FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Tomáš Rusý

Department of Probability and Mathematical Statistics

An Asset–Liability Management Stochastic Program of a Leasing Company

25th January, 2018

Outline

- Introduction
  - Motivation and the story behind
- Stochastic programming model
  - Business model of a leasing company
  - Stochastic programming formulation
  - Scenario generation
  - Risk management, constraints & interpretation
- Results
  - Sensitivity analysis
  - Stress test
- Comments and questions

Introduction

We present real and practical application of stochastic programming.

The work

- aims to find the optimal solution,
- of a real–world problem,
- while allowing its flexible modifications,
- with input parameters obtained from real data.

Business Model of a Leasing Company

- It lends money to clients.
- It borrows money from a bank.
- Current practise is that the leasing company (LC) closes mirror deal with a bank as it has with a client (the same amount, the same length)
  - This means that the LC closes its position.
  - Income is generated by different rates on each loan.
- The main idea of the problem is to suggest a better strategy for borrowing from the bank, so that we control the amount of IR risk the company faces.

Stochastic Programming Formulation

- Mathematical notation for related quantities is introduced.
- The dynamics of cash–flows is specified by a set of equations.
- These depend on random quantities (demand, interest rate) as well as on decision variables.
- The objective of the model is to maximise the value of a portfolio of loans at the time horizon n.
- Symbol $V_n(\bar{\omega}_n^s)$ denotes the value of a strategy at the investment horizon n for scenario $\bar{\omega}_n^s$, while $V_n^0(\bar{\omega}_n^s)$ denotes the value of the benchmark strategy.

Structure of Scenarios

- We assumed decision times to be equidistant, with one year gap. Time horizon was chosen to be $n = 6$ years.
- Borrowing was only possible with maximum time to maturity $m = 5$ years.
- The adopted structure of the tree was $8 - 4 - 2 - 2 - 2 - 2$ nodes from every scenario in the corresponding stage.
- Scenarios of interest rate were generated from the Hull – White model as quantiles of the interest rate distribution.
- Values of charged rates by bank/LC were determined from data of a company CS Autoleasing.

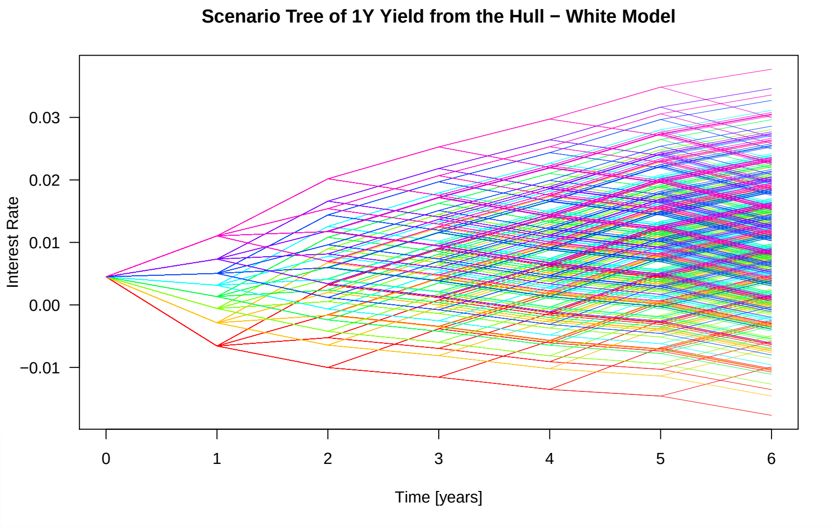


Figure: Scenario values of a one year interest rate in the Czech market in the tree structure used in the optimisation problem.

