Viacnásobne použiteľné oblasti spoľahlivosti pre viacrozmernú kalibráciu

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Obsah

Predpoklady a model

$$\boldsymbol{Y_x} = \boldsymbol{B}_0 + \boldsymbol{Bx} + \epsilon, \ \epsilon \sim N_q(\mathbf{0}, \Sigma)$$

- $\boldsymbol{Y_x}$ - je $(q \times 1)$-rozmerný vektor
- $\boldsymbol{x}$ - je $(m \times 1)$-rozmerný vektor
- musí platiť $q \geq m$
- $\boldsymbol{B}_0$, $\boldsymbol{B}$, $\boldsymbol{\Sigma}$ - neznáme parametre modelu
- $\boldsymbol{B}_0$, $(q \times 1)$-rozmerný vektor (intercept)
- $\boldsymbol{B}$ je $(q \times m)$-rozmerná matica
- $\boldsymbol{\Sigma}$ je $(q \times q)$-rozmerná pozitívne definitná matica

Viacrozmerná štatistická kalibračná úloha

- $\boldsymbol{Y_x} = \boldsymbol{B}_0 + \boldsymbol{Bx} + \boldsymbol{\epsilon},\ \boldsymbol{\epsilon} \sim N_q(\boldsymbol{0}, \Sigma)$
- inferencia pre $\boldsymbol{x}$, náročné získať jej hodnotu narozdiel od získanie hodnoty $\boldsymbol{Y_x}$ pre subjekt
- konfidenčné oblasti nazývame viacnásobne použiteľné oblasti spoľahlivosti (ang. multiple use confidence regions - MUCR) alebo simultánne kalibračné oblasti
- MUCR sú skonštruované na základe jediného $\mathcal{E}_n$, z ktorého získame odhady neznámych parametrov modelu $\boldsymbol{B}_0, \boldsymbol{B}, \Sigma$
- konštrukcia konfidenčných oblastí pre $\boldsymbol{x}_{n+1}, \boldsymbol{x}_{n+2}, \dots, \boldsymbol{x}_{n+K}$ prislúchajúcich ku K budúcim pozorovaniam $\boldsymbol{Y}_{x_{n+1}}, \boldsymbol{Y}_{x_{n+2}}, \dots, \boldsymbol{Y}_{x_{n+K}}$, $K \to \infty$
- aspoň γ skonštruovaných MUCR pre postupnosť $\boldsymbol{Y}_{x_{n+1}}, \boldsymbol{Y}_{x_{n+2}}, \dots, \boldsymbol{Y}_{x_{n+K}}$ pokrýva skutočnú prislúchajúcu hodnotu vysvetľujúcej premennej $\boldsymbol{x}$ so spoľahlivosťou $1 - \alpha$

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Motivačný príklad

1. Odhad týždňa tehotenstva z meraní dĺžok 2 fetálnych kostí.

- príklad uvedený v Mathew, Sharma a Nordström (1998)
- dáta z Oman a Wax (1984)

Týždeň tehotenstva ozn. x
Dĺžky kostí ozn. $\boldsymbol{Y}_x = (l_1, l_2)^T$
Uvažovaný model v Oman a Wax (1984)

$$\boldsymbol{Y}_x \sim N(\boldsymbol{B}_0 + \boldsymbol{B}\boldsymbol{h}(x), \boldsymbol{\Sigma}),\ \text{kde } \boldsymbol{h}(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix},$$

$\boldsymbol{B}_0$ je 2×1 neznámy intercept, $\boldsymbol{B}$ je 2×2 neznáma matica koeficientov a $\boldsymbol{\Sigma}$ je neznáma 2×2 pozitívne definitná matica. Priestor možných hodnôt x je interval $[14, 41]$ (v týždňoch). Dáta v Oman a Wax (1984) pozostávajú z 1114 meraní $(l_1, l_2)^T$ u žien, kde doba x bola presne známa.

