HOMEWORK 4

due date: November 1, 2017

a) Compute the limit
$$\lim_{n\to\infty} \sqrt[n]{2n^7 + n^6 \sin n}.$$

b) Compute the limit (justify the orders of growth argument)
$$\lim_{n\to\infty} \frac{8^{2n+1}}{3^{n^2-n}}.$$

Solution:

a) Take out the biggest term split the limit into two
$$\lim \sqrt[n]{n^7} \cdot \lim \sqrt[n]{2 + \frac{\sin n}{n}} = 1 \cdot 1 = 1.$$

Here the first limit is 1, since it is $\lim \sqrt[n]{n}\,\sqrt[n]{n}\,\sqrt[n]{n}\,\sqrt[n]{n}\,\sqrt[n]{n}\,\sqrt[n]{n}\,\sqrt[n]{n} = 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1 = 1$ and the second limit is also 1 by the sandwich theorem. One possible argumentation is: $|\sin n| \le 1$ and $n \ge 1$, so $\frac{\sin n}{n} \le 1$ for all n, therefore $1 \le 2 + \frac{\sin n}{n} \le 3$ and
$$\sqrt[n]{1} \le \sqrt[n]{2 + \frac{\sin n}{n}} \le \sqrt[n]{3},$$
where $\sqrt[n]{1}$, $\sqrt[n]{3} \to 1$. Another argumentation is: $\frac{1}{n}\sin n \to 0$ (vanishing times bounded), so $-\varepsilon < \frac{\sin n}{n} < \varepsilon$ for all n LARGER THAN SOME n_0. Then we have (for large n's)
$$\sqrt[n]{2-\varepsilon} \le \sqrt[n]{2 + \frac{\sin n}{n}} \le \sqrt[n]{2+\varepsilon},$$
where $\sqrt[n]{2-\varepsilon}$, $\sqrt[n]{2+\varepsilon} \to 1$.

b) The limit is 0. We justify it by showing that $\lim \frac{a_{n+1}}{a_n} < 1$ and applying the convergence criterion (Lemma 14).
$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{8^{2(n+1)+1}}{3^{(n+1)^2-(n+1)}}}{\frac{8^{2n+1}}{3^{n^2-n}}} = \lim \frac{8^{2n+3-(2n+1)}}{3^{n^2+n-(n^2-n)}} = \lim \frac{8^2}{3^{2n}} = \lim \frac{64}{9^n} = 0 < 1.$$