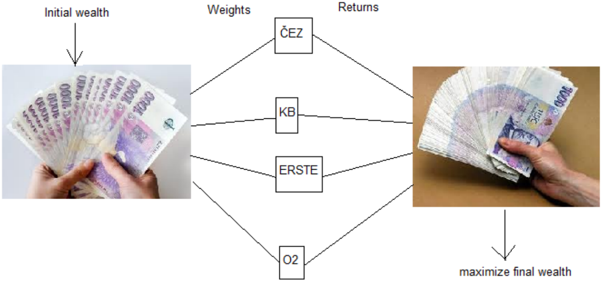
Portfolio optimization with Stochastic Dominance constraints

Thierry Post and Miloš Kopa,
Charles University, Faculty of Mathematics and Physics
Sokolovská 83, 186 75 Praha 8

January 23, 2018

Motivation

Thierry Post and Miloš Kopa, Charles University, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8



Motivation

- one criterion (expected wealth maximization) is not enough
- we do care about risk of the decision (investment)
- we want our decision to be better than some given decision (benchmark)
- the outcomes of decisions are random - **how to compare random variables**?
  - expected values - too weak
  - several characteristics - better, but still too weak
  - all realizations - too strong
  - compromise - STOCHASTIC DOMINANCE (SD)

Basic definitions

Let:

- $F_X(x)$ be CDF of random variable X.
- $F_X^{(2)}(y) = \int_{-\infty}^{y} F_X(x)dx$
- $F_X^{(3)}(t) = \int_{-\infty}^{t} F_X^{(2)}(y)dx$

Definition

Random variable X dominates random variable Y with respect to:

- FIRST ORDER SD if $F_X(x) \leq F_Y(x)$ for all $x \in \mathbb{R}$ and $\exists x_0 \in \mathbb{R}$: $F_X(x_0) < F_Y(x_0)$
- SECOND ORDER SD if $F_X^{(2)}(y) \leq F_Y^{(2)}(y)$ for all $y \in \mathbb{R}$ and $\exists y_0 \in \mathbb{R}$: $F_X^{(2)}(y_0) < F_Y^{(2)}(y_0)$
- THIRD ORDER SD if $\mathbb{E}X \geq \mathbb{E}Y$ and $F_X^{(3)}(t) \leq F_Y^{(3)}(t)$ for all $t \in \mathbb{R}$ and $\exists t_0 \in \mathbb{R}$: $F_X^{(3)}(t_0) < F_Y^{(3)}(t_0)$.

Summary

- Optimization method for building portfolios that dominate a benchmark by the third order stochastic dominance (TSD).
- Relies on the properties of semivariance, 'super-convex' dominance and QCP.
- Applied to stock market data using an industry momentum strategy.
- Important performance improvements compared with MV dominance and SSD.
- Average out-of-sample outperformance $\approx 7\%$ p.a. with less downside risk, quarterly rebalancing and no short selling.

Contents

Stochastic Dominance

- Model-free decision rules for DMuR
- Based on general regularity conditions about risk preferences
- Related to majorization in mathematical order theory
- Under Gaussianity, similar to M-V analysis
- Applies more generally for any probability distribution

Genealogy

Crit.	Reference	Distr	Utility	Risk
MV	Mark52 *JF*	Normal	$x + bx^2$	$\mathbb{E}[(x-\mu)^2]$
FSD	QuiSap62 *RES*	Any	$u' \geq 0$	$\mathbb{P}[x \leq z]$
SSD	HadRus69 *AER* HanLev69 *RES* RusSeo70 *JET*	Any	$u'' \leq 0$	$\mathbb{E}[(z-x)\mathbb{I}(x \leq z)]$
TSD	Whit70 *AER* Mark59 Ch.IX	Any	$u''' \geq 0$	$\mathbb{E}[(z-x)^2\mathbb{I}(x \leq z)]$

Numerical example

	$\tau = 1$	$\tau = 2$	$\tau = 3$	μ	σ	sk
ν_1	0.90	1.10	1.30	1.10	0.16	0.00
ν_2	0.97	1.10	1.41	1.16	0.18	1.11