Motivačný príklad

2. Meranie maratónskeho okruhu - "Bicyklová metóda"

- príklad z Mathew a Zha (1997)
- dáta z Smith a Corbett (1987)

Dĺžka úseku ozn. x
Namerané dĺžky 13 jazdcami ozn. $\boldsymbol{Y}_x = (Y_{x,1}, Y_{x,2}, \ldots, Y_{x,13})^T$
Uvažovaný (po úprave) model v Mathew a Zha (1997)

$$\boldsymbol{Y}_x \sim N_{13}(\boldsymbol{B}x, \sigma^2 \boldsymbol{I}_{13}),$$

kde $\boldsymbol{B}$ je $(q \times 1)$ rozmerná neznáma matica a $\sigma^2 > 0$ je neznámy skalár.
Dáta Smith a Corbett (1987) pozostávajú z meraní 12 vzdialeností.

Kalibračný experiment

$$\mathcal{E}_n = \{(\boldsymbol{x}_i, \boldsymbol{Y}_i), i = 1, 2, \ldots, n\}$$

- $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ - známe m-rozmerné vektory
- $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_n$ - q-rozmerné pozorovania

V 1. kroku kalibrácie získame odhady neznámych parametrov $\boldsymbol{B}_0$, $\boldsymbol{B}$, $\boldsymbol{\Sigma}$

$$\hat{\boldsymbol{B}}_0 = \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\bar{\boldsymbol{x}}, \ \hat{\boldsymbol{B}} = \boldsymbol{Y}\boldsymbol{P}_n\boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{P}_n\boldsymbol{X}^T)^{-1}, \ \boldsymbol{P}_n = \left(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{1}_n\boldsymbol{1}_n^T\right)$$

$$\hat{\boldsymbol{\Sigma}} = \frac{\boldsymbol{A}}{n-m-1}, \ \boldsymbol{A} = \boldsymbol{Y}[\boldsymbol{I}_n - \boldsymbol{P}_n\boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{P}_n\boldsymbol{X}^T)^{-1}\boldsymbol{X}\boldsymbol{P}_n]\boldsymbol{Y}^T,$$

$$\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_n), \ \boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$$

kde $\bar{\boldsymbol{Y}} = \sum_{i=1}^n \boldsymbol{Y}_i/n$, $\bar{\boldsymbol{x}} = \sum_{i=1}^n \boldsymbol{x}_i/n$, pričom platí $n - m - 1 \geq q$ a matica $[1\boldsymbol{X}']$ má hodnosť $m + 1$.

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$$\hat{\boldsymbol{\Sigma}} = \frac{\boldsymbol{A}}{n-m-1},\ \boldsymbol{A} = \boldsymbol{Y}[\boldsymbol{I}_n - \boldsymbol{P}_n\boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{P}_n\boldsymbol{X}^T)^{-1}\boldsymbol{X}\boldsymbol{P}_n]\boldsymbol{Y}^T,$$

$$\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots, \boldsymbol{Y}_n),\ \boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n)$$

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Viacrozmerná štatistická kalibrácia

Konštrukcia konfidenčných oblastí (MUCR) pre $\boldsymbol{x}_{n+1}, \boldsymbol{x}_{n+2}, \ldots, \boldsymbol{x}_{n+K}$ prislúchajúcich ku K budúcim pozorovaniam $\boldsymbol{Y}_{\boldsymbol{x}_{n+1}}, \boldsymbol{Y}_{\boldsymbol{x}_{n+2}}, \ldots, \boldsymbol{Y}_{\boldsymbol{x}_{n+K}}$, $K \to \infty$ na základe jediného kalibračného experimentu $\mathcal{E}_n$ (raz odhadnuté parametre $\bar{\boldsymbol{Y}}, \boldsymbol{B}, \Sigma$)

Viacnásobne použiteľná oblasť spoľahlivosti má tvar

$$r_{\boldsymbol{Y}_{\boldsymbol{x}}|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}} = \{\boldsymbol{x} \in \mathcal{X} : \; T(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})\},$$

kde $T(\boldsymbol{x})$ je premenná použitá pre konštrukciu MUCR, $\lambda(\boldsymbol{x})$ je funkciou $\boldsymbol{x}$ a je určená z podmienky MUCR, $\mathcal{X}$ oblasť možných hodnôt $\boldsymbol{x}$.

