ROBUST 2018

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Robust Principal Component Analysis

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Joint work with Christian Kümmerle and Felix Krahmer

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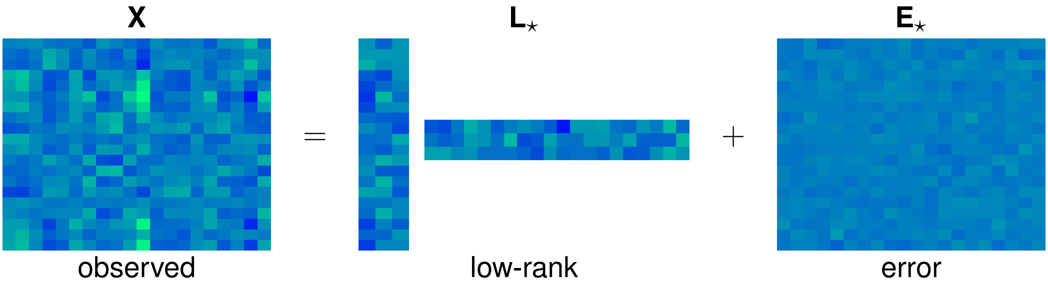
Principal Component Analysis (PCA)

Standard PCA



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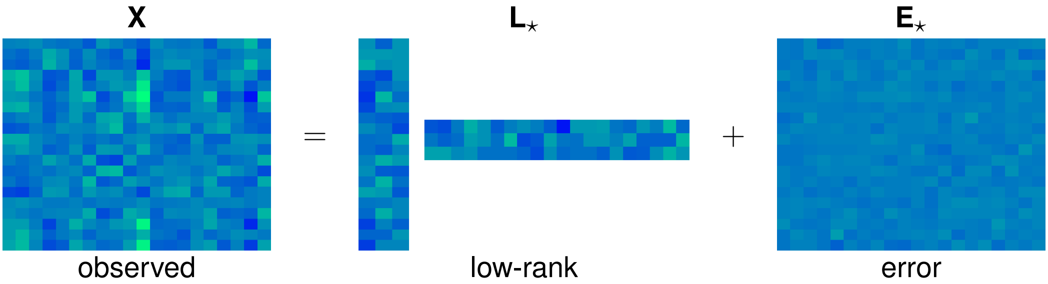
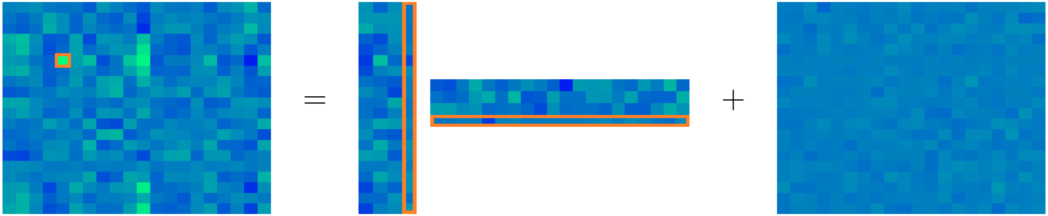


Standard PCA with outliers



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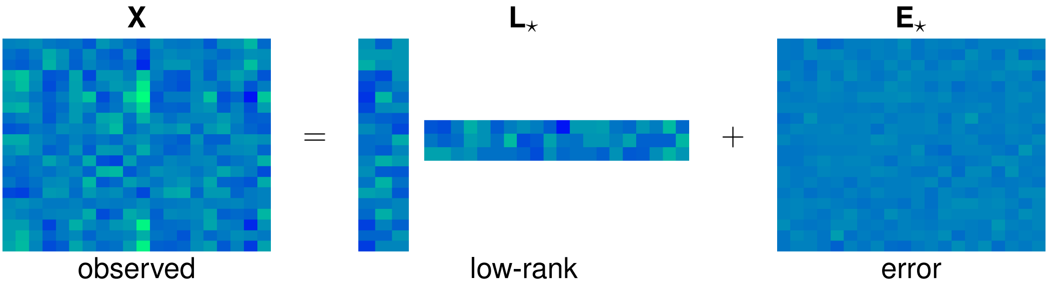

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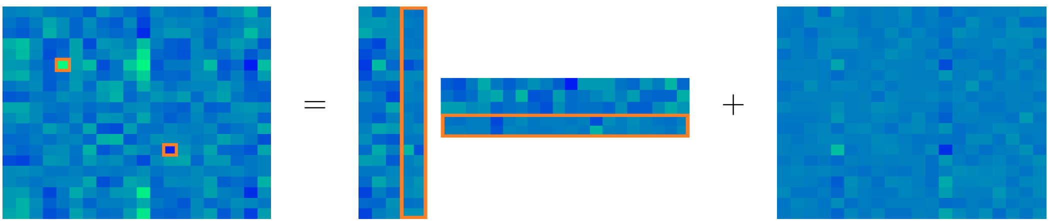