$$
\begin{aligned}
\max_{x_{i,j}(\bar\omega_i^s)} \quad & \frac{1}{|S_n|}\sum_{s\in S_n} V_n(\bar\omega_n^s) \\
\text{s.t.}\ \ R_k(\bar\omega_{k-1}^s) &= \sum_{i=[k-m]^+}^{k-1}\ \sum_{j=k-i}^{m} \frac{d_{i,j}(a_i(\bar\omega_{k-1}^s))}{\bar r_{i,j}^s}, \qquad 1\le k\le n, s\in S_{k-1}, \\
Q_k(\bar\omega_{k-1}^s) &= \sum_{i=[k-m]^+}^{k-1}\ \sum_{j=k-i}^{m} \frac{x_{i,j}(a_i(\bar\omega_{k-1}^s))}{\bar s_{i,j}^s}, \qquad 1\le k\le n, s\in S_{k-1}, \\
D_k(\bar\omega_k^s) &= \sum_{j=1}^{m} d_{k,j}(\bar\omega_k^s),\ \ X_k(\bar\omega_k^s) = \sum_{j=1}^{m} x_{k,j}(\bar\omega_k^s),\ \ 0\le k<n, s\in S_k, \\
B_0 &= X_0(\omega_0^s) - Q_0(\omega_0^s), \quad s\in S_0, \\
B_k(\bar\omega_k^s) &= \frac{B_{k-1}(a_{k-1}(\bar\omega_k^s))}{p_{k-1,1}(a_{k-1}(\bar\omega_k^s))} - E_{k-1} + X_k(\bar\omega_k^s) - Q_k(a_{k-1}(\bar\omega_k^s)) \\
&\quad + R_k(a_{k-1}(\bar\omega_k^s)) - D_k(\bar\omega_k^s), \quad 1\le k<n,\ \ s\in S_k, \\
B_n(\bar\omega_{n-1}^s) &= \frac{B_{n-1}(\bar\omega_{n-1}^s)}{p_{n-1,1}(\bar\omega_{n-1}^s)} - E_{n-1} + R_n(\bar\omega_{n-1}^s) - Q_n(\bar\omega_{n-1}^s),\ \ s\in S_{n-1}, \\
A_n(\bar\omega_n^s) &= \sum_{i=[n-m+1]^+}^{n-1}\ \sum_{j=n-i+1}^{m}\ \sum_{l=n-i+1}^{j} p_{n,l+i-n}(\bar\omega_n^s)\frac{d_{i,j}(a_i(\bar\omega_n^s))}{\bar r_{i,j}^s}, \quad s\in S_n, \\
L_n(\bar\omega_n^s) &= \sum_{i=[n-m+1]^+}^{n-1}\ \sum_{j=n-i+1}^{m}\ \sum_{l=n-i+1}^{j} p_{n,l+i-n}(\bar\omega_n^s)\frac{x_{i,j}(a_i(\bar\omega_n^s))}{\bar s_{i,j}^s}, \quad s\in S_n, \\
V_n(\bar\omega_n^s) &= A_n(\bar\omega_n^s) - L_n(\bar\omega_n^s) + B_n(a_{n-1}(\bar\omega_n^s)), \quad s\in S_n, \\
B_k(\bar\omega_k^s) &\ge 0, \quad 0\le k<n,\ \ s\in S_k, \qquad x_{i,j}^s \ge 0, \quad 0\le i\le j,\ \ s\in S_i.
\end{aligned}
$$

Risk Management

To control for the risk faced by opening the company's interest rate position, the following approaches were considered:

- Chance constraint — forces the solution to beat the benchmark with a probability $1-\alpha, \alpha \in \langle 0, 1 \rangle$.
- VaR constraint — restricts the $1-\alpha$ quantile of the portfolio value to be greater than or equal to a given limit $-u_\alpha$.
- CVaR constraint — sets the average of $1-\alpha \cdot 100\%$ of the worst values to be greater than or equal to a given limit $-v_\alpha$.
- SSD constraint — forces the optimal strategy to dominate the benchmark strategy by a second order stochastic dominance.

