Exam test (sample)

for Mathematics 1, WS 2017/18

1. (15 points) Compute the limit

$$\lim_{x \to 1} \frac{e^{x+1} - e^{2\sqrt{x}}}{\sin(\pi x) \cos(\frac{\pi}{2} x)}.$$

2. (20 points) Investigate the function

$$f(x) = \frac{x}{2} + \operatorname{arctg}(1 - x)$$

(find local extrema, intervals of monotonicity, convexity, inflections, limits in endpoints of D_f, asymptotes and draw graph of f).

3. (15 points) Investigate the function

$$g(x) = \begin{cases} \frac{\sin x}{x} & x > 0, \\ \sqrt{x^2 + x + 2} & x \le 0 \end{cases}$$

in a neighborhood of 0 (compute limits of $g(x)$ and $g'(x)$ as $x \to 0+$, $x \to 0-$, decide, whether g is continuous in 0, and draw graph of g in a neighborhood of 0) and compute $g'_+(0)$, $g'_-(0)$.

Solution.

1. The numerator and the denominator both converge to zero. Both terms in the denominator converge to zero, so let us compute limits

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x-1} \qquad \text{and} \qquad \lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{x-1}$$

using the l'Hospital rule. In both limits numerator and denominator tend to zero, so we can use the l'Hospital rule and obtain

$$\lim_{x\to 1} \frac{\sin(\pi x)}{x-1} = \lim_{x\to 1} \frac{\pi\cos(\pi x)}{1} = -\pi$$

$$\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{x-1} = \lim_{x\to 1} \frac{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)}{1} = -\frac{\pi}{2}.$$

The numerator we can rewrite as

$$e^{x+1} - e^{2\sqrt{x}} = e^{2\sqrt{x}}\left(e^{x+1-2\sqrt{x}} - 1\right) = e^{2\sqrt{x}}\,\frac{e^{x+1-2\sqrt{x}}-1}{x+1-2\sqrt{x}}\,(x+1-2\sqrt{x}).$$

So, using arithmetics of limits, we can write

$$\lim_{x\to 1}\frac{e^{x+1}-e^{2\sqrt{x}}}{\sin(\pi x)\cos(\frac{\pi}{2}x)} = \lim_{x\to 1} e^{2\sqrt{x}} \cdot \lim_{x\to 1}\frac{e^{x+1-2\sqrt{x}}-1}{x+1-2\sqrt{x}} \cdot \lim_{x\to 1}\frac{x+1-2\sqrt{x}}{(x-1)^2} \cdot \lim_{x\to 1}\frac{1}{\frac{\sin(\pi x)}{x-1}\,\frac{\cos(\frac{\pi}{2}x)}{x-1}}.$$

On the right-hand side, the first limit is e^2 (inserting $x = 1$), the last limit is equal to $\frac{2}{\pi^2}$ (by computations above) and the second limit is equal to 1 by CFT (inner function $g(x) = x+1-2\sqrt{x} \to 0$ as $x \to 1$, the outer function $f(y) = \frac{e^y-1}{y} \to 1$ as $y \to 0$, condition (I) holds). So, we have

$$\lim_{x\to 1}\frac{e^{x+1}-e^{2\sqrt{x}}}{\sin(\pi x)\cos(\frac{\pi}{2}x)} = \frac{2e^2}{\pi^2}\lim_{x\to 1}\frac{x+1-2\sqrt{x}}{(x-1)^2}.$$

The last fraction can is equal to

$$\frac{x+1-2\sqrt{x}}{(x-1)^2} = \frac{(\sqrt{x}-1)^2}{(x-1)^2} = \frac{(\sqrt{x}-1)^2}{(\sqrt{x}-1)^2(\sqrt{x}+1)^2} = \frac{1}{(\sqrt{x}+1)^2} \to \frac{1}{(1+1)^2}.$$

So, the answer is

$$\lim_{x\to 1}\frac{e^{x+1}-e^{2\sqrt{x}}}{\sin(\pi x)\cos(\frac{\pi}{2}x)} = \frac{2e^2}{\pi^2}\cdot\frac{1}{4} = \frac{e^2}{2\pi^2}.$$

Grading. denominator — 4 pts, extracting $e^{2\sqrt{x}}$ — 2 pts, using $\frac{e^y-1}{y}$ — 2 pts, CFT with details — 3 pts, $(\sqrt{x}-1)^2$ — 3 pts, correct result — 1 pt.
Further: Partial limit -2 pts, secondary school errors -0.5 pt.

2. Obviously $D_f = \mathrm{R}$. Let us compute the limits in $\pm\infty$

$$\lim_{x\to+\infty} f(x) = +\infty + (-\frac{\pi}{2}) = +\infty, \qquad \lim_{x\to-\infty} f(x) = -\infty + \frac{\pi}{2} = -\infty.$$

Let us compute the asymptote in $+\infty$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \frac{1}{2} + 0, \qquad \lim_{x\to+\infty} f(x) - \frac{1}{2}x = -\frac{\pi}{2},$$

so the asymptote in $+\infty$ is $y = \frac{1}{2}x - \frac{\pi}{2}$. Similarly,

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \frac{1}{2} + 0, \qquad \lim_{x\to+\infty} f(x) - \frac{1}{2}x = \frac{\pi}{2},$$