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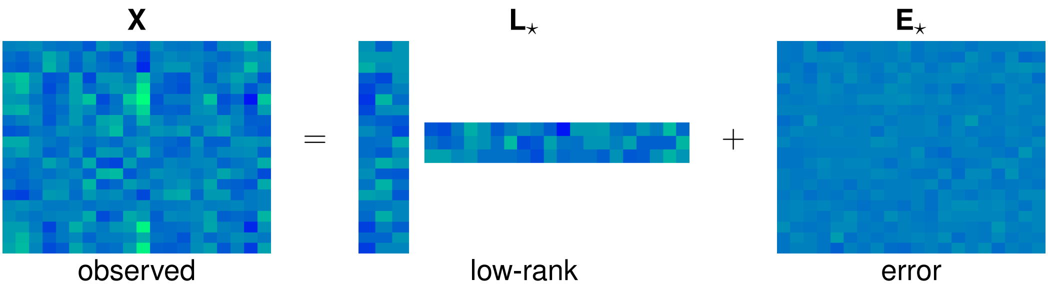

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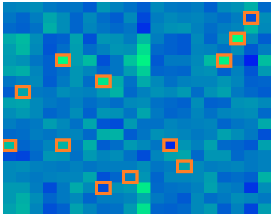
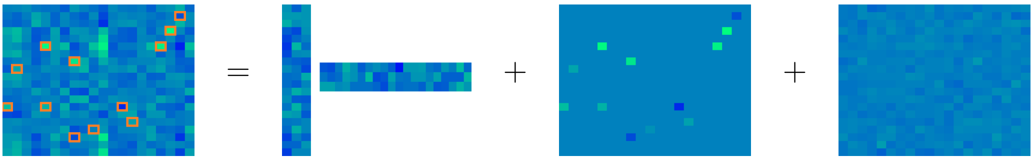


$= \quad ? \quad + \quad ?$

Robust PCA:

$\mathbf{X} = \mathbf{L}_\star + \mathbf{S}_\star + \mathbf{E}_\star$



observed = low-rank + sparse + error

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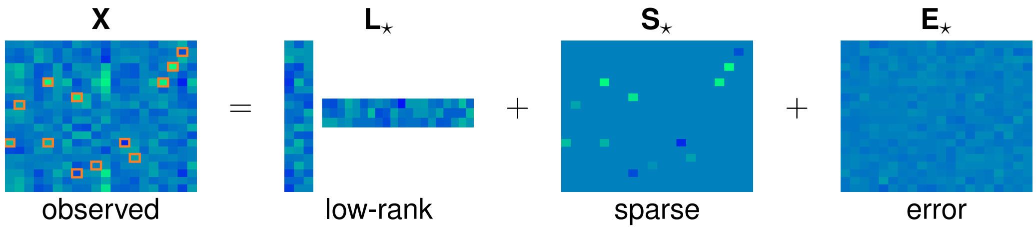
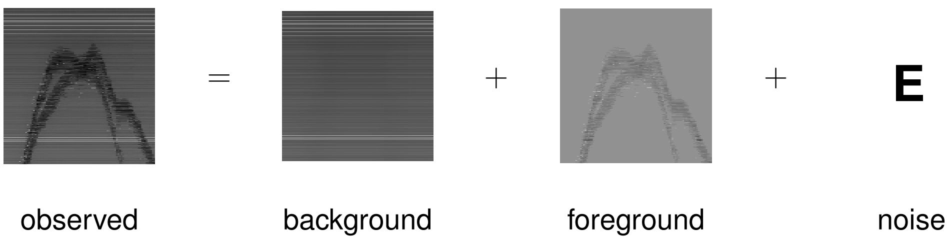


Application: video surveillance



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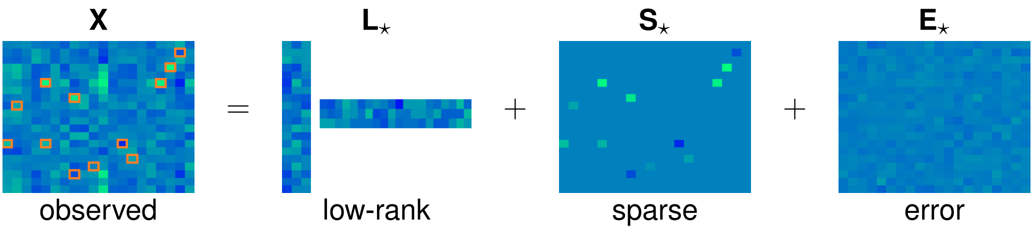