MUCR

Viacnásobne použiteľné oblasti spoľahlivosti sú skonštruované na základe kalibračných dát z jedneho kalibračného experimentu $(\bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A})$ tak, aby aspoň γ časť z nich pokryla skutočnú hodnotu $\boldsymbol{x}$ prisluchajúcu k budúcemu pozorovaniu $\boldsymbol{Y}_{\boldsymbol{x}}$ so spoľahlivosťou $1-\alpha$. Funkcia $\lambda(\boldsymbol{x})$ vyhovuje rovnosti

$$P_{\bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A}}\left[\frac{1}{K}\sum_{i=1}^{K} C(\boldsymbol{x}_{n+i}; \bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A}) \geq \gamma\right] = 1-\alpha$$

pre ľubovoľnú postupnosť $\{\boldsymbol{x}_i\}$, $i = 1, 2, \ldots, K$ a

$$C(\boldsymbol{x}; \bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A}) = P_{\boldsymbol{Y}_{\boldsymbol{x}}}\{T(\boldsymbol{x}) \leq \lambda(\boldsymbol{x}) | \bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A}\}$$

označuje pokrytie MUCR, pri daných $\bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A}$.

$$\bar{\boldsymbol{Y}} \sim N(\boldsymbol{B}_0 + \boldsymbol{B}\bar{\boldsymbol{x}}, \boldsymbol{\Sigma}/n),\ \hat{\boldsymbol{B}} \sim N(\boldsymbol{B}, \{\boldsymbol{X}\boldsymbol{P}_n\boldsymbol{X}^T\}^{-1} \otimes \boldsymbol{\Sigma}),\ \boldsymbol{A} \sim W_q(\boldsymbol{\Sigma}, n-m-1)$$

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$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[\min_{\boldsymbol{x}\in\mathcal{X}} C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}) \geq \gamma\right] = 1-\alpha$$

pre ľubovoľnú postupnosť $\{\boldsymbol{x}_i\}$, $i=1,2,\ldots,K$ a

$$C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}) = P_{\boldsymbol{Y}_{\boldsymbol{x}}}\{T(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\}$$

označuje pokrytie MUCR, pri daných $\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}$.

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Približné MUCR

Pre každé $\boldsymbol{x}$ sa spočíta hodnota $\lambda(\boldsymbol{x})$ z rovnosti

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[P_{\boldsymbol{Y}_{\boldsymbol{x}}}\{T(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\} \geq \gamma\right] = 1 - \alpha.$$

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Ako zvoliť $T(\boldsymbol{x})$?

Myšlienka z Mathew, Sharma a Nordström (1998)

Ak sú $\boldsymbol{B}_0, \boldsymbol{B}, \boldsymbol{\Sigma}$ známe, odhad $\boldsymbol{x}$ pre $\boldsymbol{Y}_{\boldsymbol{x}} \sim N(\boldsymbol{B}_0 + \boldsymbol{B}\boldsymbol{x}, \boldsymbol{\Sigma})$ je

$$\widetilde{\boldsymbol{x}} = (\boldsymbol{B}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{B})^{-1} \boldsymbol{B}^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{Y}_{\boldsymbol{x}} - \boldsymbol{B}_0)$$

s kovariančnou maticou $Var(\widetilde{\boldsymbol{x}}) = (\boldsymbol{B}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{B})^{-1}$.

Vhodná premenná na konštrukciu konfidenčnej oblasti pre $\boldsymbol{x}$ je

$$T(\boldsymbol{x}) = (\widetilde{\boldsymbol{x}} - \boldsymbol{x})^T \boldsymbol{B}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{B} (\widetilde{\boldsymbol{x}} - \boldsymbol{x})$$

Myšlienka z Mathew, Sharma a Nordström (1998)

Ak sú $\boldsymbol{B}_0, \boldsymbol{B}, \Sigma$ známe, vhodný pivot na konštrukciu oblasti pre $\boldsymbol{x}$ je

$$(\widetilde{\boldsymbol{x}} - \boldsymbol{x})^T \boldsymbol{B}^T \Sigma^{-1} \boldsymbol{B} (\widetilde{\boldsymbol{x}} - \boldsymbol{x}),$$

kde $\widetilde{\boldsymbol{x}} = (\boldsymbol{B}^T \Sigma^{-1} \boldsymbol{B})^{-1} \boldsymbol{B}^T \Sigma^{-1} (\boldsymbol{Y}_{\boldsymbol{x}} - \boldsymbol{B}_0)$, $E(\widetilde{\boldsymbol{x}}) = \boldsymbol{x}$ a $Var(\widetilde{\boldsymbol{x}}) = (\boldsymbol{B}^T \Sigma^{-1} \boldsymbol{B})^{-1}$.