Numerical example

	$\tau = 1$	$\tau = 2$	$\tau = 3$	μ	σ	sk
ν_1	0.90	1.10	1.30	1.10	0.16	0.00
ν_2	0.97	1.04	1.41	1.14	0.19	1.56

Numerical example

	$\tau = 1$	$\tau = 2$	$\tau = 3$	μ	σ	sk
ν_1	0.90	1.10	1.30	1.10	0.16	0.00
ν_2	0.97	1.00	1.40	1.12	0.20	1.70

FSD portfolio analysis

- Kuosmanen (2001, 2004) formulates FSD optimization as a large MILP problem
- Subsequent OR studies by Ruszczynski c.s. analyze various algorithms and approximations
- The combinatorial optimization problem is computationally expensive
- Solved for numerical examples and choice experiments but not for realistic data dimensions
- Presumably, FSD is also too weak to detect many risk arbitrage opportunties
- Kopa and Post (2009) distinguish between admissibility and optimality

First order stochastic dominance (FSD) - notation

$\mathbf{r}$... random vector of assets' returns
$\boldsymbol{\lambda}$... portfolio weights
$\mathbf{r}'\boldsymbol{\lambda}$... random return of portfolio $\boldsymbol{\lambda}$
$F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$... cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$.

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the first-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$) if

$$F_{\mathbf{r}'\boldsymbol{\lambda}}(x) \leq F_{\mathbf{r}'\boldsymbol{\tau}}(x) \qquad \forall x \in \mathbb{R}.$$

First order stochastic dominance (FSD) - interpretation

Other equivalent definitions: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all utility functions.
- No non-satiable decision maker prefers portfolio $\boldsymbol{\tau}$ to portfolio $\boldsymbol{\lambda}$.
- $F^{-1}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \geq F^{-1}_{\mathbf{r}'\boldsymbol{\tau}}(y) \;\; \forall y \in [0,1]$.
- $\mathrm{VaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) \leq \mathrm{VaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) \;\; \forall \alpha \in [0,1]$.

In general, FSD relation is expressed by infinitely many inequalities. However, under assumption of equiprobable scenarios, the number of inequalities is equal to the number of scenarios. FSD can be verified even easier for some other distributions. (normal, uniform, log-normal, exponential...)

First order stochastic dominance (FSD) - discrete distribution

- Let X be a matrix of scenarios of asset returns. Then $X\boldsymbol{\lambda}$ are returns of portfolio $\boldsymbol{\lambda}$ and $X\boldsymbol{\tau}$ of portfolio $\boldsymbol{\tau}$
- Let $a_1 \leq a_2 \leq ... \leq a_N$ be the returns of portfolio $\boldsymbol{\lambda}$ and $b_1 \leq b_2 \leq ... \leq b_N$ be the returns of portfolio $\boldsymbol{\tau}$. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $a_i \geq b_i$, $i = 1, ..., N$.
- equivalently $X\boldsymbol{\lambda} \geq PX\boldsymbol{\tau}$ for at least one permutation matrix P, that is, binary matrix with all row sums and all column sums equal 1, under assumption of equiprobable scenarios.

First order stochastic dominance (FSD) - continuous distributions

- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a gaussian (normal) distribution $N(\mu_{\boldsymbol{\lambda}}, \sigma_{\boldsymbol{\lambda}})$, $N(\mu_{\boldsymbol{\tau}}, \sigma_{\boldsymbol{\tau}})$, respecively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $\mu_{\boldsymbol{\lambda}} \geq \mu_{\boldsymbol{\tau}}$ and $\sigma_{\boldsymbol{\lambda}} = \sigma_{\boldsymbol{\tau}}$
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a uniform distribution on interval $\langle a_{\boldsymbol{\lambda}}, b_{\boldsymbol{\lambda}} \rangle$, $\langle a_{\boldsymbol{\tau}}, b_{\boldsymbol{\tau}} \rangle$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $a_{\boldsymbol{\lambda}} \geq a_{\boldsymbol{\tau}}$ and $b_{\boldsymbol{\lambda}} \geq b_{\boldsymbol{\tau}}$.
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have an exponential distribution with mean value $m_{\boldsymbol{\lambda}}$, $m_{\boldsymbol{\tau}}$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{FSD} \mathbf{r}'\boldsymbol{\tau}$ iff $m_{\boldsymbol{\lambda}} \geq m_{\boldsymbol{\tau}}$.