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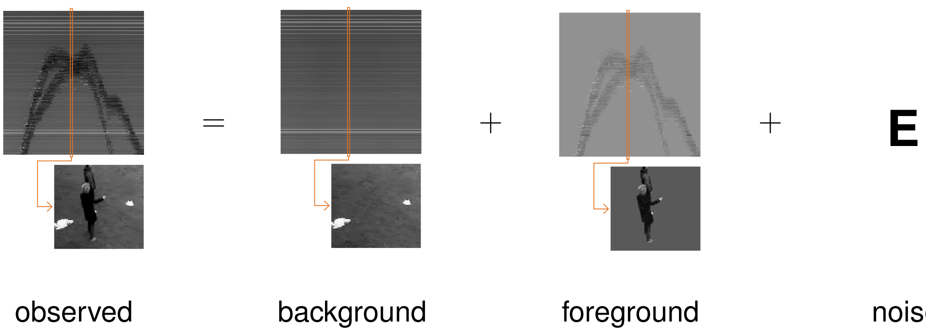

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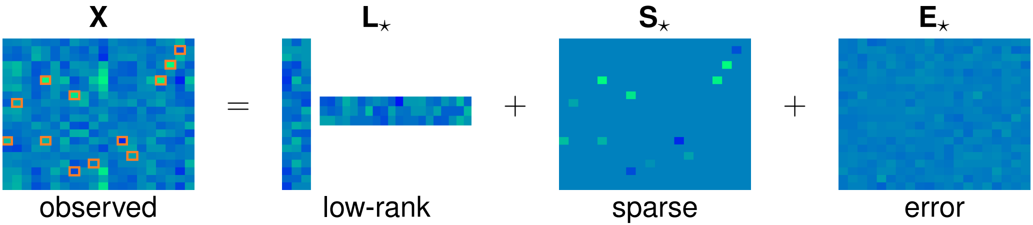

Application: video surveillance

$$= \qquad + \qquad + \quad \mathbf{E}$$

observed background foreground noise

Robust PCA

Problem: Given matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$, find a low-rank matrix $\mathbf{L}_\star$ and a sparse matrix $\mathbf{S}_\star$ such that

$$\begin{array}{rccccc} & \mathbf{X} & = & \mathbf{L}_\star & + & \mathbf{S}_\star \\ \text{df:} & n_1 n_2 & \geq & (n_1 + n_2) r & + & s \end{array}$$

where $r = \operatorname{rank} \mathbf{L}_\star$ and $s = \|(\mathbf{S}_\star)_{\text{vec}}\|_0$

Is the problem solveable?

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Natural optimization formulation:

$$\begin{aligned} (\widehat{\mathbf{L}}, \widehat{\mathbf{S}}) \quad := \quad & \underset{\mathbf{L},\mathbf{S}}{\arg\min} \quad \operatorname{rank} \mathbf{L} + \lambda \|\mathbf{S}_{\text{vec}}\|_0 \\ & \text{s.t.} \qquad \mathbf{X} = \mathbf{L} + \mathbf{S} \end{aligned}$$

It is **NP-hard**.

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It is **NP-hard**.

The seminal paper of Candés et al. (2011) showed that (given certain assumptions) the separation is possible via a convex program

$$\begin{aligned} (\widehat{\mathbf{L}}, \widehat{\mathbf{S}}) := & \operatorname*{arg\,min}_{\mathbf{L},\mathbf{S}} \quad \operatorname{rank} \|\mathbf{L}\|_* + \lambda \|\mathbf{S}_{\text{vec}}\|_1 \\ & \text{s.t.} \quad \mathbf{X} = \mathbf{L} + \mathbf{S} \end{aligned}$$

Issues with the convex approach:

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Robust PCA

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- mediocre performance (especially in practical applications, where the assumptions are typically broken)

$\Rightarrow$ new algorithms emerged in the past few years, most of them non-convex

We also propose a non-convex algorithm...

Proposed algorithm

Derivation of the algorithm in a nutshell:

1. Smooth the **non-convex objective** to become **differentiable**.
2. Express the derivative in a **special form**.
3. Solve the system of **non-linear equations** arising from the first order optimality conditions via a **fixed point scheme**.

The resulting algorithm is an instance of Iteratively Reweighted Least Squares (**IRLS**) method.

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Where is the novelty?