Results

- First, the model with no risk constraint was solved.
- The problem had 7008 variables and 4563 equations.
- The optimal strategy performed better than the benchmark in around 81% of cases.
- At 0.95 level, the VaR of the benchmark was -262.32 mil. CZK, while of the optimal strategy it was -269.1 mil. CZK.
- CVaR — the benchmark: -254.88 mil. CZK, the optimal solution: -256.71 mil. CZK.
- The optimal portfolio dominated the benchmark portfolio by SSD.

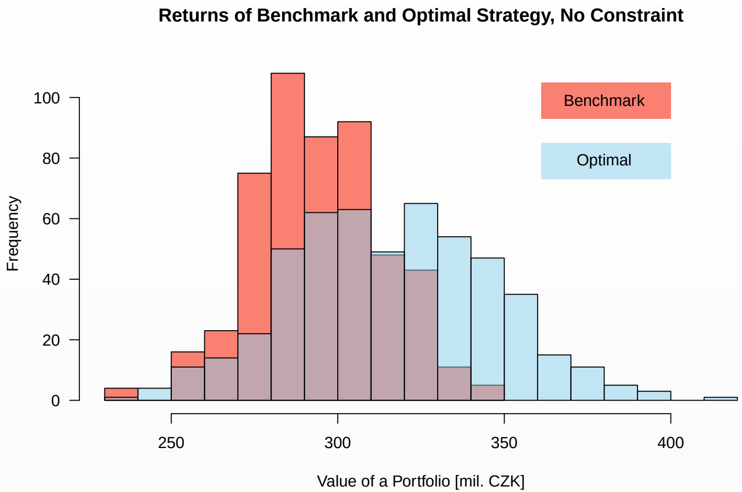


Figure: Comparison of portfolio values of the benchmark and the no–risk constraint optimal strategy.

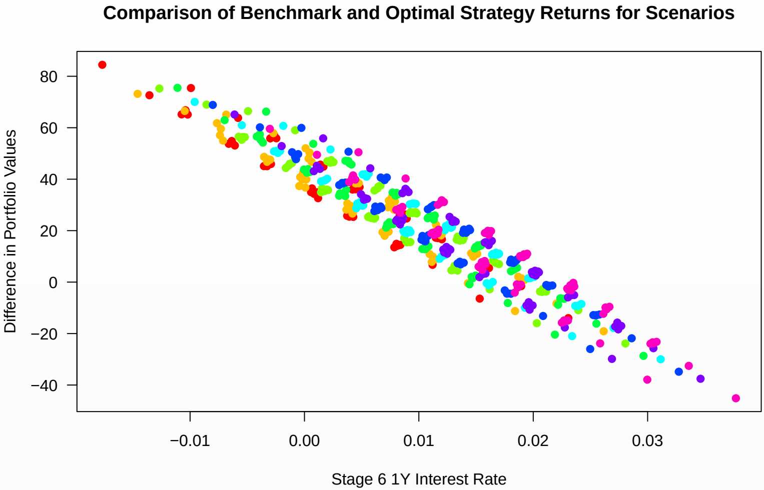


Figure: Differences: the optimal portfolio value - the benchmark portfolio value against the one year interest rate in the final stage.

CVaR Constraint

- We set limit on the 0.95 conditional Value–at–Risk of the optimal strategy:
$$\mathrm{CVaR}_\alpha(-V_{t_n}) \le v_\alpha, \qquad \alpha = 0.95.$$

- This could be expressed as
$$z^s \ge -V_n(\bar{\omega}_n^s) - a, \quad z^s \ge 0, \quad s \in S_n,$$
$$a + \frac{1}{1-\alpha}\frac{1}{|S_n|}\sum_{s\in S_n} z^s \le v_\alpha, \quad a \in \mathbb{R}.$$

- The highest reasonable CVaR limit v_α is equal to -256 mil. CZK. The lowest feasible limit was found to be -277.25 mil. CZK.