SSD portfolio analysis

- Kuosmanen (2001, 2004) and Dencheva and Ruszczynski (2003, 2004) formulate SSD optimization as a large LP problem
- Subsequent OR studies analyze various generalizations, problem reductions and algorithms; Fabian, Mitra, Roman and Zverovich (2011), Roman, Mitra, Zverovich (2013),....
- The LP problem is nowadays easy to solve for realistic data dimensions
- A recent application is Hodder, Jackwerth and Kolokolova (2015)
- SSD optimization seems superior to FSD optimization and MV optimization

Second order stochastic dominance – definitions

Let $F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$. The twice cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$ is defined as

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(y) = \int_{-\infty}^{y} F_{\mathbf{r}'\boldsymbol{\lambda}}(x)dx. \quad (1)$$

Definition

Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the second-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$) if and only if

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \leq F^{(2)}_{\mathbf{r}'\boldsymbol{\tau}}(y) \qquad \forall y \in \mathbb{R}.$$

Second order stochastic dominance – interpretation

Other equivalent definitions of SSD relation: $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if

- $Eu(\mathbf{r}'\boldsymbol{\lambda}) \geq Eu(\mathbf{r}'\boldsymbol{\tau})$ for all concave utility functions.
- No non-satiable and risk averse decision maker prefers portfolio $\boldsymbol{\tau}$ to portfolio $\boldsymbol{\lambda}$.
- $F^{-2}_{\mathbf{r}'\boldsymbol{\lambda}}(y) \leq F^{-2}_{\mathbf{r}'\boldsymbol{\tau}}(y) \quad \forall y \in [0,1]$, where $F^{-2}_{\mathbf{r}'\boldsymbol{\lambda}}$ is a cumulated quantile function.
- $\mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) \leq \mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\tau}) \quad \forall \alpha \in [0,1]$, where

$$\begin{aligned}
\mathrm{CVaR}_\alpha(-\mathbf{r}'\boldsymbol{\lambda}) = \min_{v\in\mathbb{R}, z_t\in\mathbb{R}^+} \quad & v + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T} z_t \\
\text{s.t.} \quad & z_t \geq -\mathbf{x}^t\boldsymbol{\lambda} - v, \quad t = 1,2,\ldots
\end{aligned}$$

In general, also SSD relation is expressed by infinitely many inequalities. However, again, one can simplify it for particular distributions (discrete, normal, uniform, exponential, log-normal,...)

Second order stochastic dominance (SSD) - discrete distribution

- Let X be a matrix of scenarios of asset returns. Then $X\boldsymbol{\lambda}$ are returns of portfolio $\boldsymbol{\lambda}$ and $X\boldsymbol{\tau}$ of portfolio $\boldsymbol{\tau}$
- Let $a_1 \leq a_2 \leq \ldots \leq a_N$ be the returns of portfolio $\boldsymbol{\lambda}$ and $b_1 \leq b_2 \leq \ldots \leq b_N$ be the returns of portfolio $\boldsymbol{\tau}$. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $\sum_{j=1}^{i} a_j \geq \sum_{j=1}^{i} b_j,\ i = 1, \ldots, N.$
- equivalently $X\boldsymbol{\lambda} \geq WX\boldsymbol{\tau}$ for at least one double stochastic matrix W, that is, non-negative matrix with all row sums and all column sums equal 1, under assumption of equiprobable scenarios.

