Stochastické optimalizační schéma s hodinkami

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Basic setup and notation

We consider an oriented graph $\mathrm{G} = (V, H)$, where V is a set of nodes, H is a set of oriented edges, and, A is a set of possible actions. Each node $v \in V$ is equipped with a random time $\tau_v : \Omega \to [0, +\infty)$ and with a non-empty set of allowed actions $\mathrm{A}_v \subset \mathrm{A}$.
V, A are finite sets, consequently, H, A_v for $v \in V$ are finite sets, also.

We denote an edge leading from node v to node w by $\overrightarrow{v\,w}$.
For a given node $v \in V$, we will employ a set of all its parents
$\partial_- (v) = \left\{w \in V \;:\; \overrightarrow{w\,v} \in H\right\}$ and its children
$\partial_+ (v) = \left\{w \in V \;:\; \overrightarrow{v\,w} \in H\right\}$.

Basic setup and notation

Lemma

Any oriented graph $\mathsf{G} = (V, H)$ without any oriented circle can be equivalently described as a partial ordering on the set of nodes $(V, \leq_{\mathsf{G}})$. For $v, w \in V$, the ordering is defined by $v \leq_{\mathsf{G}} w$ iff there is a path $u_0, u_1, \ldots, u_k \in V$ for some $k \in \mathbb{N}_0$ such that $\overrightarrow{u_0\, u_1}, \overrightarrow{u_1\, u_2}, \ldots, \overrightarrow{u_{k-1}\, u_k} \in H$ and $u_0 = v$, $u_k = w$.

Basic setup and notation

Using this equivalent description, we are adding a notation. For a given node $v \in V$, we will employ a set of all its ancestors $\partial_{<}(v) = \{w \in V \,:\, w \leq_{\mathsf{G}} v, w \neq v\}$ and its offspring $\partial_{>}(v) = \{w \in V \,:\, w \geq_{\mathsf{G}} v, w \neq v\}$.

Basic setup and notation

Assumptions:

- Considered oriented graph $\mathsf{G} = (V, H)$ is without any oriented circle.
- There is precisely one node $v_0 \in V$ with $\partial_- (v_0) = \emptyset$. This node is called the root of graph G.
- Any node $v \in V$ with $\partial_+ (v) = \emptyset$ is called a leaf of graph G.
- For each $v \in V$ we have $\tau_v \geq 0$.
- We have $\tau_{v_0} = 0$.
- For each $v, w \in V$, $\overrightarrow{v\,w} \in H$ we have $\tau_v < \tau_w$.

Basic setup and notation

We consider a movement on nodes V expressed as a function $\varphi : [0, +\infty) \to V$.
We say φ is non-decreasing if $\varphi(s) \leq_{\mathsf{G}} \varphi(t)$ whenever $0 \leq s \leq t$.
We say φ is right-continuous if $\varphi(t) = \varphi(t+)$ for each $0 \leq t$.
Let us denote
$\Phi = \left\{ \varphi \in V^{[0,+\infty)} : \varphi \text{ is non-decreasing and right-continuous} \right\}$.

Basic setup and notation

We control profit by a policy expressed as a right-continuous function $a : [0, +\infty) \to \mathsf{A}$.
Let us denote $\mathcal{A} = \left\{ a \in \mathsf{A}^{[0,+\infty)} \; : \; a \text{ is right-continuous} \right\}$.

Basic setup and notation

Given $\omega \in \Omega$, we call $\varphi \in \Phi$ admissible if $\varphi(0) = v_0$ and $\tau_{\varphi(t)}(\omega) \leq t$ for each $t \geq 0$.
Given $\omega \in \Omega$, $\varphi \in \Phi$, $a \in \mathcal{A}$, the couple (φ, a) is called admissible if φ is admissible, $a(t) \in \mathrm{A}_{\varphi(t)}$ for each $t \geq 0$ and if $a(t) \neq a(t-)$ then $\varphi(t) \neq \varphi(t-)$ for each $t > 0$.

Basic setup and notation

We would like to optimize a profit at a given horizon $\mathrm{T} > 0$. Thus, we have to consider a utility function $\mathrm{U} : [0, \mathrm{T}] \times V \times \mathrm{A} \to \mathbb{R}$.
Given $\omega \in \Omega$, $\varphi \in \Phi$, $a \in \mathcal{A}$, the couple (φ, a) is admissible, we are receiving a profit $\int_{[0,\mathrm{T}]} \mathrm{U}\left(t, \varphi\left(t\right), a\left(t\right)\right) \mathrm{d}t$.

Optimal solution

Our task is to find an optimal solution of the problem.
We assume a horizon $T > 0$ at which we want to optimize.

Optimal solution

Start with optimization on each time segment.
Take a node $v \in V$ and times $0 \leq s < t$. We maximize our profit

$$\mathrm{F}_{s,t}(v) = \max\left\{ \int_s^t \mathrm{U}(u, v, a)\, \mathrm{d}u \;:\; a \in \mathrm{A}_v \right\} = \int_s^t \mathrm{U}(u, v, \widehat{a}_{s,t}(v))\, \mathrm{d}u.$$

Optimal solution

Consider a movement $\varphi \in \Phi$, then φ possesses a finite number of jumps at the time interval $[0, \mathrm{T}]$, say $t_0 = 0 < t_1 < t_2 < \cdots < t_k = \mathrm{T}$, and optimal profit from the movement would be

$$\begin{aligned}
\widehat{\mathrm{F}}(\varphi) &= \sup\left\{\int_0^{\mathrm{T}} \mathrm{U}\left(u, \varphi(u), a(u)\right) \mathrm{d}u \ : \ a \in \mathcal{A}\right\} \\
&= \sup\left\{\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \mathrm{U}\left(u, \varphi(u), a(u)\right) \mathrm{d}u \ : \ a \in \mathcal{A}\right\} \\
&= \sup\left\{\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \mathrm{U}\left(u, \varphi(t_i), a(t_i)\right) \mathrm{d}u \ : \ a(t_i) \in \mathrm{A}_{\varphi(t_i)}\right\} \\
&= \sum_{i=0}^{k-1} \mathrm{F}_{t_i, t_{i+1}}\left(\varphi(t_i)\right).
\end{aligned}$$

Optimal solution

Corresponding optimal policy is

$$\widehat{a}(t) \quad = \quad \widehat{a}_{t_i,t_{i+1}}(\varphi(t_i)) \text{ if } t_i \leq t < t_{i+1}.$$

Optimal solution

The last step is to determine optimal or ε-optimal policy for $\varepsilon > 0$.

$$\widehat{\mathsf{F}} = \sup\left\{\widehat{\mathsf{F}}\left(\varphi\right) \,:\, \varphi \in \Phi \text{ admissible}\right\},$$

$$\widehat{\varphi}_\varepsilon \in \Phi \text{ admissible with } \widehat{\mathsf{F}}\left(\widehat{\varphi}_\varepsilon\right) < \widehat{\mathsf{F}} + \varepsilon.$$

Thanks for your attention !