^x}+1+\frac{x^7}{3^x}))}{\log(2^x(1+\frac{\cos(x^2)}{2^x}))} = \frac{\log(3^x)+\log(\frac{2^x}{3^x}+1+\frac{x^7}{3^x})}{\log(2^x)+\log(1+\frac{\cos(x^2)}{2^x})} = \frac{x\log 3+\log(\frac{2^x}{3^x}+1+\frac{x^7}{3^x})}{x\log 2+\log(1+\frac{\cos(x^2)}{2^x})}.$$

Now, take out x in the numerator and denominator and get

$$\frac{x\left(\log 3+\frac{\log(\frac{2^x}{3^x}+1+\frac{x^7}{3^x})}{x}\right)}{x\left(\log 2+\frac{\log(1+\frac{\cos(x^2)}{2^x})}{x}\right)} = \frac{\log 3+\frac{\log(\frac{2^x}{3^x}+1+\frac{x^7}{3^x})}{x}}{\log 2+\frac{\log(1+\frac{\cos(x^2)}{2^x})}{x}}.$$

So, by arithmetics of limits, by $\lim_{x\to+\infty}\frac{2^x}{3^x}=0=\lim_{x\to+\infty}\frac{x^7}{3^x}=\lim_{x\to+\infty}\frac{\cos(x^2)}{2^x}$ and by continuity of logarithm we have

$$\lim_{x\to+\infty}\frac{\log(2^x+3^x+x^7)}{\log(2^x+\cos(x^2))} = \lim_{x\to+\infty}\frac{\log 3+\frac{\log(\frac{2^x}{3^x}+1+\frac{x^7}{3^x})}{x}}{\log 2+\frac{\log(1+\frac{\cos(x^2)}{2^x})}{x}} = \frac{\log 3+\log 1}{\log 2+\log 1} = \frac{\log 3}{\log 2}.$$

In fact, here we used CFT for $f(y)=\log y$, $g(x)=\frac{2^x}{3^x}+1+\frac{x^7}{3^x})$, resp. $g(x)=1+\frac{\cos(x^2)}{2^x}$ with $c=+\infty$, $A=1$, $B=0$ and condition (C) is satisfied since $\log y$ is continuous at $y=1$.