Ak sú $\boldsymbol{B}_0, \boldsymbol{B}, \Sigma$ neznáme, použijú sa odhady

$$\begin{aligned}
T(\boldsymbol{x}) &= \frac{n-m-q}{m} (\hat{\boldsymbol{x}} - \boldsymbol{x})^T \hat{\boldsymbol{B}}^T \boldsymbol{A}^{-1} \hat{\boldsymbol{B}} (\hat{\boldsymbol{x}} - \boldsymbol{x}) \\
&= \frac{n-m-q}{m} [\boldsymbol{Y}_{\boldsymbol{x}} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\boldsymbol{x}]^T \boldsymbol{A}^{-1} \hat{\boldsymbol{B}} (\hat{\boldsymbol{B}}^T \boldsymbol{A}^{-1} \hat{\boldsymbol{B}})^{-1} \hat{\boldsymbol{B}}^T \boldsymbol{A}^{-1} [\boldsymbol{Y}_{\boldsymbol{x}} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\boldsymbol{x}] \\
\hat{\boldsymbol{x}} &= (\hat{\boldsymbol{B}}^T \hat{\boldsymbol{A}}^{-1} \hat{\boldsymbol{B}})^{-1} \hat{\boldsymbol{B}}^T \hat{\boldsymbol{A}}^{-1} (\boldsymbol{Y}_{\boldsymbol{x}} - \hat{\boldsymbol{B}}_0)
\end{aligned}$$

MUCR - Mathew, Sharma a Nordström (1998)

Viacnásobne použiteľná oblasť spoľahlivosti má tvar

$$r_{\boldsymbol{Y_x}|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}} = \{\boldsymbol{x} \in \mathcal{X} : \ T_{MSN}(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})\},$$

kde

$$T_{MSN}(\boldsymbol{x}) = \frac{n-m-q}{m}[\boldsymbol{Y_x} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\boldsymbol{x}]^T \boldsymbol{A}^{-1}\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}^T\boldsymbol{A}^{-1}\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^T\boldsymbol{A}^{-1}[\boldsymbol{Y_x} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\boldsymbol{x}]$$

a $\lambda(\boldsymbol{x})$ pre každé $\boldsymbol{x}$ vyhovuje rovnosti

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[P_{Y_{\boldsymbol{x}}}\{T_{MSN}(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\} \geq \gamma\right] = 1-\alpha$$

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a $\lambda(\boldsymbol{x})$ pre každé $\boldsymbol{x}$ vyhovuje rovnosti

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}) \geq \gamma\right] = 1-\alpha$$

$$C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}) = P_{Y_{\boldsymbol{x}}}\{T_{MSN}(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\}$$

$$\bar{\boldsymbol{Y}} \sim N(\boldsymbol{B}_0 + \boldsymbol{B}\bar{\boldsymbol{x}}, \boldsymbol{\Sigma}/n), \ \hat{\boldsymbol{B}} \sim N(\boldsymbol{B}, \{\boldsymbol{X}\boldsymbol{P}_n\boldsymbol{X}^T\}^{-1} \otimes \boldsymbol{\Sigma}), \ \boldsymbol{A} \sim W_q(\boldsymbol{\Sigma}, n-m-1)$$

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$$T_{MSN}(\boldsymbol{x}) = \frac{n-m-q}{m}[\boldsymbol{Y_x} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\boldsymbol{x}]^T \boldsymbol{A}^{-1}\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}^T\boldsymbol{A}^{-1}\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^T\boldsymbol{A}^{-1}[\boldsymbol{Y_x} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}\boldsymbol{x}]$$

a $\lambda(\boldsymbol{x})$ pre každé $\boldsymbol{x}$ vyhovuje rovnosti

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}) \geq \gamma\right] \geq P_{\boldsymbol{\nu},\boldsymbol{v}}\left[C^*(\boldsymbol{x};\boldsymbol{\nu},\boldsymbol{v}) \geq \gamma\right] = 1-\alpha$$

$$C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}) = P_{Y_{\boldsymbol{x}}}\{T_{MSN}(\boldsymbol{x}) \leq \lambda(\boldsymbol{x})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\} \geq C^*(\boldsymbol{x};\boldsymbol{\nu},\boldsymbol{v})$$

Rozdelenie premenných nezávisí od neznámych parametrov.