– special form of the derivative $\rightarrow$ a **local quadratic convergence rate**
– a competitive performance

Proposed algorithm

Our algorithm is **not**

1. fastest – the algorithm of Yi et al. (2016) attains the time complexity of the standard PCA and thus is hard to beat
2. statistically most accurate – the algorithm of Oh et al. (2016) is hard to beat
3. numerically most accurate

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1. fastest – the algorithm of Yi et al. (2016) attains the time complexity of the standard PCA and thus is hard to beat
2. statistically most accurate – the algorithm of Oh et al. (2016) is hard to beat
3. numerically most accurate – actually, **it is**

Small Comparison

Take $\mathbf{X} = \mathbf{A}\mathbf{B}^\top$ with $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{200 \times \mathbf{r}}$ and replace **s** random entries by random corruptions. Algorithm succeeds if $\|\widehat{\mathbf{L}} - \mathbf{A}\mathbf{B}^\top\|_F / \|\mathbf{A}\mathbf{B}^\top\|_F < 0.01$.

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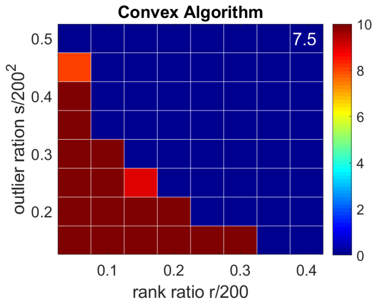

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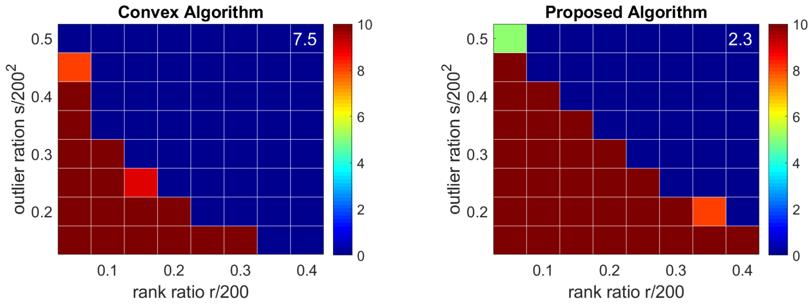

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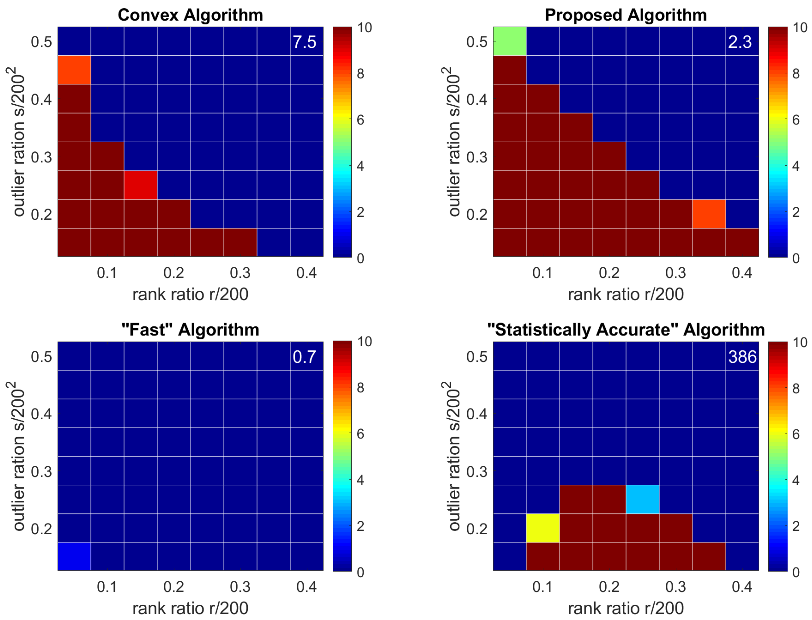

Conclusions

- A new algorithm for Robust PCA proposed.
  - the only competitive algorithm among the IRLS class
  - the only algortihm with super-linear convergence rate
  - the only algorithm uniformly outperforming the convex approach (disclaimer: subjective)
- The local quadratic convergence rate proved.

References

- Candès, E. J., Li, X., Ma, Y., & Wright, J. (2011). Robust principal component analysis?. *Journal of the ACM (JACM)*, 58(3), 11.
- Oh, T. H., Matsushita, Y., Kweon, I., & Wipf, D. (2016). A pseudo-bayesian algorithm for robust PCA. In *Advances in Neural Information Processing Systems* (pp. 1390-1398).
- Yi, X., Park, D., Chen, Y., & Caramanis, C. (2016). Fast algorithms for robust PCA via gradient descent. In *Advances in neural information processing systems* (pp. 4152-4160).