Second order stochastic dominance (SSD) - continuous distributions

- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a gaussian (normal) distribution $N(\mu_{\boldsymbol{\lambda}}, \sigma_{\boldsymbol{\lambda}})$, $N(\mu_{\boldsymbol{\tau}}, \sigma_{\boldsymbol{\tau}})$, respecively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $\mu_{\boldsymbol{\lambda}} \geq \mu_{\boldsymbol{\tau}}$ and $\sigma_{\boldsymbol{\lambda}} \leq \sigma_{\boldsymbol{\tau}}$
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have a uniform distribution on interval $\langle a_{\boldsymbol{\lambda}}, b_{\boldsymbol{\lambda}}\rangle$, $\langle a_{\boldsymbol{\tau}}, b_{\boldsymbol{\tau}}\rangle$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $a_{\boldsymbol{\lambda}} \geq a_{\boldsymbol{\tau}}$ and $a_{\boldsymbol{\lambda}} - a_{\boldsymbol{\tau}} \geq -b_{\boldsymbol{\lambda}} + b_{\boldsymbol{\tau}}$.
- Assume that returns of portfolio $\boldsymbol{\lambda}$, $\boldsymbol{\tau}$ have an exponential distribution with mean value $m_{\boldsymbol{\lambda}}$, $m_{\boldsymbol{\tau}}$, respectively. Then $\mathbf{r}'\boldsymbol{\lambda} \succeq_{SSD} \mathbf{r}'\boldsymbol{\tau}$ iff $m_{\boldsymbol{\lambda}} \geq m_{\boldsymbol{\tau}}$.

TSD portfolio analysis

Thierry Post and Miloš Kopa, Charles University, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8

- 1970s literature on algorithms for pairwise analysis
- Bawa c.s. (1985) develop an LP problem for comparison with a discrete choice set
- Gotoh and Konno (2000) develop a mean-risk model
- Post and Versijp (2007) develop a GMM test to detect incremental improvement possibilities
- Armbruster and Delage (2015) develop an XXL LP problem to approximate TSD optimization
- A tractable approach for realistic data dimensions and applications do not exist

Problem definition

- Let $\mathcal{S}^2_{\boldsymbol{\lambda}}(z) := \mathbb{E}[(z - \mathbf{x}^{\mathrm{T}}\boldsymbol{\lambda})^2 \mathbb{I}(\mathbf{x}^{\mathrm{T}}\boldsymbol{\lambda} \leq z)]$ for portfolio $\boldsymbol{\lambda} \in \Lambda$ and threshold $z \in [a, b]$
- **Definition 2.2:** *Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates the benchmark $\boldsymbol{\tau} \in \Lambda$ by third-degree stochastic dominance (TSD), or $\boldsymbol{\lambda} \succeq_{TSD} \boldsymbol{\tau}$, if*

$$\mathcal{S}^2_{\boldsymbol{\lambda}}(z) \leq \mathcal{S}^2_{\boldsymbol{\tau}}(z), \forall z \in [a, b]; \tag{2}$$
$$\mathbb{E}[\mathbf{x}^{\mathrm{T}}\boldsymbol{\lambda}] \geq \mathbb{E}[\mathbf{x}^{\mathrm{T}}\boldsymbol{\tau}].$$

- Analytical challenges:
  - Infinitely many threshold levels $z \in [a, b]$ - unlike in SSD
  - Truncation at the threshold requires binary 0-1 variables

Our solution in 5 steps

1. Discrete state-dependent distribution
2. 'Super-convexity' conditions
3. Quadratic problem for $S_{\lambda}^{2}(z)$ given λ and z
4. One large convex QCP problem
5. Problem reduction by fixing the values of most 0-1 variables

Step 1: Discrete distribution

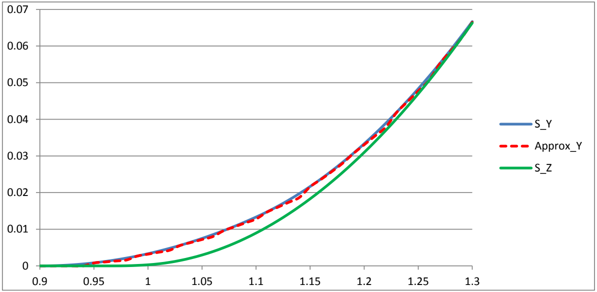
- To obtain a tractable problem of finite dimensions, we assume a discrete state-dependent distribution
- Scenarios with realizations $\boldsymbol{X}_t := (X_{1,t} \cdots X_{K,t})^{\mathrm{T}}$ and probabilities p_t: $=\mathbb{P}[\boldsymbol{x} = \boldsymbol{X}_t]$, $t = 1, \cdots, T$
- Probs can be estimated using historical freqs, GMM/GEL implies probs or Bayes posterior probs
- Flexibility to include realistic multivariate scenarios of market sell-offs and momentum crashes
- Continuous distributions can be approximated using a finite number of random draws (MC sim)
- SV becomes a non-decreasing, convex, piece-wise quadratic function: $\mathcal{S}^2_{\boldsymbol{\lambda}}(x) := \sum_{t=1}^{T} p_t (x - \boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\lambda})^2 \mathbb{I}(\boldsymbol{X}_t^{\mathrm{T}} \boldsymbol{\lambda} \leq x)$

