Linear Filtering of general Gaussian processes

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Stochastic differential equation example - Vasicek model

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- $\mathrm{d}X_t = \lambda(\mu - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad X_0 = x, \quad \lambda, \sigma > 0$

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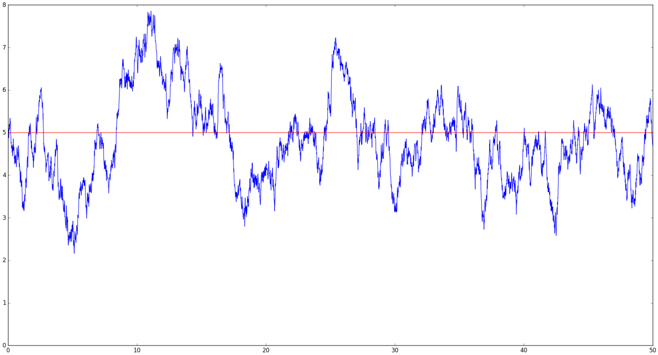

Vasicek model with parameters: $\mu = 5$, $\lambda = 0.5$, $\sigma = 1$ and $x = 20$

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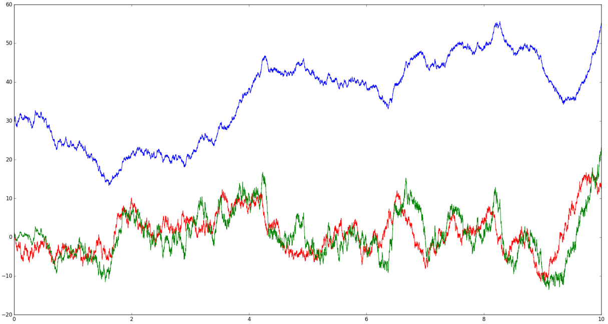
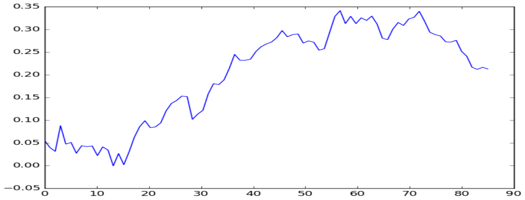


Blue is the observation process Y, red is its drift process X (the signal) and green is the estimate of the drift process (the filter). Parameters are $h = 3$, $\sigma_1 = 5$, $a = 2$, $\sigma = 10$, $y = 30$ and $x = 0$.

One - dimensional linear filtering example - stock prediction



Stock evolution in time.
Source: Kaggle - Two Sigma Financial Modelling Challenge, stock id 816.

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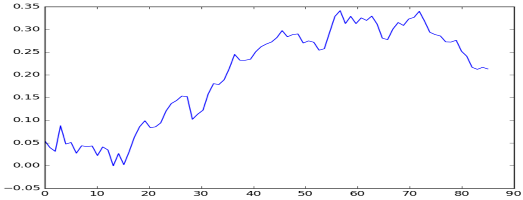

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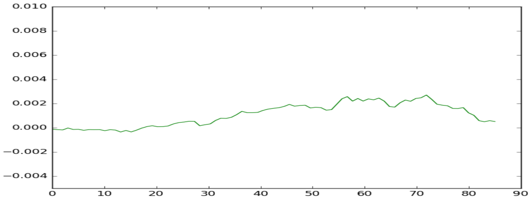

Slope of the stock modelled by Kalman bucy filter. Parameters of the modell: proces: $h = 0.01$ and $\sigma_1 = 5$, slope: $a = 2$ and $\sigma = 20$.

Theorem

Let us assume V *is a general separable Hilbert space, signal* $\{X_t, t \geq 0\}$ *is a general centered Gaussian process in* V*, the real - valued observation process* $\{Y_t, t \geq 0\}$ *is given as*

$$Y_t = \int_0^t A(s) X_s \, \mathrm{d}s + W_t, \; Y_0 = y,$$

where A *is a bounded linear operator from* V *to* $\mathbb{R}$ *and* $\{W_t, t \geq 0\}$ *is a real - valued Wiener process independent of the signal* X*. Let* $\{F_t^Y, t \geq 0\}$ *be the sigma algebra generated by observation process* Y*.*
Then the filter $\widehat{X}_t = \mathbb{E}[X_t | F_t^Y]$ *is given by stochastic integral equation:*

$$\widehat{X}_t = \int_0^t \Phi(t,s) A^*(s) \left(\mathrm{d}Y_s - (s) A(s) \widehat{X}_s \, \mathrm{d}s \right),$$

where for all $0 \leq s \leq t$

$$\Phi(t,s) = \mathbb{E}[X_t X_s^*] - \int_0^s \Phi(t,r) A^*(r) A(r) \Phi^*(s,r) \, \mathrm{d}r$$

and

$$\Phi(t,t) = \mathbb{E}[(X_t - \widehat{X}_t)(X_t - \widehat{X}_t)^*].$$

Gaussian signal with values in space of continuous functions

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- Heat equation with gaussian noise:

$$X = \{X(t,x), t \geq 0, x \in D\}$$

$$\mathrm{d}X(t,x) = \Delta X(t,x)dt + \mathrm{d}W_t, \quad \Delta X(t,x) = \sum_{i=1}^{n} \frac{\mathrm{d}^2 X(t,x)}{\mathrm{d}^2 x_i}$$

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- Musiela equation - forward rate curve evolution:

$$X = \{X(t,x), t \geq 0, x \geq 0\}$$

$$\mathrm{d}X(t,x) = \left(\frac{\mathrm{d}X(t,x)}{\mathrm{d}x} + \sigma(t,x) \int_0^x \sigma(t,y)\,\mathrm{d}y \right) dt + \sigma(t,x)\,\mathrm{d}W_t$$

Thank you for your attention

References.

[1] Kalman, R. E., & Bucy, R. S. (1961). *New results in linear filtering and prediction theory*. Journal of basic engineering, 83(1), 95-108.

[2] Kleptsyna, M. L., Kloeden, P. E., & Anh, V. V. (1998). *Linear filtering with fractional Brownian motion*. Stochastic Analysis and Applications, 16(5), 907-914.

[3] Kleptsyna, M. L., & Le Breton A. (2001). *Optimal linear filtering of general multidimensional Gaussian processes and its application to Laplace transforms of quadratic functionals*. Journal of Applied Mathematics and Stochastic Analysis, 14(3), 215-226.