HOMEWORK 5

due date: November 8, 2017

Compute the limit

$$\lim_{n\to\infty}\left(\frac{\sqrt{4^n+3^n}-\sqrt{4^n-3^n}-\left(\frac{4}{3}\right)^n}{\sqrt{4^n+2^n}-2^n}+2^n\left[\frac{2^n+3^n\sin^2 n}{3^n+2^n+1}\right]\right),$$

here $[\dots]$ means the integer part. If you are not able to compute the whole limit, try to compute some parts of it and write some ideas what to do.

Solution. Let us split the limit into two (arithmetics of limits)

$$\lim_{n\to\infty}\frac{\sqrt{4^n+3^n}-\sqrt{4^n-3^n}-\left(\frac{4}{3}\right)^n}{\sqrt{4^n+2^n}-2^n}+\lim_{n\to\infty}2^n\left[\frac{2^n+3^n\sin^2 n}{3^n+2^n+1}\right] \tag{1}$$

and let us start with the second one. Since

$$0\le\frac{2^n+3^n\sin^2 n}{3^n+2^n+1}\le\frac{2^n+3^n\sin^2 n}{3^n+2^n+1}<1$$

holds for all $n\in\mathbb{N}$ and the last inequality is strict, we have

$$\left[\frac{2^n+3^n\sin^2 n}{3^n+2^n+1}\right]=0 \quad\text{and}\quad 2^n\left[\frac{2^n+3^n\sin^2 n}{3^n+2^n+1}\right]=0,$$

therefore the second limit in (1) is zero (limit of the sequence of zeros).
Let us now focus on the first limit in (1). The denominator is equal to ($2^n=\sqrt{4^n}$, so we have subtraction of two square roots)

$$\frac{4^n+2^n-4^n}{\sqrt{4^n+2^n}+\sqrt{4^n}}=\frac{2^n}{2^n\left(\sqrt{1+\frac{1}{2^n}}+1\right)}=\frac{1}{\sqrt{1+\frac{1}{2^n}}+1}.$$

Concerning the numerator, we have

$$\sqrt{4^n+3^n}-\sqrt{4^n-3^n}=\frac{4^n+3^n-4^n+3^n}{\sqrt{4^n+3^n}+\sqrt{4^n-3^n}}=\frac{2\cdot 3^n}{2^n\left(\sqrt{1+\frac{3^n}{4^n}}+\sqrt{1-\frac{3^n}{4^n}}\right)}=\left(\frac{3}{2}\right)^n\frac{2}{\sqrt{1+\frac{3^n}{4^n}}+\sqrt{1-\frac{3^n}{4^n}}}.$$

So, in the numerator of the first limit in (1) we have

$$\left(\frac{3}{2}\right)^n\frac{2}{\sqrt{1+\frac{3^n}{4^n}}+\sqrt{1-\frac{3^n}{4^n}}}-\left(\frac{4}{3}\right)^n.$$

Since $\frac{3}{2}>\frac{4}{3}$ we take out the biggest term and obtain

$$\left(\frac{3}{2}\right)^n\left(\frac{2}{\sqrt{1+\frac{3^n}{4^n}}+\sqrt{1-\frac{3^n}{4^n}}}-\left(\frac{8}{9}\right)^n\right)$$

since $\frac{2}{3}\cdot\frac{4}{3}=\frac{8}{9}$. Now we can insert everything to the first limit in (1) and we may split it into three limits (arithmetics) without causing any harm (no undefined operations)

$$\lim_{n\to\infty}\left(\frac{3}{2}\right)^n\cdot\frac{\lim\limits_{n\to\infty}\frac{2}{\sqrt{1+\frac{3^n}{4^n}}+\sqrt{1-\frac{3^n}{4^n}}}-\left(\frac{8}{9}\right)^n}{\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{2^n}}+1}}=+\infty\cdot\frac{\frac{2}{1+1}-0}{\frac{1}{1+1}}=+\infty\cdot 2=+\infty.$$

The result is $+\infty+0=+\infty$.