Step 2: Super-convexity

- BBRS (1985): 'super-convex' TSD if the SV restrictions hold with sufficient slack at grid points: $(1+\varepsilon)\mathcal{S}^2_{\boldsymbol{\lambda}}(z_s) \leq \mathcal{S}^2_{\boldsymbol{\tau}}(z_s)$, $s = 1, \cdots, T$, with $\varepsilon > 0$ such that $(1+\varepsilon)\mathcal{S}^2_{\boldsymbol{\tau}}(z_s) \geq \mathcal{S}^2_{\boldsymbol{\tau}}(z_{s+1})$, $s = 1, ..., T-1,$.
- We refine this condition to $(1+\varepsilon_s)\mathcal{S}^2_{\boldsymbol{\lambda}}(z_s) \leq \mathcal{S}^2_{\boldsymbol{\tau}}(z_s)$, $s = 1, \cdots, S$, with $\varepsilon_s = f(\mathcal{S}^2_{\boldsymbol{\tau}}(z_{s-1}), \mathcal{S}^2_{\boldsymbol{\tau}}(z_s), \mathcal{E}_{\boldsymbol{\tau}}(z_{s-1}))$
- This amounts to using a piece-wise linear convex lower envelope for the SV function
- The approximation achieves machine precision for relatively rough partitions

Step 2: Super-convexity - simple example

	$s=1$	$s=2$	$s=3$	μ	σ	sk
$\mathbb{P}[s]$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
Y	0.90	1.10	1.30	1.10	0.16	0.00
Z	0.97	1.00	1.40	1.12	0.20	1.70



Step 2: Formalization

- **Definition**: *Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates the benchmark $\boldsymbol{\tau} \in \Lambda$ by super-convex third-degree stochastic dominance (SCTSD), or $\boldsymbol{\lambda} \succeq_{SCTSD} \boldsymbol{\tau}$, if*

$$\mathcal{S}^2_{\boldsymbol{\lambda}}(\mathbf{y}_s) \leq \frac{\mathcal{S}^2_{\boldsymbol{\tau}}(\mathbf{y}_s)}{(1+\varepsilon_s)}, \; s = 1, \cdots, T;$$

$$\sum_{t=1}^{T} p_t \mathbf{X}_t^{\mathrm{T}} \boldsymbol{\lambda} \geq \sum_{t=1}^{T} p_t \mathbf{y}_t.$$

- **Proposition**:*If portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by SCTSD, then $\boldsymbol{\lambda}$ also dominates $\boldsymbol{\tau}$ by TSD: $(\boldsymbol{\lambda} \succeq_{SCTSD} \boldsymbol{\tau}) \Rightarrow (\boldsymbol{\lambda} \succeq_{TSD} \boldsymbol{\tau})$.*

Step 3: Quadratic form SV

- Rockafellar and Uryasev (2000) derive an LP problem for expected shortfall of a given portfolio and threshold
- Similarly, we can formulate the restriction $(1+\varepsilon_s)\mathcal{S}^2_{\boldsymbol{\lambda}}(y_s) \leq \mathcal{S}^2_{\boldsymbol{\tau}}(y_s)$, for given $\boldsymbol{\lambda} \in \Lambda$ and $s = 1, \cdots, T$, by the following convex quadratic system:

$$(1+\varepsilon_s)\sum_{t=1}^{T} p_t \theta_t^2 \leq \mathcal{S}^2_{\boldsymbol{\tau}}(y_s); \tag{3}$$

$$\theta_t \geq y_s - \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\lambda},\ t = 1, \cdots, T; \tag{4}$$

$$\theta_t \geq 0,\ t = 1, \cdots, T. \tag{5}$$

- This formulation avoids binary variables and is linear in $\boldsymbol{\lambda}$ which appears as the RHS of linear constraints