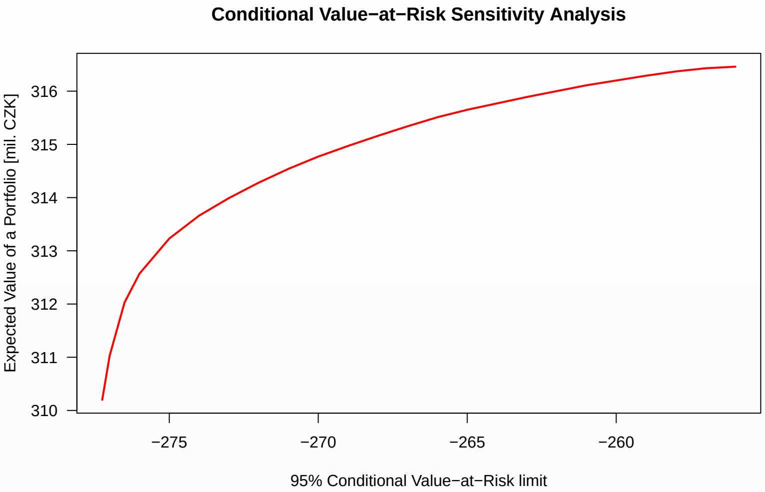
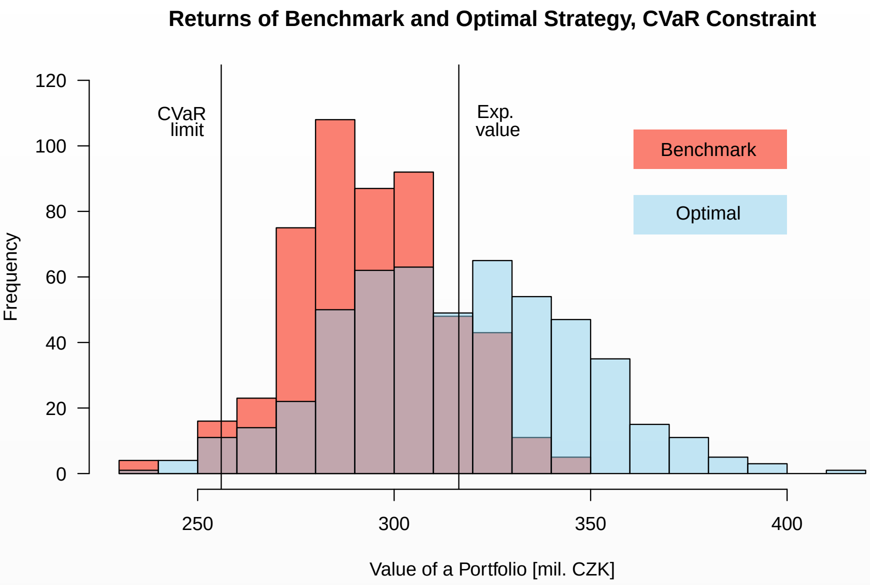


Figure: Dependence of the optimal expected value on the value of the limit v_α in the conditional Value–at–Risk constraint.


Returns of Benchmark and Optimal Strategy, CVaR Constraint
CVaR limit
Exp. value
Benchmark
Optimal
Frequency
0
20
40
60
80
100
120
250
300
350
400
Value of a Portfolio [mil. CZK]

Comparison of Optimal Strategies

- Multiple risk constraints were analysed, but we wish to know what are their suggestions for "today's" decision
- Risk limits were chosen in a way that the corresponding risk measure ranks "just better" the optimal portfolio.

	Benchmark	Opt.	C. C.	VaR	CVaR	SSD
$\mathbb{E}V$	294.47	316.46	308.84	316.46	316.46	316.46
$x_{0,1}$	294.97	1552.2	1042.7	1552.2	1552.2	1552.2
$x_{0,2}$	330.50	0	0	0	0	0
$x_{0,3}$	314.24	0	0	0	0	0
$x_{0,4}$	290.21	0	0	0	0	0
$x_{0,5}$	322.26	0	509.5	0	0	0

Table: Table with the mean value of portfolios in the final stage and the here and now decisions of how to borrow money.