Podmienka MUCR a podmienka tolerančnej oblasti

Hodnotu $\lambda(\boldsymbol{x})$ sa počíta v každom $\boldsymbol{x}$ ako

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A})\geq\gamma\right]=1-\alpha,$$

kde

$$C(\boldsymbol{x};\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A})=P_{Y_{\boldsymbol{x}}}\{T(\boldsymbol{x})\leq\lambda(\boldsymbol{x})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\}$$

Premenná pre tolerančnú oblasť

$$T_{TO}(\boldsymbol{x})=(n-m-1)[\boldsymbol{Y}_{\boldsymbol{x}}-\bar{\boldsymbol{Y}}-\hat{\boldsymbol{B}}(\boldsymbol{x}-\bar{\boldsymbol{x}})]^T\boldsymbol{A}^{-1}[\boldsymbol{Y}_{\boldsymbol{x}}-\bar{\boldsymbol{Y}}-\hat{\boldsymbol{B}}(\boldsymbol{x}-\bar{\boldsymbol{x}})]$$

Tolerančná oblasť

Označme

$$\begin{aligned} \boldsymbol{Z} &= \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{Y}_{\boldsymbol{\theta}} - \boldsymbol{B}_0 - \boldsymbol{B}\boldsymbol{\theta}) \sim N(\boldsymbol{0}, \boldsymbol{I}_q) \\ \boldsymbol{H} &= \boldsymbol{\Sigma}^{-1/2}[\bar{\boldsymbol{Y}} - \boldsymbol{B}_0 - \boldsymbol{B}\bar{\boldsymbol{x}} - (\hat{\boldsymbol{B}} - \boldsymbol{B})(\boldsymbol{x} - \bar{x})] \sim N(\boldsymbol{0}, d^2(\boldsymbol{x})\boldsymbol{I}_q) \\ d^2(\boldsymbol{x}) &= 1/n + (\boldsymbol{x} - \bar{\boldsymbol{x}})^T\{\boldsymbol{X}\boldsymbol{P}_n\boldsymbol{X}^T\}^{-1}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \\ \boldsymbol{V} &= \boldsymbol{\Sigma}^{-1/2}\boldsymbol{A}\boldsymbol{\Sigma}^{-1/2} \sim W_q(\boldsymbol{I}_q, n - m - 1). \end{aligned}$$

potom $\lambda(\boldsymbol{x})$ pre každé $\boldsymbol{x}$ spĺňa rovnosť

$$P_{\boldsymbol{H},\boldsymbol{V}}\left[P_{\boldsymbol{Z}}\left\{(n-m-1)(\boldsymbol{Z}-\boldsymbol{H})^T\boldsymbol{V}^{-1}(\boldsymbol{Z}-\boldsymbol{H}) \leq \lambda(\boldsymbol{x})|\boldsymbol{H},\boldsymbol{V}\right\} \geq \gamma\right] = 1-\alpha$$

MUCR z tolerančnej oblasti

$$r_{\boldsymbol{Y}_{\boldsymbol{x}}|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}} = \{\boldsymbol{x} \in \mathcal{X} : (n-m-1)[\boldsymbol{Y}_{\boldsymbol{x}} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}(\boldsymbol{x}-\bar{\boldsymbol{x}})]^T\boldsymbol{A}^{-1}[\boldsymbol{Y}_{\boldsymbol{x}} - \bar{\boldsymbol{Y}} - \hat{\boldsymbol{B}}(\boldsymbol{x}-\bar{\boldsymbol{x}})] \leq \lambda(\boldsymbol{x})\}$$

Odhad spoľahlivosti približných MUCR

Pre MUCR $\lambda(.)$ vyhovuje rovnosti

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[\frac{1}{K}\sum_{i=1}^{K}P_{\boldsymbol{Y}_{\mathbf{x}_{n+i}}}\{T(\boldsymbol{x}_{n+i})\leq\lambda(\boldsymbol{x}_{n+i})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\}\geq\gamma\right]=1-\alpha$$

pre ľubovoľnú postupnosť $\{\boldsymbol{x}_{n+i}\}$, $i=1,2,\ldots,K$.