Step 4: QCP problem

- We apply the system for every y_s, $s = 1, \cdots, T$, and endogenize the portfolio weights:

$$(1+\varepsilon_s)\sum_{t=1}^{T} p_t \theta_{s,t}^2 \leq \mathcal{S}_{\boldsymbol{\tau}}^2(y_s),\; s = 1, \cdots, T; \tag{6}$$

$$-\theta_{s,t} - \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\lambda} \leq -y_s,\;\; s, t = 1, \cdots, T;$$

$$-\sum_{t=1}^{T} p_t \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\lambda} \leq -\sum_{t=1}^{T} p_t y_t;$$

$$\mathbf{1}_K^{\mathrm{T}}\boldsymbol{\lambda} = 1;$$

$$\theta_{s,t} \geq 0,\;\; s, t = 1, \cdots, T;$$

$$\lambda_k \geq 0,\;\; k = 1, \cdots, K.$$

- Maximizing $g(\boldsymbol{\lambda}) := \mathbb{E}[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda}] - \sum_{s=1}^{T} w_s \mathcal{S}_{\boldsymbol{\lambda}}^2(y_s)$, $w_s \geq 0$, $s = 1, \cdots, T$, s.t. this system is a QCP problem

Step 5: Problem reduction

- In MV analysis, #var and #constr are $\mathcal{O}(K)$; in our case, $\mathcal{O}(T^2)$
- The large size stems from relaxation of all binary vars $\mathbb{I}(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\lambda} \leq \boldsymbol{y}_s)$, $t = 1, \cdots, T$; $s = 1, \cdots, T$
- A preliminary analysis can determine the value of most binary vars $\mathbb{I}(\boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\lambda} \leq \boldsymbol{y}_s)$, $t = 1, \cdots, T$; $s = 1, \cdots, T$
- Our optimal portfolio must be an element the polytope $\Omega := \left\{\lambda \in \Lambda : \left(\sum_{t=1}^{T} p_t \boldsymbol{X}_t^{\mathrm{T}}\boldsymbol{\lambda}\right) \geq \left(\sum_{t=1}^{T} p_t \boldsymbol{y}_t\right); \boldsymbol{X}_1^{\mathrm{T}}\boldsymbol{\lambda} \geq \boldsymbol{y}_1\right\}$
- For every scenario $s = 1, \cdots, T$, , we compute the min and max return for portfolios $\boldsymbol{\lambda} \in \Omega$
- This allows us to fix most binary vars and eliminate the corresponding vars and constrs

Step 5: Problem size

- In our application ($K = 49, T > 250$), the original problem has >62,500 vars and >62,500 constrs
- The reduced problem typically has <15,625 vars and <15,625 constrs
- We solved it on a desktop PC (Intel i7; 2.93 GHz; 16GB) with the IPOPT 3.12.3 solver in GAMS
- The median run time (using the reduction) was about two minutes
- Further reductions obtained through lessening the partition

Step 5: Problem size

	Relative error	Computer time(s)
S=1000	0%	475
S=800	0.001%	420
S=600	0.002%	280
S=400	0.004%	182
S=200	0.020%	102
S=100	0.080%	63
S=50	0.323%	28
S=25	1.165%	14
SSD	3.140%	85

Table: Numerical results using GAMS software with solver IPOPT

Motivation

- Stock price momentum was documented first by Jegadeesh and Titman (1993)
- It appears also for industries; Moskowitz and Grinblatt (1999)
- Typical momentum strategies rely on heuristics such as buy D10 and sell D01
- It seems interesting to use decision theory and optimization to improve on such heuristics
- Hodder, Jackwerth and Kolokolova (2015) use SSD enhancement
- Concentration in winner industries creates positive skewness
- To exploit skew, we apply TSD enhancement

Data

- Benchmark = CRSP all-share index
- Base assets = 49 vw industry portfolios from Ken French' library
- No concentration in individual stocks & no short positions
- Daily excess returns 1927-2014
- Same data as Hodder, Jackworth and Kolokolova (2015)
- Other data sets are work in progress (IND10, 5MEx5BtM)