Stress Test

- One can ask what happens when things do not go as we expected (=modelled in our scenario tree).
- Opening the interest rate position can have fatal consequences in case of interest rate increase/decrease.
- Assume that we fix our here and now decision, borrow according to it, but then, a crisis comes.
- We create crisis scenarios by a rapid increase in interest rate.
- Thereafter, we readjust our strategy and learn the new expected value of the portfolio.

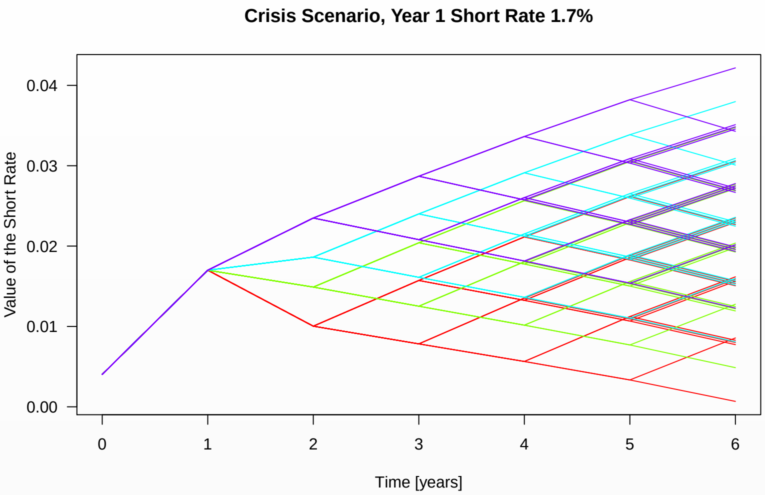


Figure: Scenario tree for stress test with the value of the short rate 1.7%

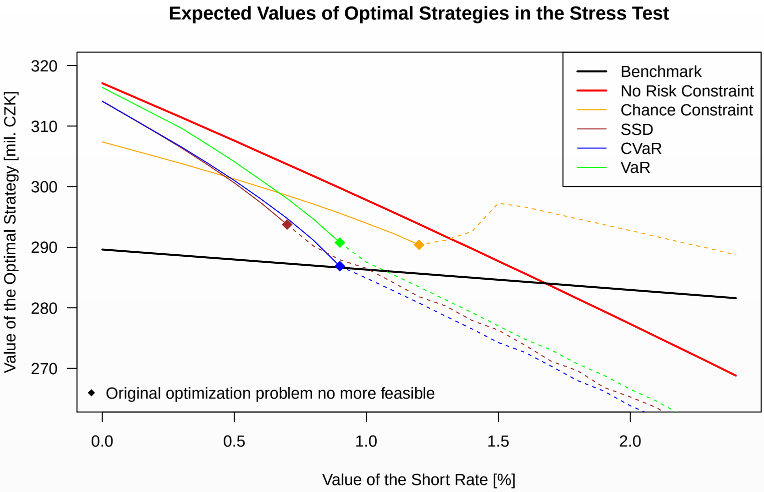


Figure: The expected value of the optimal solution for different level of stress test and various strategies. Dashed lines show optimal values of the programs after relaxing the (infeasible) risk constraints.

Conclusion

- Stochastic program within an asset–liability model was developed.
- A lot of attention has been paid to creating realistic inputs (e.g. improved calibration procedure of the Hull – White model, market data about leasing loans and rates).
- Four different risk constraints were introduced to offer a possibility to manage interest rate risk.
- Stress test was proposed to asses the effect of unconsidered scenarios on the optimal solutions.
- We found the benchmark strategy to be largely inferior to all optimal strategies in most aspects.

Thank you for your attention.