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Pre MUCR $\lambda(.)$ vyhovuje rovnosti

$$P_{\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}}\left[\frac{1}{K}\sum_{i=1}^{K} P_{\boldsymbol{Y}_{\boldsymbol{x}_{n+i}}}\{T(\boldsymbol{x}_{n+i}) \leq \lambda(\boldsymbol{x}_{n+i})|\bar{\boldsymbol{Y}},\hat{\boldsymbol{B}},\boldsymbol{A}\} \geq \gamma\right] = 1-\alpha$$

pre ľubovoľnú postupnosť $\{\boldsymbol{x}_{n+i}\}$, $i = 1, 2, \ldots, K$.

- 100 000 simulácii $\bar{\boldsymbol{Y}}, \hat{\boldsymbol{B}}, \boldsymbol{A}$
- $x_{n+i} \sim Ro(\mathcal{X})$, $i = 1, 2, \ldots, 10\,000$, $\mathcal{X}$ množina možných hodnôt $\boldsymbol{x}$
- pre každé x_{n+i} 10 000 simulácii Y_{n+i}

Odhad spoľahlivosti približných MUCR

Model $$\boldsymbol{Y}_x \sim N(\boldsymbol{B}x, \sigma^2 \boldsymbol{I}_q)$$

$\boldsymbol{Y}_x$ je $(q \times 1)$-rozmerné pozorovanie
x je skalár $(m = 1)$

$$\hat{\boldsymbol{B}} = \boldsymbol{Y}\boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{X}^T)^{-1} \sim N(\boldsymbol{B}, \{\boldsymbol{X}\boldsymbol{X}^T\}^{-1} \otimes \sigma^2 \boldsymbol{I})$$
$$s^2 = tr\{\boldsymbol{Y}[\boldsymbol{I}_n - \boldsymbol{X}^T(\boldsymbol{X}\boldsymbol{X}^T)^{-1}\boldsymbol{X}]\boldsymbol{Y}^T\},\ s^2/\sigma^2 \sim \chi^2_{q(n-m)}$$
$$\boldsymbol{X} = (x_1, x_2, \ldots, x_n),\ \boldsymbol{Y} = (\boldsymbol{Y}_{x_1}, \boldsymbol{Y}_{x_2}, \ldots, \boldsymbol{Y}_{x_n})$$

$$\boldsymbol{X}\boldsymbol{X}^T = \sum x_i^2 = s_x^2$$
$$\boldsymbol{B}_1 = \hat{\boldsymbol{B}} s_x \sim N(\boldsymbol{B} s_x, \sigma^2 \boldsymbol{I}_q),\ \theta_1 = x/s_x,\ \boldsymbol{Y}_{\theta_1} \sim N(\boldsymbol{B}_1\theta_1, \sigma^2 \boldsymbol{I}_q)$$
$$\lambda(\boldsymbol{x}) \to \lambda(\theta_1^2),\ \theta_1^2 = x^2/s_x^2$$

Odhad spoľahlivosti približných MUCR

$n = \{10, 30, 50\}$, $q = \{3, 5\}$
Vypočítali sme 136 hodnôt $\lambda(\theta_1^2)$ pre $\boldsymbol{\theta}_1^2 \in \{0, 0.01, 0.02, \ldots, 1.35\}$
a zafitovali $\lambda(\theta_1^2)$ na $\mathcal{X} = [0, \tau]$, kde $\tau = \{0.05, 0.15, 0.45, 0.75, 1.05, 1.35\}$.

Odhad spoľahlivosti je založený na 100 000 simuláciach:
$\boldsymbol{B}_1 \sim N(c\mathbf{1}_q^T, \sigma^2)$, kde $c = \{0.1, 1\}$ a $s^2/\sigma^2 \sim \chi^2_{q(n-m)}$.