Enhanced portfolios

- We compare 4 alternative enhanced portfolios:
  - Top-15 = EWA of 15 recent winner industries
  - 3 optimized portfolios maximize the mean s.t. benchmark risk restrictions:
    1. MV (variance)
    2. SSD (expected shortfall)
    3. SCTSD (semi-variance)
- Formation period = a 12-month trailing window of daily returns ($T > 250$)
- Portfolios are held for 3 months and then rebalanced (Jan - Apr - Jul - Oct)

Performance Evaluation

- We illustrate the features of the method using in-sample performance
- Out-of-sample performance is evaluated on an annual basis ($N = 87$ Jan - Dec returns)
- We focus on the raw outperformance ($X - X_{Bench}$) of annual returns
- We do not report alphas of factor models:
  - The market betas of the portfolios are smaller than 1
  - The SMB and HML loadings are limited (dynamic & diversified)
  - Even MOM explains only part of the outperformance (industry-level & no short)

Performance Evaluation

- We decompose the outperformance $(X_{SCTSD} - X_{Bench})$ in components of
  1. $(X_{Top15} - X_{Bench})$
  2. $(X_{MV} - X_{Top15})$
  3. $(X_{SSD} - X_{MV})$
  4. $(X_{SCTSD} - X_{SSD})$
- We report t-stats for statistical significance
- We report also certainty equivalents (using logarithmic utility function)

Performance Summary 1/3

	In-sample			
	Daily			Remarks
X	$\overline{(\bar{X})}$	$\overline{(s_X)}$	$\overline{(sk_X)}$	
$X_{Bench} - X_{Bond}$	0.028	0.943	-0.325	Downs correl
$X_{Top15} - X_{Bond}$	0.091	0.981	-0.434	Excess risk
$X_{MV} - X_{Bond}$	0.128	0.923	-0.253	Max Sharpe
$X_{SSD} - X_{Bond}$	0.131	0.965	-0.019	$\sigma \neq$risk
$X_{SCTSD} - X_{Bond}$	0.134	0.984	0.032	Upside pot
$X_{Top15} - X_{Bench}$	0.063	0.352	-0.059	Form&Hold
$X_{MV} - X_{Top15}$	0.038	0.553	0.150	Risk constr
$X_{SSD} - X_{MV}$	0.003	0.213	0.353	Downs risk
$X_{SCTSD} - X_{SSD}$	0.003	0.080	0.183	Skewness
$X_{SCTSD} - X_{Bench}$	0.106	0.662	0.400	Hindsight

Performance Summary 2/3

In-sample				
	Annual			Remarks
X	$\bar{X}$	$t_{\bar{X}}$	CE	
$X_{Bench} - X_{Bond}$	8.16	3.70	6.19	
$X_{Top15} - X_{Bond}$	29.17	8.41	25.90	
$X_{MV} - X_{Bond}$	41.54	15.21	39.51	
$X_{SSD} - X_{Bond}$	42.69	15.12	40.50	
$X_{SCTSD} - X_{Bond}$	43.73	15.17	41.45	
$X_{Top15} - X_{Bench}$	21.00	10.03	19.70	Form&Hold
$X_{MV} - X_{Top15}$	12.37	7.13	13.62	Risk constr
$X_{SSD} - X_{MV}$	1.15	2.22	0.99	Downs risk
$X_{SCTSD} - X_{SSD}$	1.04	6.40	0.95	Skewness
$X_{SCTSD} - X_{Bench}$	35.56	18.63	35.26	Hindsight

Performance Summary 3/3

Out-of-sample				
	Annual			Remarks
X	$\bar{X}$	$t_{\bar{X}}$	CE	
$X_{Bench} - X_{Bond}$	8.16	3.70	6.19	
$X_{Top15} - X_{Bond}$	12.66	4.84	10.10	
$X_{MV} - X_{Bond}$	14.55	6.33	12.62	
$X_{SSD} - X_{Bond}$	14.79	6.18	12.71	
$X_{SCTSD} - X_{Bond}$	14.98	6.19	12.86	
$X_{Top15} - X_{Bench}$	4.50	4.58	3.91	Form&Hold
$X_{MV} - X_{Top15}$	1.88	1.65	2.53	Risk constr
$X_{SSD} - X_{MV}$	0.24	0.42	0.09	Downs risk
$X_{SCTSD} - X_{SSD}$	0.19	0.88	0.15	Skewness
$X_{SCTSD} - X_{Bench}$	6.81	6.58	6.67	No short, Hold =

