HOMEWORK 10

due date: December 13, 2017

a) Draw graph of

$$f(x) = \left[\sqrt{x^2+3}\right] \log\left(\frac{x}{x+1}\right)$$

on a neighborhood of $x = 1$ ($[\dots]$ means the integer part).
b) Compute the limit

$$\lim_{x\to 0+} \frac{\arcsin\left(\frac{1}{\ln x}\right)}{\sqrt[3]{x}}.$$

Solution. a) We have

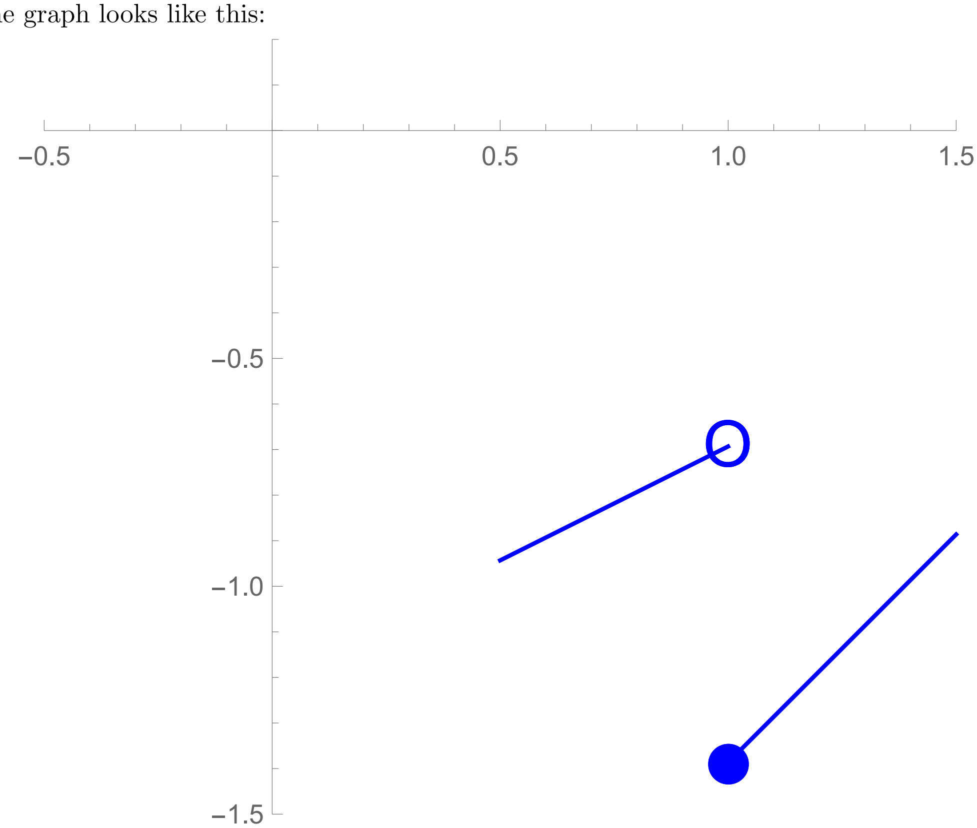
$$f(x) = \begin{cases} 2\log\left(\frac{x}{x+1}\right) & x \in (1, 1+\varepsilon) \\ 2\log\frac{1}{2} & x = 1 \\ \log\left(\frac{x}{x+1}\right) & x \in (1-\varepsilon, 1), \end{cases}$$

where $2\log\frac{1}{2} < \log\frac{1}{2} < 0$. We can see that f is right-continuous in 1 but not left-continuous. We compute limits of derivatives:

$$\lim_{x\to 1+} f'(x) = \lim_{x\to 1+} 2 \cdot \frac{x+1}{x} \cdot \frac{x+1-x}{(x+1)^2} = \lim_{x\to 1+} \frac{2}{x(x+1)} = 1,$$

$$\lim_{x\to 1-} f'(x) = \lim_{x\to 1-} \frac{x+1}{x} \cdot \frac{x+1-x}{(x+1)^2} = \lim_{x\to 1-} \frac{1}{x(x+1)} = \frac{1}{2}.$$

So, the graph looks like this:



b) We can use the l'Hospital rule ("$\frac{0}{0}$") to get

$$\lim_{x\to 0+} \frac{\arcsin\left(\frac{1}{\log x}\right)}{\sqrt[3]{x}} = \lim_{x\to 0+} \frac{\frac{1}{\sqrt{1-\frac{1}{\log^2 x}}}\cdot\left(\frac{1}{\log x}\right)'}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x\to 0+} 3x^{\frac{2}{3}}\frac{1}{\sqrt{1-\frac{1}{\ln^2 x}}}\cdot\left(-\frac{1}{x\log^2 x}\right)$$

$$= -3\lim_{x\to 0+}\frac{1}{\sqrt{1-\frac{1}{\log^2 x}}}\cdot\lim_{x\to 0+}\frac{1}{x^{\frac{1}{3}}\log^2 x}.$$

Obiously, the first limit in the last expression equals 1 (by CFT with outer function $f(y)=\frac{1}{\sqrt{1-y}}$ and inner function $g(x)=\frac{1}{\log^2 x}\to 0$) and the second limit in the last expression is $+\infty$ by the following computation. First,

$$\lim_{x\to 0+}\frac{1}{x^{\frac{1}{3}}\log^2 x} = \lim_{x\to 0+}\frac{1}{x^{\frac{1}{6}}\log x}\lim_{x\to 0+}\frac{1}{x^{\frac{1}{6}}\log x}.$$

Second,

$$\lim_{x\to 0+}\frac{1}{x^{\frac{1}{6}}\log x} = \lim_{x\to 0+}\frac{x^{-\frac{1}{6}}}{\log x} = \lim_{x\to 0+}\frac{-\frac{1}{6}x^{-\frac{7}{6}}}{x^{-1}} = \lim_{x\to 0+} -\frac{1}{6}x^{-\frac{1}{6}} = \lim_{x\to 0+} -\frac{1}{6\sqrt[6]{x}} = -\infty,$$

where we used the l'Hospital rule in the second equality (type "$\frac{\pm\infty}{-\infty}$").
Together we have

$$\lim_{x\to 0+}\frac{\arcsin\left(\frac{1}{\log x}\right)}{\sqrt[3]{x}} = -3\cdot 1\cdot(+\infty) = -\infty.$$