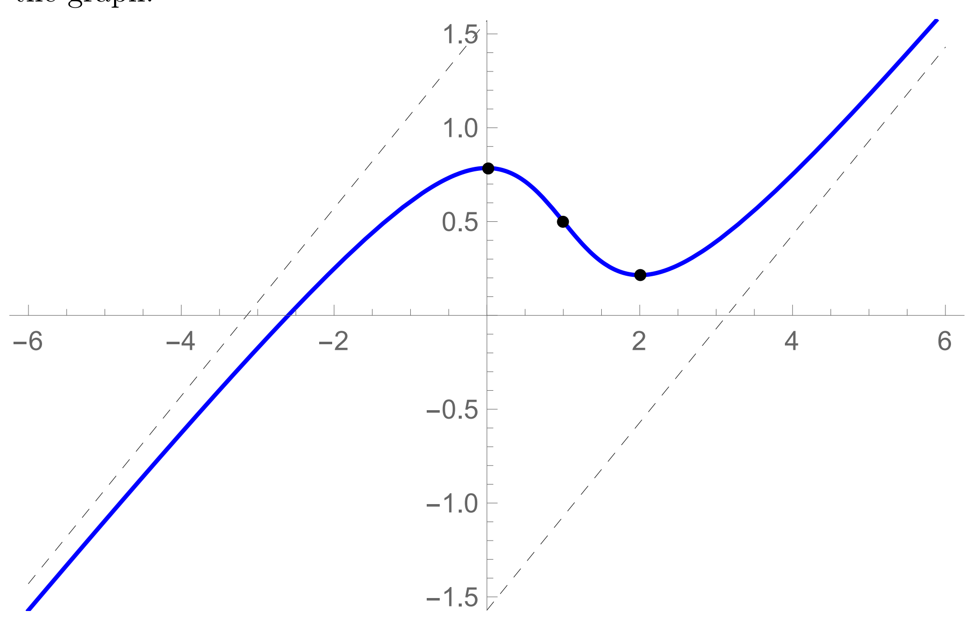
so the asymptote in $-\infty$ is $y = \frac{1}{2}x - \frac{\pi}{2}$.
Let us compute derivatives

$$f'(x) = \frac{1}{2} + \frac{1}{1+(1-x)^2}(-1) = \frac{2-2x+x^2-2}{2(2-2x+x^2)} = \frac{x(x-2)}{2(2-2x+x^2)}.$$

The denominator is always positive (since $2-2x+x^2 = (1-x)^2+1 > 0$), so f is increasing ($f' > 0$) on $(2,+\infty)$ and $(-\infty, 0)$ and decreasing ($f' < 0$) on $(0,2)$. Therefore, f has a local maximum in 0 and local minimum in 2.
We have

$$f''(x) = \frac{1}{2} \cdot \frac{(2x-2)(2-2x+x^2) - (x^2-2x)(2x-2)}{(2-2x+x^2)^2} = \frac{2(x-1)}{(2-2x+x^2)^2}.$$

We can see that f is convex ($f'' > 0$) on $(1,+\infty)$ and concave ($f'' < 0$) on $(-\infty, 1)$ and there is an inflection in 1.
It remains to evaluate f in the interesting points $f(0) = \frac{\pi}{4}$, $f(2) = 1 - \frac{\pi}{4} > 0$, $f(1) = \frac{1}{2}$ and draw the graph.



Grading. limits and asymptotes — 3 pts, f' — 2 pts, monotonicity and extrema — 4 pts, f'' — 2 pts, convexity and inflection — 4 pts, graph — 5 pts.

3. We first compute the limits

$$\lim_{x\to 0-} g(x) = \lim_{x\to 0-} \sqrt{x^2+x+2} = 2, \qquad \lim_{x\to 0+} g(x) = \lim_{x\to 0+} \frac{\sin x}{x} = 1$$

and conclude that g is not continuous in 0. Now, let us compute the derivative

$$g'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & x > 0, \\ \frac{2x+1}{2\sqrt{x^2+x+2}} & x < 0. \end{cases}$$

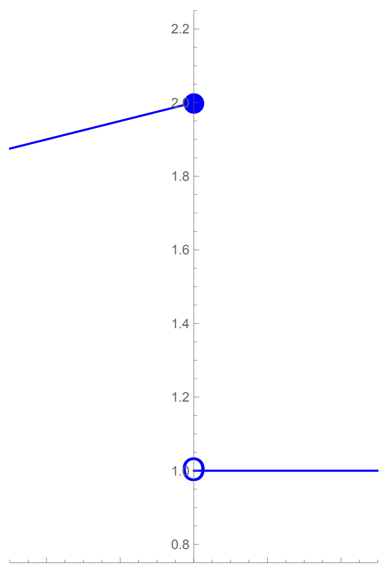
So, the limits are

$$\lim_{x\to 0-} g'(x) = \lim_{x\to 0-} \frac{2x+1}{2\sqrt{x^2+x+2}} = \frac{1}{4}$$

and the second using the l'Hospital rule

$$\lim_{x\to 0+} g'(x) = \lim_{x\to 0+} \frac{x\cos x - \sin x}{x^2} = \lim_{x\to 0+} \frac{\cos x - x\sin x - \cos x}{2x} = 0$$

and the graph looks as follows.



Let us conclude that the function is continuous from the left in 0, so $g'_-(0) = \frac{1}{4}$. On the other hand, g is not continuous from the right and

$$g'_+(0) = \lim_{x\to 0+} \frac{g(x) - g(0)}{x} = -\infty$$

("division by negative zero").

Grading. limits and continuity — 3 pts, g' — 2 pts, limits of g' — 3 pts, graph — 5 pts, $g'(0)$ — 2pts.