Close-up of 2013 in-sample results

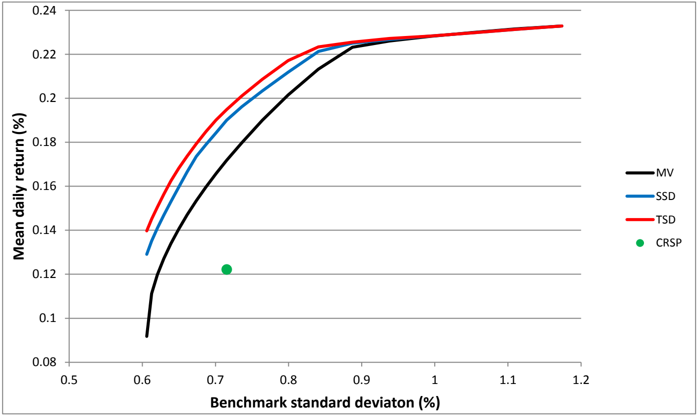
Thierry Post and Miloš Kopa, Charles University, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8

- We formed 21 benchmark portfolios:
  - Market portfolio
  - 10 convex combinations of global minimal variance portfolio and market portfolio
  - 10 convex combinations of global maximal mean portfolio and market portfolio
- We found MV, SSD and TSD solution portfolios for each benchmark
- We present the results in mean-st.dev. figure

Close-up of 2013 in-sample results

Portfolio optimization with Stochastic Dominance constraints

Thierry Post and Miloš Kopa, Charles University, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8



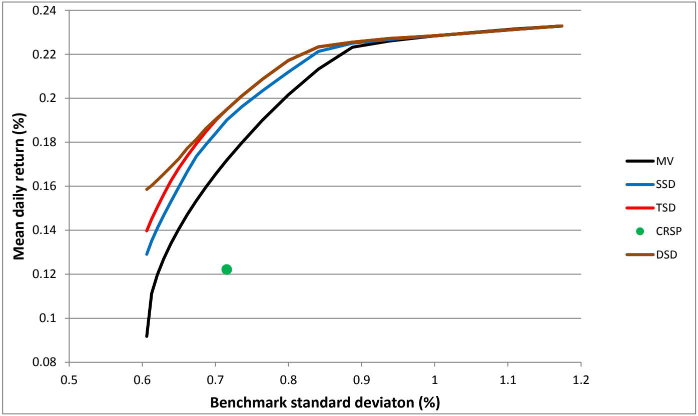
Conclusions

- Our contributions:
  1. Refinement of SCTSD
  2. QP for SV
  3. CQP for SCTSD enhancement
  4. Outperformance of Bench, Top15, MV and SSD in application
- Follow-up ideas:
  1. Better estimates:
     1. Conditioning on business cycle and market conditions
     2. GMM/GEL implied probabilities
     3. Bayesian posterior distribution
  2. More data sets (IND30, 5MEx5BtM)
  3. Consider only decreasing absolute risk aversion utility functions

DARA SD

- Arrow-Pratt coefficient of absolute risk aversion: $r(x) = -\frac{u''(x)}{u'(x)}$
- $r(x)$ is typically decreasing (non-increasing)
- A recent analysis of DARA SD in Post, Fang, Kopa (2015)
- TSD implies DARA SD
- DARA SD enhancement - mean maximization over larger set of portfolios (more portfolios dominates the benchmark)
- DARA SD outperforms TSD mainly for the benchmarks with relatively low returns

Close-up of 2013 in-sample results + DARA



Thank you

Post, T., and Kopa, M.: Portfolio Choice based on Third-degree Stochastic Dominance. Forthcoming in *Management Science*.
http://dx.doi.org/10.1287/mnsc.2016.2506

e-mail: kopa@karlin.mff.cuni.cz
www.karlin.mff.cuni.cz/ kopa