Mathew, Sharma a Nordström (1998)

$$T_{MSN}(\theta_1) = \frac{q(n-m)}{m} \frac{[\boldsymbol{Y}_{\theta_1} - \boldsymbol{B}_1\theta_1]^T \boldsymbol{B}_1(\boldsymbol{B}_1^T\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1^T[\boldsymbol{Y}_{\theta_1} - \boldsymbol{B}_1\theta_1]}{s^2} \tag{1}$$

Tolerančná oblasť

$$T_{TO}(\theta_1) = q(n-m)\frac{[\boldsymbol{Y}_{\theta_1} - \boldsymbol{B}_1\theta_1]^T[\boldsymbol{Y}_{\theta_1} - \boldsymbol{B}_1\theta_1]}{s^2} \tag{2}$$

Výsledky simulačnej štúdie

			τ					
c	n	q	0.05	0.10	0.15	0.45	0.75	1.05
0.1	10	3	0.9489	0.9485	0.9485	0.9515	0.9517	0.951
		5	0.9537	0.9549	0.9544	0.9545	0.9536	0.9524
	30	3	0.9512	0.953	0.9536	0.952	0.9523	0.9509
		5	0.9557	0.9556	0.9543	0.9556	0.9554	0.9551
	50	3	0.9529	0.9559	0.9554	0.9554	0.9534	0.9533
		5	0.952	0.9534	0.9523	0.95	0.9503	0.9486
1	10	3	0.9571	0.9625	0.9656	0.9748	0.9752	0.9744
		5	0.9683	0.9773	0.9795	0.9831	0.9838	0.9833
	30	3	0.9648	0.972	0.9755	0.9776	0.9762	0.9765
		5	0.9797	0.9845	0.9848	0.9865	0.9866	0.9862
	50	3	0.9682	0.9749	0.9766	0.9773	0.9775	0.9775
		5	0.9796	0.9842	0.9843	0.9843	0.9842	0.9845

Odhadnutá spoľahlivosť približných MUCR skonštruovaných na základe Mathew, Sharma a Nordstrom oblasti, požadovaná úroveň spoľahlivosti $1 - \alpha = 0.95$.

Výsledky simulačnej štúdie

			τ					
c	n	q	0.05	0.10	0.15	0.45	0.75	1.05
0.1	10	3	0.9486	0.9492	0.9496	0.948	0.9484	0.9492
		5	0.9508	0.9511	0.9512	0.9527	0.9533	0.952
	30	3	0.9494	0.9515	0.9513	0.9551	0.9535	0.9527
		5	0.9501	0.9538	0.9539	0.954	0.9555	0.9543
	50	3	0.9536	0.9545	0.9557	0.9569	0.9561	0.9549
		5	0.9501	0.9509	0.9521	0.9546	0.9535	0.951
1	10	3	0.9482	0.949	0.9494	0.9489	0.9499	0.9498
		5	0.9514	0.9506	0.951	0.951	0.9522	0.9521
	30	3	0.9483	0.9508	0.9522	0.9522	0.9531	0.9531
		5	0.95	0.9497	0.9529	0.9549	0.9538	0.9535
	50	3	0.9543	0.9553	0.9549	0.956	0.9541	0.9532
		5	0.9488	0.9526	0.952	0.953	0.9527	0.9503

Odhadnutá spoľahlivosť približných MUCR skonštruovaných na základe toleranačných oblastí, požadovaná úroveň spoľahlivosti

$$1 - \alpha = 0.95.$$

Meranie maratónskeho okruhu - “Bicycle method”

- príklad Mathew a Zha (1997)
- dáta Smith a Corbett (1987)
- 12 dĺžok $x_1, x_2, \ldots, x_{12}$
- každá dĺžka bola meraná 13 cyklistami

Model $\quad \boldsymbol{Y}_x \sim N(\boldsymbol{B}_1\theta_1, \sigma^2 \boldsymbol{I}_{13})$

$\lambda(\boldsymbol{x}) \to \lambda(\theta_1^2)$, kde $\theta_1 = x/s_x$

Hodnota $\lambda(\theta_1^2)$ spočítaná pre $\theta_1^2 \in \{0.05, 0.06, \ldots, 0.15\}$ a $\lambda(\theta_1^2)$ zafitovaná na $\mathcal{X} = [0.05, 0.15]$

x	MSN - MUCR	TO - MUCR
601.258	$[600.980, 601.968]$	$[600.582, 602.367]$
1000	$[998.334, 999.757]$	$[997.926, 1000.166]$

Ďakujem za pozornosť!

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